# Edexcel International A Level (IAL) Maths: Statistics 2

Your notes

## Continuous Random Variables

### Contents

Your notes

# Calculating Probabilities using PDF

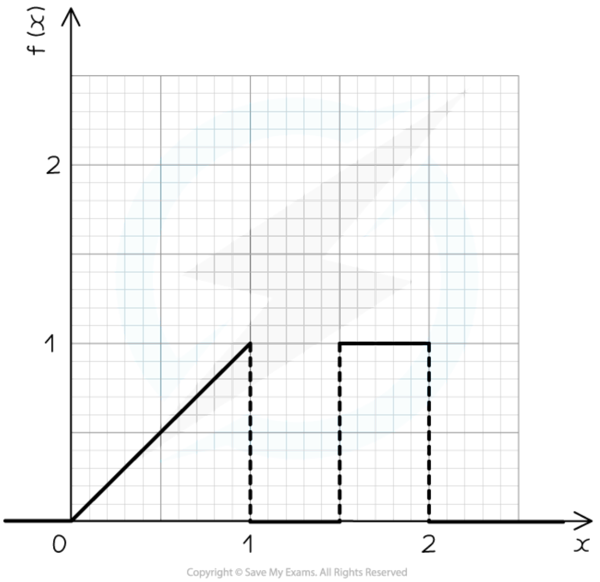
## What is a probability density function (p.d.f.)?

- For a **continuous random variable**, it is often possible to model probabilities using a function
  - This function is called a **probability density function (p.d.f.)**
  - For the continuous random variable, $X$ , it would usually be denoted as a function of x (such as f(x) or g(x) ) and is usually given **piecewise**

e.g.

$$\mathrm{f}(x)=\begin{cases} x & 0 \le x \le 1 \\ 1 & 1.5 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

- f(x) should be defined for all values of $x \in \mathbb{R}$
- The distribution (or **density**) of probabilities can be illustrated by the graph of f(x)
- The graph does not need to start and end on the x-axis
- The graph does not have to be continuous




- For f(x) to **represent** a p.d.f. the following conditions must apply
  - $\mathrm{f}(x) \ge 0$ for **all** values of x
    - This is the equivalent to $P(X=x) \ge 0$ for a discrete random variable

- The **area under the graph** must total 1

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

- This is equivalent to $\sum P(X = x) = 1$ for a discrete random variable

## How do I find probabilities using a probability density function (p.d.f.)?

- The probability that the continuous random variable *X* lies in the interval $a \le X \le b$, where *X* has the probability density function f(x) , is given by

$$P(a \le X \le b) = \int_{a}^{b} f(x)\, dx$$

- As with the **normal distribution** $P(a \le X \le b) = P(a < X < b)$
  - for any continuous random variable, $P(X = n) = 0$ for all values of n
  - One way to think of this is that $a = b$ in the integral above
- **Piecewise Function** are often used as the p.d.f. is not often a single function of x
  - Finding a probability may involve splitting the area across more than one piece of the function
  - This will depend on the limits a and b in $P(a \le X \le b)$ that is being found

## How do I solve problems using the PDF?

- Some questions may ask for justification of the use of a given function for a probability density function
  - In such cases check that the function meets the two conditions $f(x) \ge 0$ for all values of $X$ and the total area under the graph is 1
- If asked to find a probability
  - STEP 1
    As the **probability density function**, f(x), is usually given **piecewise** make sure you are clear about the values of x for which each part applies

e.g. $f(x) = \begin{cases} x - 1 & 1 \le x \le 2 \\ 3 - x & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$

- STEP 2

If simple to do so, **sketching** the **graph** of y= f(x) may help to find the probability

- Look for **basic shapes** such as **triangles** or **rectangles**; finding **areas** of these is easy and avoids integration
- Look for **symmetry** in the graph that may make the problem easier

Your notes

- STEP 3
  - Identify the **range** of X, particularly noting if it is split across different parts of the p.d.f.
  - Find the required **area (probability)**, either by **basic shapes** or **integrate f(x)** and evaluate it between the two limits, splitting if necessary
- Trickier problems may involve finding a limit of the integral given its value
  - i.e. one of the values in the range of X, given the probability
    e.g. Find the value of given $\mathrm{P}(0 \le X \le a) = 0.09$

Your notes

### Worked Example

The continuous random variable, $X$, has probability density function

$$\mathrm{f}(x) = \begin{cases} 0.4(x-1) & 1 \le x \le 2 \\ 0.1(6-x) & 2 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $\mathrm{f}(x)$ can represent a probability density function.

(b) Find

(i) $\mathrm{P}(X > 4)$

(ii) $\mathrm{P}(X = 3.2)$

(iii) $\mathrm{P}(1.5 \le X \le 2.5)$

**Answer:**

a) There are two conditions for a function to be a p.d.f.

$$\int_{-\infty}^{\infty} \mathrm{f}(x)\,dx = \int_{1}^{2} 0.4(x-1)\,dx + \int_{2}^{6} 0.1(6-x)\,dx$$

$$= 0.4\,[0.5x^2 - x]_1^2 + 0.1\,[6x - 0.5x^2]_2^6$$

$$= 0.4\,(0 - -0.5) + 0.1\,(18 - 10)$$

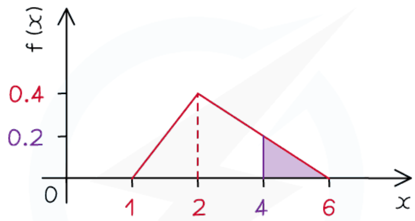
$$= 0.2 + 0.8$$

$$\therefore \int_{-\infty}^{\infty} \mathrm{f}(x)\,dx = 1$$

Also, $\mathrm{f}(x) \geq 0$ for all values of $x$ so $\mathrm{f}(x)$ meets both conditions to represent a probability density function.

b) STEP 1 p.d.f. is given piecewise with
$1 \leqslant x \leqslant 2$ and $2 \leqslant x \leqslant 6$

STEP 2 Graph is two straight lines, so sketch




(i) STEP 3 Identify the range: $> 4$

This is a triangle so use "$\frac{1}{2}$ bh"

$$P(X > 4) = \frac{1}{2} \times 2 \times 0.2$$

$\boxed{1 - P(X < 4) \text{ is also valid but would require more work!}}$

$$P(X > 4) = 0.2$$



(ii) $P(X = 3.2) = 0$

$\boxed{P(X = n) = 0 \text{ for all values of } n}$

(iii) STEP 3 Range is 1.5 – 2.5 so is split

Shapes are trapeziums, so could use "$\frac{1}{2}h(a + b)$"

but not especially easier than integration

$$P(1.5 \leqslant X \leqslant 2.5) = \int_{1.5}^{2} 0.4(x - 1)\,dx + \int_{2}^{2.5} 0.1(6 - x)\,dx$$

$$= 0.4\left[0.5x^2 - x\right]_{1.5}^{2} + 0.1\left[6x - 0.5x^2\right]_{2}^{2.5}$$

$$= 0.4(0 - -0.375) + 0.1(11.875 - 10)$$

$$= 0.15 + 0.1875$$

$$P(1.5 \leqslant X \leqslant 2.5) = 0.3375$$



## Examiner Tips and Tricks

- If the graph is easy to draw, then a sketch of f(x) is helpful
  - Some p.d.f. graphs have **symmetry**, common shapes such as **triangles** or **rectangles** so areas are easier to find, avoiding the need for integration

Your notes

- Always keep an eye for probabilities that are split across different parts of a piecewise function

Your notes

# Median and Mode of a CRV

## What is meant by the median of a continuous random variable?

- The **median**, ***m***, of a continuous random variable, $X$, with **probability density function f(x)** is defined as the value of the **continuous random variable** $X$, such that

$$P(X < m) = P(X > m) = 0.5$$

- Since $P(X = m) = 0$ this can also be written as $P(X \le m) = P(X \ge m) = 0.5$
- If the p.d.f. is **symmetrical** (i.e. the graph of y = f(x) is symmetrical) then the median will be halfway between the lower and upper limits of x
  - In such cases the graph of y=f(x) has axis of symmetry in the line x = m

## How do I find the median of a continuous random variable?

- By solving one of the equations to find $m$

$$\int_{-\infty}^{m} f(x)\, dx = 0.5$$

and

$$\int_{m}^{\infty} f(x)\, dx = 0.5$$

- If the graph of $y = f(x)$ is **symmetrical**, symmetry may be used to deduce the median
- For **piecewise functions**, you will need to determine which part of the function the median lies within to determine which equation to use
  - If there are more than two (non-zero) parts to a function then the integration may need splitting
- You can also use the **cumulative distribution function** to find the median

## How do I find quartiles (or percentiles) of a continuous random variable?

- In a similar way to finding the median
  - The lower quartile will be the value $L$ such that $P(X \le L) = 0.25$ or $P(X \ge L) = 0.75$
  - The upper quartile will be the value $U$ such that $P(X \le U) = 0.75$ or $P(X \ge U) = 0.25$
- Percentiles can be find in the same way

- The 15th percentile will be the value $k$ such that $P(X \le k) = 0.15$ or $P(X \ge k) = 0.85$

- In all cases start by determining which part(s) of the function are involved

## What is meant by the mode of a continuous random variable?

- The **mode** of a continuous random variable, $X$ , with **probability density function f(x)** is the value of x that produces the greatest value of f(x) .

## How do I find the mode of a PDF?

- This will depend on the type of function f(x); the easiest way to find the mode is by considering the shape of the graph of y= f(x)

- If the graph is a curve with a (local) **maximum point**, the mode can be found by **differentiating** and solving the equation f'(x) = 0
  - If there is more than one solution to f'(x) = 0 , further work may be needed to deduce which answer is the mode
    - Look for valid values of $X$ from the **definition** of the p.d.f.
    - Use the **second derivative** (f'' (x) ) to deduce the nature of each **stationary point**
    - You may need to check the values of f(x) at the **endpoints** too

### Worked Example

The continuous random variable $X$ has probability density function $\mathrm{f}(x)$ defined as

$$\mathrm{f}(x) = \begin{cases} \frac{1}{64}x(16 - x^2) & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

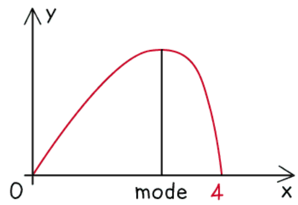
(a) Find the median of $X$, giving your answer to three significant figures

(b) Find the *exact* value of the mode of $X$

**Answer:**

Your notes

a) Sketch the graph




- Third root is $x = -4$
- Negative cubic:
- Not necessarily symmetrical
- (Local) maximum

For the median, solve "$\int_0^m f(x)\ dx = 0.5$"

$$\frac{1}{64}\int_0^m (16x - x^3)\,dx = \frac{1}{2}$$

$$\left[8x^2 - \frac{1}{4}x^4\right]_0^m = 32$$

$$8m^2 - \frac{1}{4}m^4 = 32$$

$$m^4 - 32m^2 + 128 = 0$$

This is a 'hidden quadratic' equation in $m^2$
Using the formula

$$m^2 = 16 \pm 8\sqrt{2}$$

$\therefore\ m = 5.226\ 251...$ Out of range

or $m = 2.164\ 764...$

$\therefore$ Median $m = 2.16$ (3sf)



b) Differentiate, solving $f'(x) = 0$ to find the mode

$$f'(x) = \frac{1}{64}(16 - 3x^2)$$

$$\therefore 16 - 3x^2 = 0$$

$$x^2 = \frac{16}{3}$$

$x = \pm\frac{4\sqrt{3}}{3}$ Reject $x = -\frac{4\sqrt{3}}{3}$ since $0 \leqslant x \leqslant 4$

$\therefore$ Mode $= \frac{4}{3}\sqrt{3}$ Clearly from sketch of graph $x = \frac{4}{3}\sqrt{3}$ is a (local) maximum



## Examiner Tips and Tricks

- Avoid spending too long sketching the graph of $y = f(x)$, only do this if the graph is straightforward as finding the median and mode by other means can be just as quick

Your notes

# E(X) & Var(X) (Continuous)

## What are E(X) and Var(X)?

- E(X)is the **expected value**, or **mean**, of a **random variable X**
  - E(X) is the same as the population mean so can also be denoted by $\mu$
- **Var (X)** is the **variance** of the continuous random variable $X$
  - **Standard deviation** is the **square root** of the **variance**

## How do I find the mean and variance of a continuous random variable?

- The **mean**, for a **continuous random variable** ***X*** is given by

$$\mathrm{E}(X) = \mu = \int_{-\infty}^{\infty} x\mathrm{f}(x)\ \mathrm{d}x$$

- This is equivalent to $\Sigma x\mathrm{P}(X = x)$ for discrete random variables
- If the graph of $y = \mathrm{f}(x)$ has **axis** of **symmetry,** x = a , then E(X) = a
- The **variance** is given by

$$\mathrm{Var}(X) = \sigma^2 = \int_{-\infty}^{\infty} x^2\mathrm{f}(x)\mathrm{d}x - \mu^2$$

- This is equivalent to $\Sigma x^2\mathrm{P}(X = x) - \mu^2$ for discrete random variables
- Be careful about confusing $E(X^2)$ and $[E(X)]^2$
  - $E(X^2) = \int_{-\infty}^{\infty} x^2\mathrm{f}(x)\ \mathrm{d}x$ “mean of the squares”
  - $[E(X)]^2 = \left[\int_{-\infty}^{\infty} x\ \mathrm{f}(x)\ \mathrm{d}x\right]^2$ “square of the mean”
  - If you are happy with the difference between these and how to calculate them the variance formula becomes very straightforward

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

## How do I calculate E(g(X))?

- $\mathrm{E}(g(X)) = \int_{-\infty}^{\infty} g(x)\mathrm{f}(x)\ \mathrm{d}x$

- In particular:
  - $\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 \mathrm{f}(x)\, \mathrm{d}x$ as seen above
- If $g(X) = aX + b$ (a linear function) then
  - $\mathrm{E}(g(X)) = \mathrm{E}(aX + b) = a\mathrm{E}(X) + b$
  - $\mathrm{Var}(g(X)) = \mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$

Your notes

## Worked Example

A continuous random variable, *X*, is modelled by the probability distribution function f(x), such that

$$\mathrm{f}(x) = \begin{cases} 1.5x^2(1 - 0.5x) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $\mathrm{E}(X)$.

(b) Find $\mathrm{Var}(X)$.

**Answer:**

(a) "$\mathrm{E}(x) = \mu = \int_{-\infty}^{\infty} x\mathrm{f}(x)\, \mathrm{d}x$"

$$\mu = \int_0^2 1.5x^3(1 - 0.5x)\, \mathrm{d}x$$

$$\mu = \int_0^2 (1.5x^3 - 0.75x^4)\, \mathrm{d}x$$

$$\mu = [0.375x^4 - 0.15x^5]_0^2$$

$$\mu = 6 - 4.8$$

$$\mu = 1.2$$

(b) "$Var(x) = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) - [\mu]^2$" or "$E(x^2) - [\mu]^2$"

$E(x^2) = \int_0^2 (1.5x^4 - 0.75x^5)\, dx$

$= [0.3x^5 - 0.125x^6]_0^2 = 1.6$

$\therefore \sigma^2 = E(x^2) - [\mu]^2$

$= 1.6 - (1.2)^2$

$\sigma^2 = 0.16$



Your notes

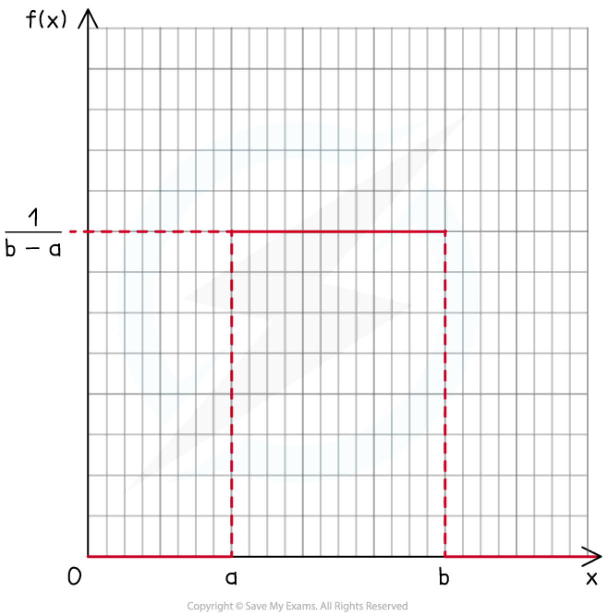
## Examiner Tips and Tricks

- A **sketch** of the graph of y = f(x) can highlight any **symmetrical** properties which can help reduce the work involved in finding the **mean** and **variance**
- Take care with awkward **values** and **negatives** – use the **memory** features on your calculator and **avoid rounding** until your **final** answer (if rounding at all!)
- The formulae for $\mathrm{E}(X) = \mu$ and $Var(X) = \sigma^2$ are given in the formulae booklet

# Continuous Uniform Distribution

## What is meant by the continuous uniform distribution?

- This is a special case of a probability density function for a continuous random variable
  - The **normal distribution** is another special case covered in S1
- The **uniform**, or **rectangular**, **distribution** is a p.d.f. that is **constant** and **non-zero** over a range of values but zero everywhere else




- Since the area under the graph has to total 1, the height of the uniform distribution would be

$$\frac{1}{b-a}$$

- Therefore the probability density function is given by

$$f(x)=\begin{cases}\dfrac{1}{b-a} & a\le x\le b\\ 0 & \text{otherwise}\end{cases}$$

## How do I find probabilities for a continuous uniform distribution?

- **Sketch** the graph of y= f(x)

Your notes

- Probabilities are the **area under** the **graph**, all such **areas** will now be **rectangles**
  - Finding the area of a rectangle is likely to be easier than integration!
- The **symmetrical** properties of rectangles may also be used to find probabilities

## How do I find the mean, median, mode and variance of a continuous uniform distribution?

- The **mean**, or expected value, is given by

$$\mathrm{E}(X) = \frac{1}{2}(\mathrm{a}+\mathrm{b})$$

- This is the (vertical) **axis** of **symmetry** of the **rectangle**
- Should the above be forgotten, $E(X) = \int_a^b x\mathrm{f}(x)\,\mathrm{d}x$ can still be applied
  - You be may asked to use this to prove the result
- The **median** can also be found by **symmetry** and will be **equal** to the **mean**
- There is **no mode** as f(x) is equal - and so at its greatest - for all values of x
- The **variance** is given by

$$\mathrm{Var}(X) = \frac{1}{12}(\mathrm{b}-\mathrm{a})^2$$

- Should the above be forgotten, $\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2\mathrm{f}(x)\,\mathrm{d}x - [E(X)]^2$ or $Var(X) = E(X^2) - [E(X)]^2$ can still be applied
  - You may be asked to use this to prove the result
  - The **standard deviation** is the **square root** of the **variance**

### Worked Example

A continuous random variable, $X$, is modelled by the uniform distribution such that $\mathrm{f}(x) = 0.4$ for $a \le x \le 4$ and $\mathrm{f}(x) = 0$ otherwise.
a is a constant.

(a) Show that the value of a is 1.5 .

(b) Find

(i) $\mathrm{P}(2.5 \le X \le 3)$

(ii) $E(X)$

(c) Find the standard deviation of $X$, giving your answer in the form $a\sqrt{3}$ , where a is a rational number.

**Answer:**

a) "$f(x) = \frac{1}{b-a}$" for all values of x

$$\frac{1}{4-a} = 0.4$$

$$1 = 0.4(4-a)$$

$$0.4a = 1.6 - 1$$

$$a = \frac{0.6}{0.4}$$

$$\therefore a = 1.5$$



b)(i) Shade your graph from earlier

$$P(2.5 \leqslant X \leqslant 3) = 0.5 \times 0.4 = 0.2$$

$$P(2.5 \leqslant X \leqslant 3) = 0.2$$

(ii) "$E(X) = \frac{1}{2}(a+b)$"

$$E(X) = \frac{1}{2}(1.5+4) = 2.75$$

$$E(X) = 2.75$$



c) "$Var(X) = \frac{1}{12}(b-a)^2$"

$$Var(X) = \frac{1}{12}(4-1.5)^2$$

$$Var(X) = \frac{6.25}{12} = \frac{25}{48}$$

$$\text{St. Dev.} = \sqrt{\frac{25}{48}} = \frac{5\sqrt{3}}{12}$$

Standard deviation is $\frac{5}{12}\sqrt{3}$



Your notes

## Examiner Tips and Tricks

- A **sketch** of the graph of a uniform distribution is quick and will highlight the **symmetry** in a uniform distribution
- Use **areas** of **rectangles** to find **probabilities** rather than integrating