Edexcel International A Level (IAL) Maths: Pure 4

Further Parametric Equations

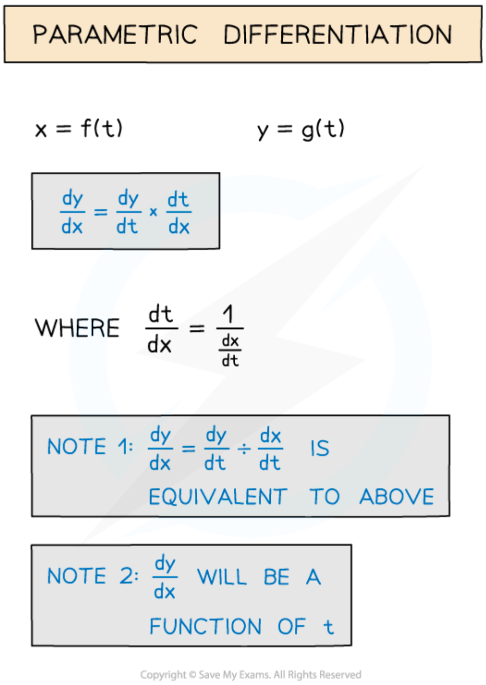
Your notes

Contents

Parametric differentiation

How do I find dy/dx from parametric equations?

- Ensure you are familiar with Parametric Equations – Basics first



- This method uses the chain rule and the reciprocal property of derivatives
    - $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$
    - $\frac{\mathrm{d}t}{\mathrm{d}x} = 1 \div \frac{\mathrm{d}x}{\mathrm{d}t}$
- Equivalently, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$
- $\frac{\mathrm{d}y}{\mathrm{d}x}$ will be in terms of *t* – this is fine
    - Questions usually involve finding gradients, tangents and normals

- The chain rule is always needed when there are three variables or more – see Connected Rates of Change

Your notes

How do I find gradients, tangents and normals from parametric equations?

To find a gradient

- **STEP 1**
  Find **dx/dt** and **dy/dt**
- **STEP 2**
  Find **dy/dx** in terms of **t**
  - Using either **dy/dx = dy/dt ÷ dx/dt**

    or **dy/dx = dy/dt × dt/dx** where **dt/dx = 1 ÷ dx/dt**
- **STEP 3**
  Find the value of **t** at the required point
- **STEP 4**
  Substitute this value of **t** into **dy/dx** to find the gradient

Your notes

e.g. FIND THE GRADIENT OF THE CURVE DEFINED PARAMETRICALLY BY

$$x = 2t - 3$$
$$y = t^3 - 4t + 1$$

AT THE POINT WHERE $x = 5$

STEP 1 FIND $\frac{dx}{dt}$ AND $\frac{dy}{dt}$

$$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = 3t^2 - 4$$

STEP 2 FIND $\frac{dy}{dt}$ (IN TERMS OF t)

$$\frac{dy}{dx} = \underbrace{(3t^2 - 4)}_{\frac{dy}{dt}} \times \underbrace{\frac{1}{2}}_{\frac{1}{\frac{dy}{dt}}}$$

$$\frac{dy}{dx} = \frac{1}{2}(3t^2 - 4)$$

STEP 3 FIND THE VALUE OF t AT THE REQUIRED POINT

AT $x = 5$, $\quad 5 = 2t - 3$

$t = 4$

STEP 4 SUBSTITUTE THIS VALUE OF t INTO $\frac{dy}{dx}$ TO FIND THE GRADIENT

AT $x = 5$, $t = 4$, $\quad \frac{dy}{dx} = \frac{1}{2}(3 \times 4^2 - 4)$

$$\frac{dy}{dx} = 22$$

To then go on to find the equation of a tangent

- **STEP 5**
  Find the **y** coordinate
- **STEP 6**
  Use the gradient and point to find the equation of the tangent

Your notes

e.g. FIND THE EQUATION OF THE TANGENT TO THE CURVE DEFINED PARAMETRICALLY BY

$$x = 2t - 3$$
$$y = t^3 - 4t + 1$$

AT THE POINT WHERE $x = 5$

TANGENT: "$y - y_1 = m(x - x_1)$"

$m = \frac{dy}{dx}$ AT $x = 5$

FROM STEP 4 → $m = 22$

STEP 5 FIND THE y−COORDINATE

TO FIND THE EQUATION OF A LINE WE NEED THE GRADIENT, m, AND A POINT (x_1, y_1) THE LINE PASSES THROUGH

AT $x = 5$, $t = 4$, $y = 4^3 - 4 \times 4 + 1 = 49$

FROM STEP 3

STEP 6 USE GRADIENT AND POINT TO FIND EQUATION OF TANGENT

$y - 49 = 22(x - 5)$

$y = 22x - 61$

To find a normal…

- **STEP 7**
  Use perpendicular lines property to find the gradient of the normal $m_1 \times m_2 = -1$
- **STEP 8**
  Use gradient and point to find the equation of the normal $y - y_1 = m(x - x_1)$

How do I solve problems using parametric differentiation?

- Questions may require use of tangents and normals as per the coordinate geometry sections
  - Find points of intersection between a tangent/normal and **x**/**y** axes
  - Find areas of basic shapes enclosed by axes and/or tangents/normals

Your notes

- Find stationary points (**dy/dx = 0**)

e.g. FIND THE STATIONARY POINTS ON THE CURVE DEFINED PARAMETRICALLY BY

$$x = 2t - 3$$
$$y = t^3 - 4t + 1$$

$$\frac{dy}{dx} = \frac{1}{2}(3t^2 - 4)$$ FROM EARLIER EXAMPLES

$$\frac{1}{2}(3t^2 - 4) = 0$$ $\frac{dy}{dx} = 0$ AT STARTING POINTS

$$t^2 = \frac{4}{3}$$

SAME AS $t = \pm\frac{2\sqrt{3}}{3}$

$$t = \pm\frac{2}{\sqrt{3}}$$

WORK OUT x & y FROM t

AT $t = \frac{2}{\sqrt{3}}$, $x = \frac{-9 + 4\sqrt{3}}{3}$ $y = \frac{9 - 16\sqrt{3}}{9}$

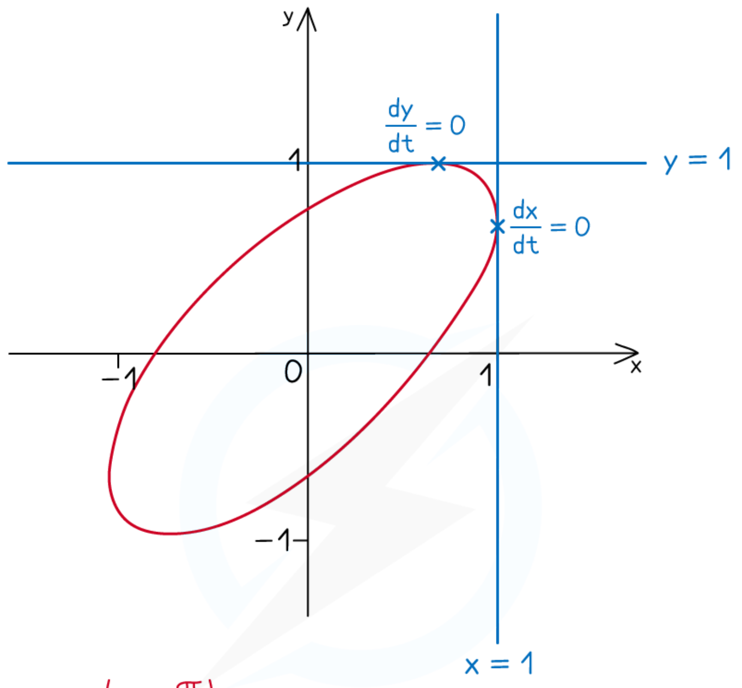
AT $t = \frac{-2}{\sqrt{3}}$, $x = \frac{-9 - 4\sqrt{3}}{3}$ $y = \frac{9 + 16\sqrt{3}}{9}$

CAREFUL WITH CALCULATION DISPLAY: $-\frac{9 + 4\sqrt{3}}{3}$

$\therefore$ THE COORDINATES OF THE STATIONARY POINTS ARE

$$\left(\frac{-9 + 4\sqrt{3}}{3}, \frac{9 - 16\sqrt{3}}{9}\right)$$ AND
$$\left(\frac{-9 - 4\sqrt{3}}{3}, \frac{9 + 16\sqrt{3}}{9}\right)$$

- You may also be asked about horizontal and vertical tangents
  - At horizontal (parallel to the **x**-axis) tangents, **dy/dt = 0**
  - At vertical (parallel to **y**-axis) tangents, **dx/dt = 0**



$x = \cos\left(2t - \frac{\pi}{4}\right)$

$y = \sin\left(2t + \frac{\pi}{2}\right)$

- $x = -1$ IS ALSO A VERTICAL TANGENT
- $y = -1$ IS ALSO A HORIZONTAL TANGENT

Worked Example

Your notes

Curve C is defined parametrically by the equations

$$x = 2\cos 3t \text{ and } y = \sin(3t + \frac{\pi}{3})$$

where $-\frac{\pi}{3} \leq t \leq 0$

The line l_1 is the normal to the curve at the point where $x = 1$.

(a) Find the equation of l_1

(b) Find the exact area of the triangle made by l_1 and the coordinate axes

a) l_1 : "$y - y_1 = m(x - x_1)$"

STEP 1 FIND $\frac{dx}{dt}$ AND $\frac{dy}{dt}$

$$\frac{dx}{dt} = -6\sin 3t$$

$$\frac{dy}{dt} = 3\cos\left(3t + \frac{\pi}{3}\right)$$

BOTH INVOLVE CHAIN RULE

STEP 2 FIND $\frac{dy}{dx}$ (IN TERMS OF t)

$$\frac{dy}{dx} = \frac{3\cos\left(3t + \frac{\pi}{3}\right)}{-6\sin 3t}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$= \frac{-\cos\left(3t + \frac{\pi}{3}\right)}{2\sin 3t}$$

STEP 3 FIND THE VALUE OF t AT $x = 1$

$$2\cos 3t = 1$$

$$\cos 3t = \frac{1}{2}$$

$$3t = -\frac{\pi}{3}$$

$$t = -\frac{\pi}{9}$$

EXTREME CARE

$$-\frac{\pi}{3} \leqslant t \leqslant 0$$

$$\therefore -\pi \leqslant 3t \leqslant 0$$

STEP 4 FIND $\frac{dy}{dx}$ AT $t = -\frac{\pi}{9}$

$$\frac{dy}{dx} = \frac{-\cos(0)}{2\sin\left(-\frac{\pi}{3}\right)} = \frac{\sqrt{3}}{3}$$

STEP 5 — FIND THE y-COORDINATE USING THE VALUE OF t

$$y = \sin\left(3 \times \left(-\frac{\pi}{9}\right) + \frac{\pi}{3}\right)$$

$$y = \sin(0) = 0$$

STEP 6 — "$m_1 \times m_2 = -1$"

GRADIENT OF NORMAL, $m = \dfrac{-1}{\frac{\sqrt{3}}{3}} = -\sqrt{3}$

STEP 7 — "$y - y_1 = m(x - x_1)$"

$$y - 0 = -\sqrt{3}(x - 1)$$

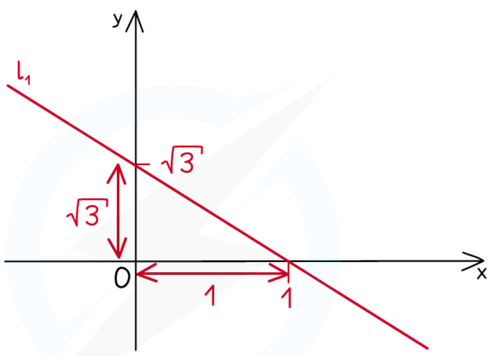
$$l_1 : y = -\sqrt{3}x + \sqrt{3}$$

b) $y = -\sqrt{3}x + \sqrt{3}$

FIND WHERE LINE INTERCEPTS AXES

AT $x = 0$, $y = \sqrt{3}$

AT $y = 0$, $x = 1$



A SKETCH MAKES THE AREA "EASY TO SEE"

$$\text{AREA} = \frac{1}{2} \times 1 \times \sqrt{3}$$

$$\text{AREA} = \frac{\sqrt{3}}{2} \text{ SQUARE UNITS}$$

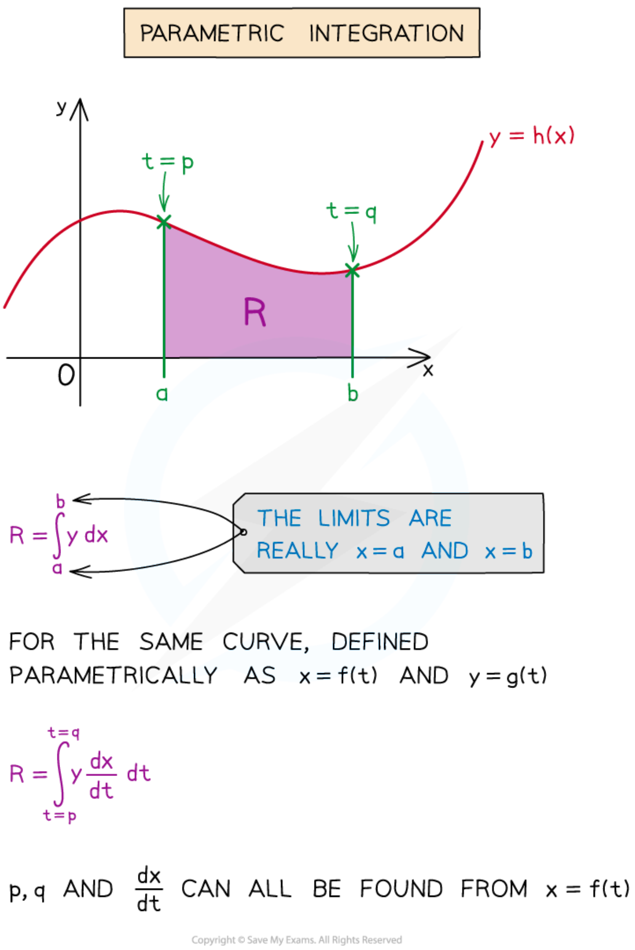
Your notes

Your notes

Parametric integration

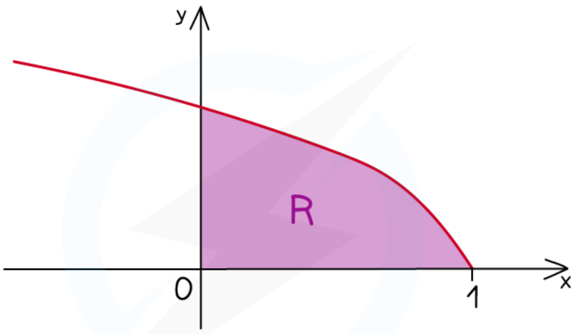
How does integration work with parametric equations?

- Integration is used to find the area under a curve where the curve has been defined by parametric equations
  - $x = f(t)$
  - $y = g(t)$



- The key point is to ensure the limits of the integral are changed to the parameter e.g. THE DIAGRAM BELOW SHOWS A SKETCH OF THE CURVE DEFINED PARAMETRICALLY BY

$$x = \cos 2\theta \qquad 0 < \theta < \pi$$
$$y = \sin\theta$$



FIND THE AREA MARKED R

STEP 1 | WRITE DOWN THE INTEGRAL IN GENERAL TERMS OF x AND y

$$R = \int_0^1 y \, dx$$

Your notes

Your notes

STEP 2 CHANGE BOTH LIMITS FROM x's TO THE PARAMETER

AT $x = 0$, $\cos 2\theta = 0$

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

AT $x = 1$, $\cos 2\theta = 1$

$$2\theta = 0$$

$$\theta = 0$$

STEP 3 FIND $\frac{dx}{d\theta}$

$$\frac{dx}{d\theta} = -2\sin 2\theta$$

STEP 4 PUT THE INTEGRAL IN TERMS OF THE PARAMETER TOGETHER

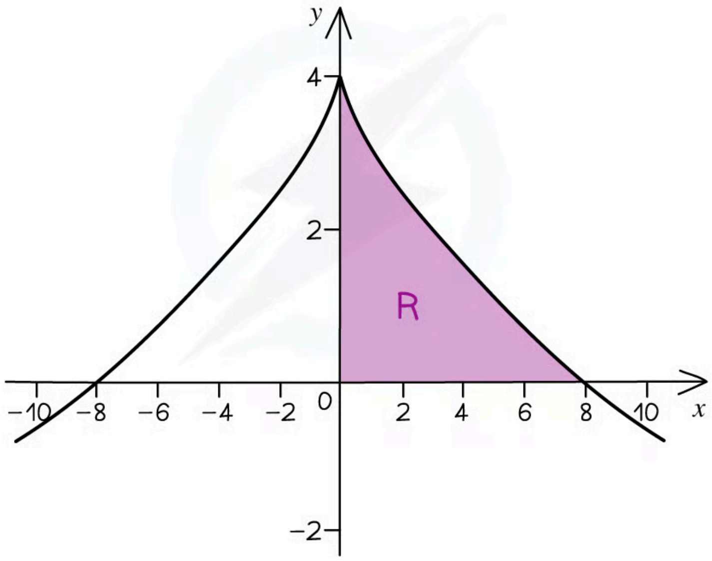
CAREFUL WITH LIMITS, KEEP THEM IN THE RIGHT ORDER

$$R = \int_{\frac{\pi}{4}}^{0} -2\sin\theta \sin 2\theta \, d\theta \qquad (*)$$

Worked Example

The sketch below shows the graph of a curve defined parametrically by

$$x = t^3 \text{ and } y = (4 - t^2)$$



Find the exact area of the region marked R.

Your notes

STEP 1 WRITE THE INTEGRAL IN TERMS OF x AND y

$$R = \int_0^8 y \, dx$$

STEP 2 CHANGE LIMITS FROM x TO t

AT $x = 0$, $t^3 = 0$
$t = 0$

AT $x = 8$, $t^3 = 8$
$t = 2$

STEP 3 FIND $\frac{dx}{dt}$

$$\frac{dx}{dt} = 3t^2$$

STEP 4 PUT THE INTEGRAL TOGETHER IN t

$$R = \int_0^2 (4 - t^2) \times 3t^2 dt$$

COULD USE CALCULATOR

$$R = \int_0^2 (12t^2 - 3t^4) dt$$

$$R = \left[4t^3 - \frac{3}{5}t^5\right]_0^2$$

$$R = \left(32 - \frac{96}{5}\right) - (0 - 0)$$

$$R = \frac{64}{5} \text{ SQUARE UNITS}$$

Parametric volumes of revolution

What is a parametric volume of revolution?

- **Solids** of **revolution** are formed by rotating functions about the x-axis
- Here though, rather than given y in terms of x, both x and y are given in terms of a parameter, t
  - $x = \mathrm{f}(t)$
  - $y = \mathrm{g}(t)$
  - Depending on the nature of the functions f and g it may not be convenient or possible to find y in terms of x

How do I find parametric volumes of revolution?

- The aim is to replace everything in the 'original' integral so that it is in terms of t
- For the 'original' integral $V = \pi \int_{x_1}^{x_2} y^2 \mathrm{d}x$ and parametric equations given in the form $x = \mathrm{f}(t)$ and $y = \mathrm{g}(t)$ use the following process
- **STEP 1:** Find dx in terms of t and dt
  - $\mathrm{d}x = \mathrm{f}'(t)\mathrm{d}t$
- **STEP 2:** If necessary, change the limits from x values to t values using
  - $x_1 = \mathrm{f}(t_1)$
  - $x_2 = \mathrm{f}(t_2)$
- **STEP 3:** Square y
  - $y^2 = [\mathrm{g}(t)]^2$
  - *Do this separately to avoid confusing when putting the integral together*
- **STEP 4:** Set up the integral, so everything is now in terms of t, simplify where possible and evaluate the integral to find the volume of revolution

$$V = \pi \int_{t_1}^{t_2} (\mathrm{g}(t))^2 \mathrm{f}'(t)\mathrm{d}t$$

Worked Example

The curve C is defined parametrically by $x = \sec t$ and $y = \sqrt{\operatorname{cosec} t}$.

C is rotated 360° about the x-axis between the values of $x = \sqrt{2}$ and $x = 2$.

Show that the volume of the solid of revolution generated by this rotation is $\pi(\sqrt{p} - q)$ cubic units where p and q are integers to be found.

STEP 1 FIND dx IN TERMS OF t AND dt

$$\frac{dx}{dt} = \sec t \tan t$$

$$dx = \sec t \tan t \, dt$$

STEP 2 IF NECCESSARY, CHANGE LIMITS

$x = \sqrt{2}$, $\sec t = \sqrt{2}$

$$t = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \quad \text{(RADIANS)}$$

$x = 2$, $\sec t = 2$

$$t = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

STEP 3 SQUARE y

$$y^2 = (\sqrt{\operatorname{cosec} t})^2 = \operatorname{cosec} t$$

STEP 4 SET UP INTEGRAL IN TERMS OF T, SIMPLIFY AND EVALUATE

$$V = \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\operatorname{cosec} t)(\sec t \tan t)\, dt$$

$$V = \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sin t} \times \frac{1}{\cos t} \times \frac{\sin t}{\cos t} \, dt$$

$$V = \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 t \, dt$$

$$V = \pi \left[\tan t\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$\therefore V = \pi [\sqrt{3} - 1] \quad (p=3, q=1)$$

Examiner Tips and Tricks

- Avoid the temptation to jump straight to STEP 4
    - There could be a lot to change and simplify in exam style problems
    - Doing each step carefully helps maintain high levels of accuracy

Your notes