Edexcel International A Level (IAL) Maths: Statistics 2

Sampling Distributions & Introduction to Hypothesis Testing

Your notes

Contents

Sampling Distributions

What key words do I need to know about sampling?

- The **population** refers to the **whole set** of things which you are interested in
  - For example: if a vet wanted to know how long a typical French bulldog slept for in a day then the population would be all the French bulldogs in the world
- A **sampling unit** is an individual member of the population
  - For example: a French bulldog would be a sampling unit in the above population
- A **sample** refers to a **subset of the population** which is used to collect data from
  - For example: the vet might take a sample of French bulldogs from different cities and record how long they sleep in a day
- A **sampling frame** is a **list** of all members of the **population**
  - For example: a list of employees' names within a company
- A **population parameter** is a **numerical value** which describes a **characteristic of the population**
  - These are usually unknown
  - For example: the mean (μ) height of all 16-year-olds in the UK
- A **sample statistic** is a **value** computed using **data from the sample**
  - These are used to estimate population parameters
  - For example: the mean ($\overline{X}$) height of 200 16-year-olds from randomly selected cities in the UK

What are the differences between a census and sampling?

- A **census** collects data about **all** the members of a **population**
  - For example: the Government in England does a national census every 10 years to collect data about every person living in England at the time
- The main advantage of a census is that it gives fully accurate results
- The disadvantages of a census are:
  - It is time consuming and expensive to carry out
  - It can destroy or use up all the members of a population when they are consumables (imagine a company testing every single firework)
- **Sampling** is used to collect data from a **subset of the population**

- The advantages of sampling are:
  - It is quicker and cheaper than a census
  - It leads to less data needing to be analysed
- The disadvantages of sampling are:
  - It might not represent the population accurately
  - It could introduce **bias**

What is a statistic and its sampling distribution?

- A **statistic** is a **random variable** that is calculated by **only** using the **data from a sample**
  - It **does not contain** any **unknown values** such as population parameters
  - $\frac{\Sigma x}{n} - \overline{X}^2$ would be a statistic as $\overline{X}$ is the mean of the sample
  - $\frac{\Sigma x}{n} - \mu^2$ would not be a statistic as μ is an unknown population parameter
- Common examples for a statistic include:
  - Sample mean, sample median, range of the sample, sample variance
- The **sampling distribution** of a statistic gives all the **possible values** of the statistic along with their **associated probabilities**

How do I find the sampling distribution of a statistic?

- **STEP 1**: **List all** the possible samples
  - Some samples can have the same combination of members but still list each one separately
    - AAB and ABA both contain two A's and one B
    - It can be helpful to group these together
  - Consider how many possible samples there are
    - If there are 2 different types of members in a population and you take a sample of 3 then there are 8 possible samples (2^3 or $2 \times 2 \times 2$)
- **STEP 2**: **Calculate** the **value** of the **statistic** for each possible sample
  - This is why grouping the samples with the same combination is helpful
- **STEP 3**: **Calculate** the **probability** of each sample being picked
  - The samples will be independent
  - If the population is described as being large then you can treat the sampling as if it were with replacement

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- The probability of selecting a type of member will be constant regardless of how many has already been selected

- **STEP 4**: **Construct** the **distribution** table
  - Add together the probabilities of the samples that result in the same value of the statistic

Worked Example

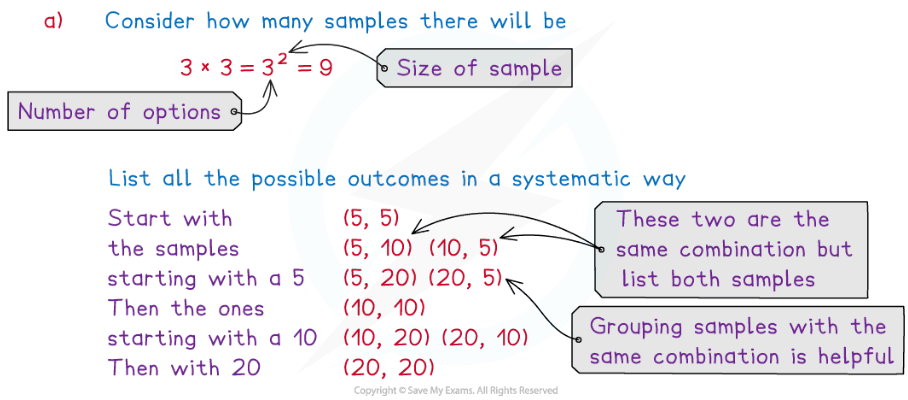
A money box contains a large number of £5, £10 and £20 notes. 50% are £5 notes, 30% are £10 notes and 20% are £20 notes.

A random sample of two notes is taken from the money box.

(a) List all the possible samples.

(b) Find the sampling distribution of the mean.

Answer:



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b) Let X be the value of the note

x	5	10	20
$P(X = x)$	0.5	0.3	0.2

Let $\bar{X}$ be the mean of the sample

Step 1 List all the samples	Step 2 Calculate the value of $\bar{X}$	Step 3 Calculate the probability of each sample
(5, 5)	$\frac{5+5}{2} = 5$	$0.5 \times 0.5 = 0.25$
(5, 10) (10, 5)	$\frac{5+10}{2} = 7.5$	$0.5 \times 0.3 = 0.15$
(5, 20) (20, 5)	$\frac{5+20}{2} = 12.5$	$0.5 \times 0.2 = 0.1$
(10, 10)	$\frac{10+10}{2} = 10$	$0.3 \times 0.3 = 0.09$
(10, 20) (20, 10)	$\frac{10+20}{2} = 15$	$0.3 \times 0.2 = 0.06$
(20, 20)	$\frac{20+20}{2} = 20$	$0.2 \times 0.2 = 0.04$

Step 4

Combine into a table

$\bar{x}$	5	7.5	10	12.5	15	20
$P(X = x)$	0.25	0.3	0.09	0.2	0.12	0.04

2 samples so 2×0.15

2 samples so 2×0.1

2 samples so 2×0.06

Check $= 0.25 + 0.3 + 0.09 + 0.2 + 0.12 + 0.04 = 1$

Examiner Tips and Tricks

- Remember the probabilities in a sampling distribution should add up to 1. You can use this to either check your answer or as a shortcut to find the final probability once you have the others.

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Language of Hypothesis Testing

What is a hypothesis test?

- A hypothesis test uses a **sample of data** in an experiment to test a **statement** made about the value of a **population parameter**
- A hypothesis test is used when the value of the assumed population parameter is questioned
- The hypothesis test will look at the which outcomes are unlikely to occur if assumed population parameter is true
- The probability found will be compared against a given **significance level** to determine whether there is **evidence to believe** that the assumed population parameter is not true

What are the key terms used in statistical hypothesis testing?

- Every hypothesis test **must** begin with a clear **null hypothesis** (what we believe to already be true) and **alternative hypothesis** (how we believe the data pattern or probability distribution might have changed)
- A **hypothesis** is an assumption that is made about a particular **population parameter**
  - A **population parameter** is a numerical characteristic which helps define a population
    - One example of a population parameter is the **probability, p** of an event occurring
    - Another example is the mean of a population
  - The **null hypothesis** is denoted **H_0** and sets out the assumed population parameter given that no change has happened
  - The **alternative hypothesis** is denoted **H_1** and sets out how we think the population parameter could have changed
  - When a hypothesis test is carried out, the null hypothesis is **assumed to be true** and this assumption will either be **accepted** or **rejected**
- A hypothesis test could be a **one-tailed** test or a **two-tailed** test
- The null hypothesis will always be **$H_0 : \theta = \ldots$**
- The alternate hypothesis will depend on if it is a one-tailed or two-tailed test
  - A one-tailed test would test to see if the **population parameter, θ,** has **either increased or decreased**
    - The **alternative hypothesis, H_1 will be $H_1 : \theta > \ldots$ or $H_1 : \theta < \ldots$**
  - A two-tailed test would test to see if the **population parameter, θ**, has **changed**

    - The **alternative hypothesis, H_1 will be** $H_1 : \theta \neq \ldots$
  - It is important to read the wording of the question carefully to decide whether your hypothesis test should be one-tailed or two-tailed
- To carry out a hypothesis test an experiment will be carried out on a **sample of data**, the result of this experiment will be the **test statistic**
  - A **sample of data** is a subset of data taken from the population
  - The **test statistic** is a numerical value calculated from the data
    - The test statistic is sometimes also referred to as the **observed value**
- A hypothesis test will always be carried out at an appropriate **significance level**
  - The significance level sets the **smallest probability** that an event could have occurred by chance.
    - Any probability smaller than the significance level would suggest that the event is unlikely to have happened by chance
  - The **significance level** must be set **before** the hypothesis test is carried out
  - The significance level will usually be 1%, 5% or 10%, however it may vary

Worked Example

A hypothesis test is carried out at the 5% level of significance to test if a normal coin is fair or not.

(i) Describe what the population parameter could be for the hypothesis test.

(ii) State whether the hypothesis test should be a one-tailed test or a two-tailed test, give a reason for your answer.

(iii) Clearly defining your population parameter, state suitable null and alternative hypotheses for the test.

Answer:

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(i) The population parameter, p, could be the probability that the coin lands on heads

Equally, this could have been the probability that the coin lands on tails.

(ii) The test should be a two-tailed test because we are testing to see if the probability that the coin lands on heads has changed

If tails had been chosen in Part (i), then tails should be used here.

(iii) The population parameter, p, is the probability the coin lands on heads

The null hypothesis is what we believe should be true:

$H_0: p = \frac{1}{2}$

Probability the coin lands on heads

The alternative hypothesis is how we believe the population parameter could have changed.

$H_1: p \neq \frac{1}{2}$

The probability the coin lands on heads is not $\frac{1}{2}$

Alternative hypothesis

$H_0: p = \frac{1}{2}$

$H_1: p \neq \frac{1}{2}$

Examiner Tips and Tricks

- Make sure you read the question carefully to determine whether the test you are carrying out is for a one-tailed or a two-tailed test.

Critical Regions

How do we decide whether to reject or accept the null hypothesis?

- The null hypothesis would be rejected if the **observed value** falls **within the critical region**
  - The **critical region** is the range of values that the observed value could take which will lead to the **null hypothesis** being **rejected**
- The **critical value** is the boundary of the critical region
  - It is the least extreme value that would lead to the rejection of the null hypothesis

- The **critical value** is determined by the **significance level**
- In a **two-tailed test** the significance level is halved and both the upper and the lower tails are tested
- For **discrete distributions** the critical value is the first value that falls within the critical region and so the probability of the observed value falling within the critical region may be lower than the given significance level
    - This probability will be known as the **actual significance level**
    - The actual significance level is the probability of incorrectly rejecting the null hypothesis
- Finding the critical region will be different for a **two-tailed test** than it is for a **one-tailed test**

- For an $\alpha\%$ significance level
    - In a one-tailed test the critical region will consist of $\alpha\%$ in the tail that is being tested for
    - In a two-tailed test the critical region will consist of $\frac{\alpha}{2}\%$ in each tail

Do we always need to find the critical region?

- In most cases the best method of conducting a hypothesis test is to find the critical region
    - It allows you to see how far the observed value is from the critical value and make decisions about whether further testing is necessary
- In some cases a hypothesis test can be carried out without finding the critical region
- The null hypothesis would be rejected if the probability of a value being **at least as extreme** as the observed value, assuming that the null hypothesis is true, is **less than the significance level**
    - If the test is looking for a decrease then extreme values are smaller than the observed value, so find the probability of less than or equal to the observed value
    - If the test is looking for an increase then extreme values are bigger than the observed value, so find the probability of greater than or equal to the observed value
    - This probability is called the "**p-value**"
- In a **two-tailed test** it is common to half the significance level and compare this with the probability found in one of the tails

Worked Example

Your notes

For the following situations, state at the 1% and 5% significance levels whether the null hypothesis should be rejected or not.

(i) The critical region is $X \leq 3$ and the observed value is 4.

(ii) Assuming the null hypothesis is true, the probability of a value being at least as extreme as the test statistic in a one-tailed hypothesis test is 0.0429.

(iii) Assuming the null hypothesis is true, the probability of a value being at least as extreme as the test statistic in a two-tailed hypothesis test is 0.00705.

Answer:

(i) The observed value is outside of the critical region so do not reject Ho

(ii) For a one-tailed test the significance level is all in one tail

0.0429 < 5% so reject Ho at the 5% level of significance

0.0429 > 1% so do not reject Ho at the 1% level of significance

(iii) For a two-tailed test use half of the significance level in each tail

5% significance level → 2.5% either side

1% significance level → 0.5% either side

0.00705 < 2.5% so reject Ho at the 5% level of significance

0.00705 > 0.5% so do not reject Ho at the 1% level of significance

Conclusions of Hypothesis Testing

How is a hypothesis test carried out?

- There are a number of ways that a hypothesis test can be carried out for different models, however the following steps should form the base for your test:
- **Step 1.** Define the test statistic and population parameter
- **Step 2.** Write the null and alternative hypotheses clearly
- **Step 3.** Calculate the **critical value(s)** or the necessary probability for the test
- **Step 4.** Compare the observed value with the critical value(s) or the probability with the significance level
- **Step 5.** Decide whether there is enough evidence to reject H_0 or whether it has to be accepted
- **Step 6.** Write a conclusion in context

How should a conclusion be written for a hypothesis test?

- Your conclusion **must** be written in the context of the question
- Use the wording in the question to help you write your conclusion

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- If rejecting the null hypothesis your conclusion should state that there is **sufficient evidence to suggest** the alternative hypothesis is true at this level of significance
- If accepting the null hypothesis your conclusion should state that there is **not enough evidence** to suggest the alternative hypothesis is true at this level of significance

- Your conclusion **must not** be definitive
  - There is a chance that the test has led to an incorrect conclusion
  - The outcome is dependent on the sample, a different sample might lead to a different outcome
- The conclusion of a two-tailed test can state if there is evidence of a change
  - You should not state whether this change is an increase or decrease

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Worked Example

A teacher carried out a hypothesis test at the 10% significance level to test if her students perform better in exams after using a new revision technique. Under the null hypothesis she calculates the probability that a value will be at least as extreme as the observed value to be 0.09142. Write a conclusion for her hypothesis test.

Answer:

The teacher is testing for an increase in exam scores so the test is a one-tailed test.

For a one-tailed test the significance level is all in one tail.

$0.09142 < 10\%$ so reject H_0 at the 10% level of significance

There is sufficient evidence to suggest at the 10% level of significance that the students did perform better after using the new revision technique.

Examiner Tips and Tricks

- It is best to use the exact wording from the question when writing your conclusion for the hypothesis test, do not be afraid to sound repetitive.