Edexcel International A Level (IAL) Maths: Pure 4

Your notes

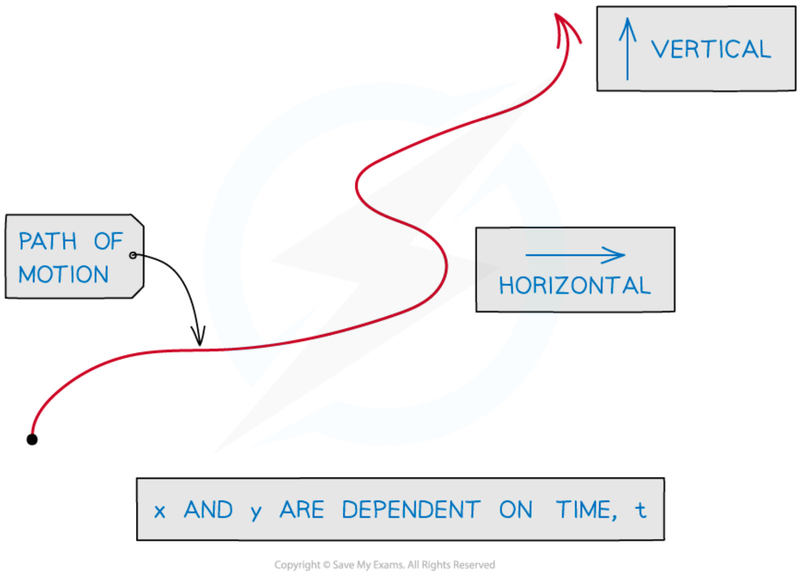
Parametric Equations

Contents

Basics of parametric equations

What are parametric equations?

- Graphs are usually described by a **Cartesian equation**
  - The equation involves **x** and **y** only
- Equations like this can sometimes be rearranged into the form, **y = f(x)**



- In **parametric equations** both **x** and **y** are dependent on a third variable
  - This is called a **parameter**
  - **t** and **θ** are often used as parameters
- A common example ...
  - **x** is the horizontal position of an object
  - **y** is the vertical position of an object
  - and the position of the object is dependent on time **t**
- **x** is a function of **t**, **y** is a function of **t**
  - **x = f(t)**
  - **y = g(t)**

How do I plot parametric equations?

- It is still possible to plot a graph of **y** against **x** from their parametric equations

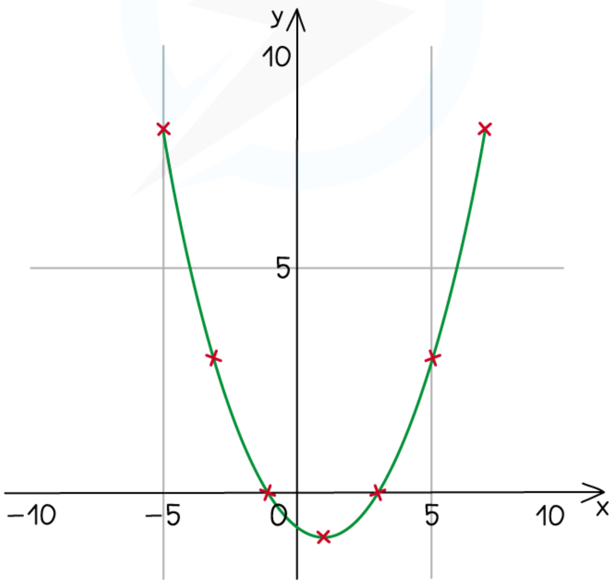
PLOTTING A GRAPH FROM PARAMETRIC EQUATIONS

e.g. PLOT THE GRAPH GIVEN BY THE PARAMETRIC EQUATIONS $x = 2t + 1$ AND $y = t^2 - 1$ FOR $-3 \leqslant t \leqslant 3$.

CONSTRUCT A TABLE OF VALUES

t	−3	−2	−1	0	1	2	3
x	−5	−3	−1	1	3	5	7
y	8	3	0	−1	0	3	8

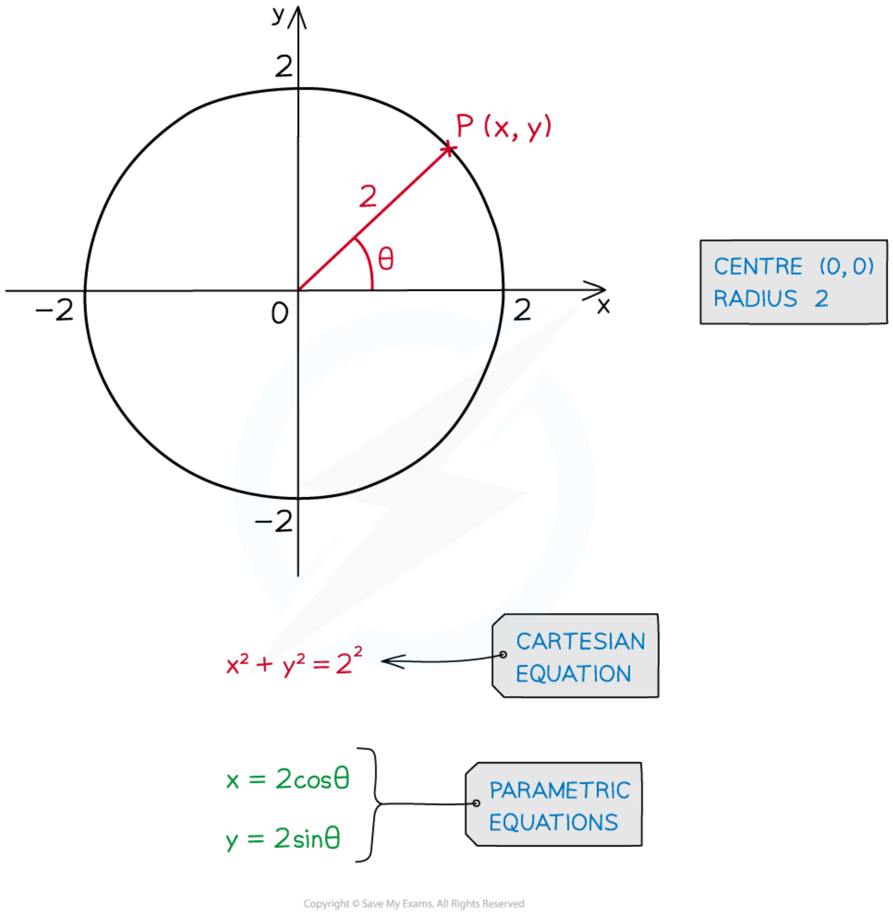
PLOT THE GRAPH OF y AGAINST x



- Also see Parametric Equations – Sketching Graphs

What are the parametric equations of a circle?

Your notes



- For a circle, centre (0, 0) and radius **r**
  - **$x = r\cos\theta$**
  - **$y = r\sin\theta$**
  - (Note that **r** is constant, this is not two parameters)
- For a circle, centre (**a**, **b**) and radius **r**
  - **$x = r\cos\theta + a$**
  - **$y = r\sin\theta + b$**

Worked Example

Your notes

(a) Write down the radius and the centre of the circle defined by the parametric equations

$$x = 3\cos\theta$$

and

$$y = 3\sin\theta - 4$$

(b) Hence write down the Cartesian equation of the circle

a) CENTRE: $(0, -4)$
RADIUS: 3

$x = r\cos\theta + a$
$y = r\sin\theta + b$

b) $x^2 + (y+4)^2 = 9$

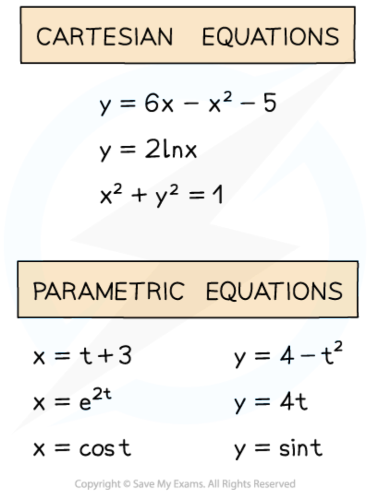
$r^2 = 3^2 = 9$

CENTRE $(0, -4)$

Your notes

Eliminating the parameter of parametric equations

What does eliminating the parameter mean?



- In parametric equations, **x = f(t)** and **y = g(t)**
- There is still a connection directly linking **x** and **y**
  - This will be the **Cartesian** equation of the graph

How do I find the Cartesian equation from parametric equations?

Your notes

e.g. FIND A CARTESIAN EQUATION FOR THE CURVE GIVEN PARAMETRICALLY AS

$$x = \frac{t-3}{4} \qquad y = e^{2t}$$

STEP 1 $\boxed{\text{REARRANGE ONE EQUATION TO MAKE } t \text{ THE SUBJECT}}$

$4x = t - 3$

$t = 4x + 3$

$\boxed{\text{BOTH EQUATIONS EASY TO REARRANGE}}$

STEP 2 $\boxed{\text{SUBSTITUTE INTO OTHER EQUATION}}$

$y = e^{2(4x+3)}$

STEP 3 $\boxed{\text{REARRANGE INTO CARTESIAN FORM}}$

$y = e^{8x+6}$

$\boxed{\text{NOT MUCH TO DO HERE!}}$

- **STEP 1**
  Rearrange one of the equations to make t the subject
  - Either **t = p(x)** or **t = q(y)**
- **STEP 2**
  Substitute into the other equation
- **STEP 3**
  Rearrange into the desired (Cartesian) form

How can I eliminate the parameter using trigonometric identities?

e.g. FIND A CARTESIAN EQUATION FOR THE CURVE GIVEN PARAMETRICALLY AS

$$x = 3 + \cos t \qquad y = 2 - \sin t$$

STEP 1 — REARRANGE BOTH EQUATIONS INTO THE FORM $\cos t = \ldots$ AND $\sin t = \ldots$

$$\cos t = x - 3$$

$$\sin t = 2 - y$$

STEP 2 — SQUARE BOTH SIDES OF BOTH EQUATIONS

$$\cos^2 t = (x-3)^2$$

$$\sin^2 t = (2-y)^2$$

STEP 3 — ADD THE EQUATIONS TOGETHER

$$\cos^2 t + \sin^2 t = (x-3)^2 + (2-y)^2$$

STEP 4 — USE THE TRIG IDENTITY "$\sin^2 x + \cos^2 x \equiv 1$" TO ELIMINATE t

$$1 = (x-3)^2 + (2-y)^2$$

STEP 5 — REARRANGED INTO DESIRED FORM

$$(x-3)^2 + (y-2)^2 = 1$$

$(y-2)^2 = (2-y)^2$

EQUATION OF A CIRCLE CENTRE: (3, 2) RADIUS: 1

- **STEP 1**
  Rearrange both equations into the forms "cos t = …" and "sin t = …"
- **STEP 2**
  Square both sides of both equations
- **STEP 3**
  Add the equations together
- **STEP 4**
  The trig identity "$\sin^2 x + \cos^2 x \equiv 1$" eliminates **t**

- **STEP 5**
  Rearrange into desired (Cartesian) form
  - This technique is seen in Trigonometric Identities

Your notes

Examiner Tips and Tricks

When choosing which equation to rearrange, aim for "as simple as possible":

- Linear equations are simpler than quadratics
  - eg Rearrange $\mathbf{x = 2t + 3}$

    or

    $\mathbf{y = 3t^2 + 3t - 4}$?
- Single exponential terms are quite easy to deal with
  - eg $\mathbf{x = e^t \rightarrow t = \ln x}$

Trig identities may be needed and remember **squared** terms are good!

- eg $\mathbf{\sin^2 x + \cos^2 x \equiv 1}$

Worked Example

Your notes

Find the Cartesian equation for the curve C defined by the parametric equations

$$x = 3\sin 2t$$

and

$$y = 2\cos 2t$$

- THIS IS A sin/cos BASED QUESTION
- AIM FOR "SQUARE AND ADD"

STEP 1 REARRANGE BOTH INTO FORM "sin t = ..." AND "cos t = ..."

$\frac{x}{3} = \sin 2t \qquad \frac{y}{2} = \cos 2t$

STEP 2 & 3 "SQUARE AND ADD"

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = \sin^2 2t + \cos^2 2t$$

STEP 4 ELIMINATE t

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

STEP 5 REARRANGE INTO DESIRED FORM

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

EQUATION OF AN ELIPSE

$$4x^2 + 9y^2 = 36$$

Sketching graphs of parametric equations

How do I sketch a graph from parametric equations?

- **Plotting** a graph is covered in Parametric Equations - Basics

e.g. SKETCH THE GRAPH OF y AGAINST x FOR THE PARAMETRIC EQUATIONS

$$x = 3t + 1 \qquad y = t^2 - 4$$

$3t + 1 = 0$

$t = -\frac{1}{3} \longrightarrow y = \left(-\frac{1}{3}\right)^2 - 4$

$\left(0, -\frac{35}{9}\right) \qquad y = -\frac{35}{9}$

y−AXIS INTERCEPT
$x = 0$

$t^2 - 4 = 0$

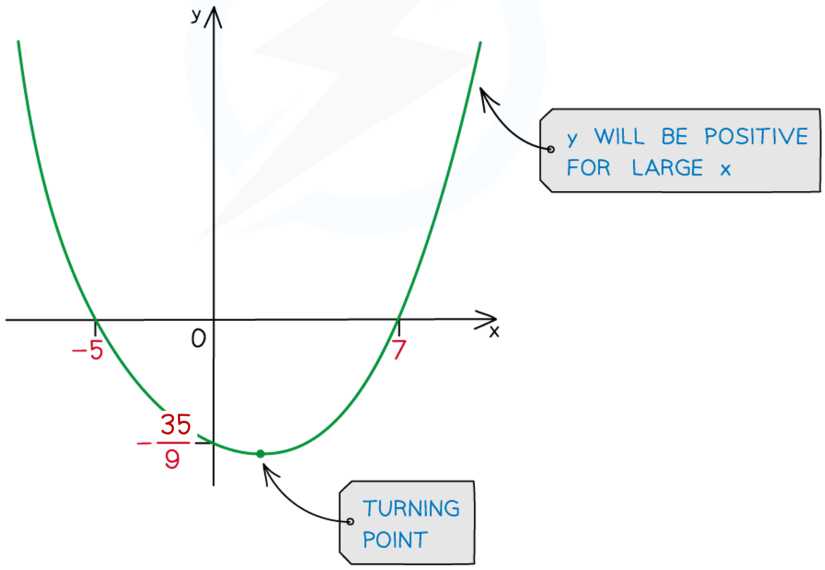
$t = \pm 2 \longrightarrow x = 3 \times 2 + 1 = 7$

$x = 3 \times -2 + 1 = -5$

$(-5, 0)$

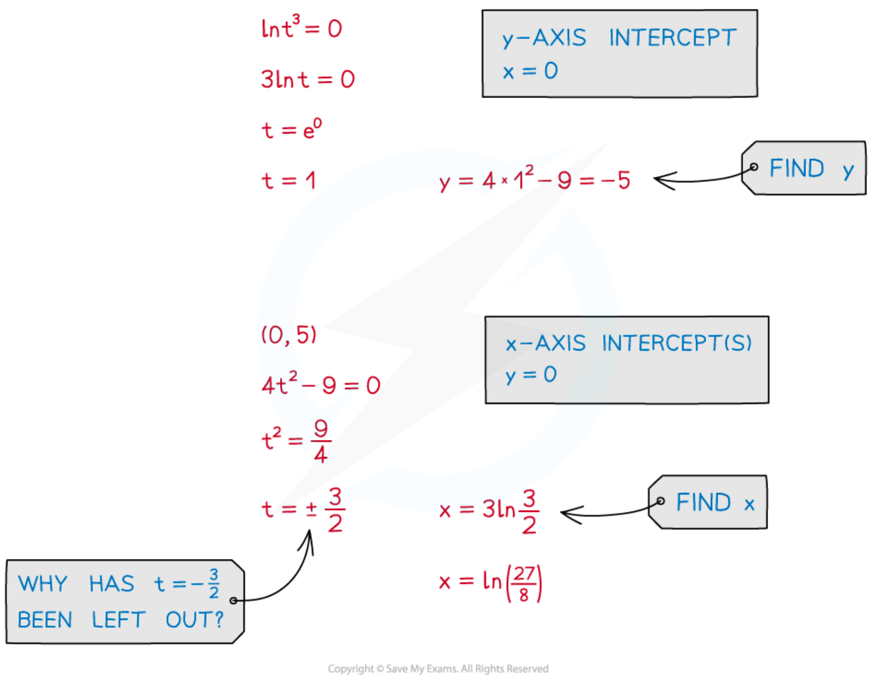
$(7, 0)$

x−AXIS INTERCEPT(S)
$y = 0$



IT IS NOT REQUIRED BUT THE CARTESIAN EQUATION IS $9y = x^2 - 2x - 35$ WHICH IS A QUADRATIC

- Still find the key features of a graph ...
- ... the **y**-axis intercept
    - ... the **x**-axis intercept(s)
    - ... asymptotes
    - ... location of (and if required coordinates of) stationary points (see Parametric Differentiation)
- **Sketch** these points and join up accordingly



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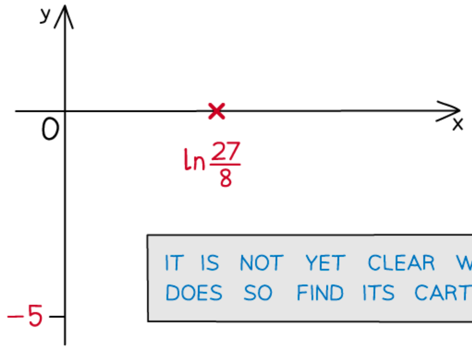


IT IS NOT YET CLEAR WHAT ELSE THE GRAPH DOES SO FIND ITS CARTESIAN EQUATION

$$x = \ln t^3$$

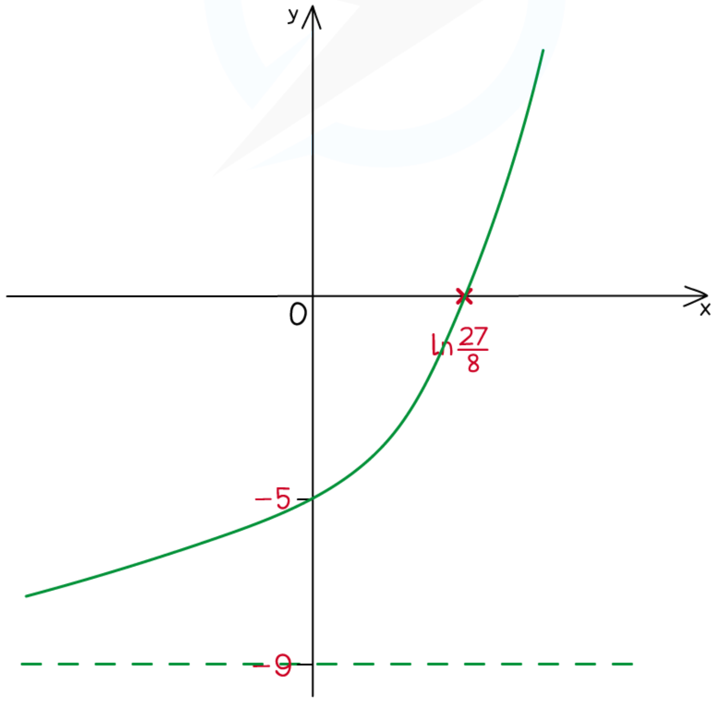
$$x = 3\ln t$$

$$\ln t = \frac{x}{3}$$

$$t = e^{\frac{x}{3}} \qquad y = 4\left(e^{\frac{x}{3}}\right)^2 - 9$$

$$y = 4e^{\frac{2}{3}x} - 9$$

A POSITIVE EXPONENTIAL GRAPH
∘ ASYMPTOTE AT $y = -9$



Your notes

Examiner Tips and Tricks

- Not all curves defined parametrically lead to familiar shaped graphs and it may be worth plotting a few extra points by calculating them.
- Your calculator may be able to produce a table of values quickly, however, remember you are sketching and not plotting.
- It may be easier to find the Cartesian equation first and draw the graph from that – this will depend on the question.
- It is only a **definite** strategy if you cannot make progress otherwise.
- If you are given the sketch of a graph it is usually only for reference and can be used to check answers for axes intercepts, etc.

Worked Example

Your notes

Sketch the graph of the curve defined by the parametric equations

$$x = e^t \text{ and } y = t^2 - t$$

stating clearly the coordinates of any points where the curve intercepts the coordinate axes.

DEDUCE $x > 0$ SO y-AXIS IS AN ASYMPTOTE

$x = e^t = 0$

NO SOLUTIONS

y-AXIS INTERCEPT

$x = 0$

$y = t^2 - t = 0$

$t(t-1) = 0$

$t = 0, \quad t = 1$

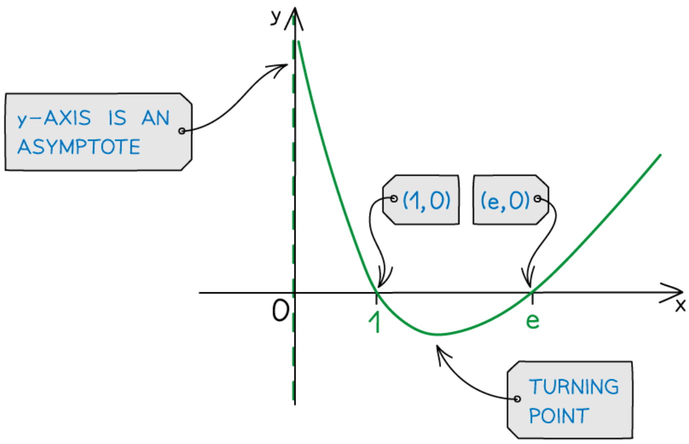
x-AXIS INTERCEPT(S)

$y = 0$

AT $t = 0, \quad x = e^0 = 1$

AT $t = 1, \quad x = e^1 = e$

THERE WILL BE A TURNING POINT BETWEEN $x = 1$ AND $x = e$



WHY IS THE GRAPH THIS WAY UP?

ANSWER: AS $t \to -\infty$, $x \to 0$, $y \to \infty$