Edexcel International A Level (IAL) Maths: Pure 4

Your notes

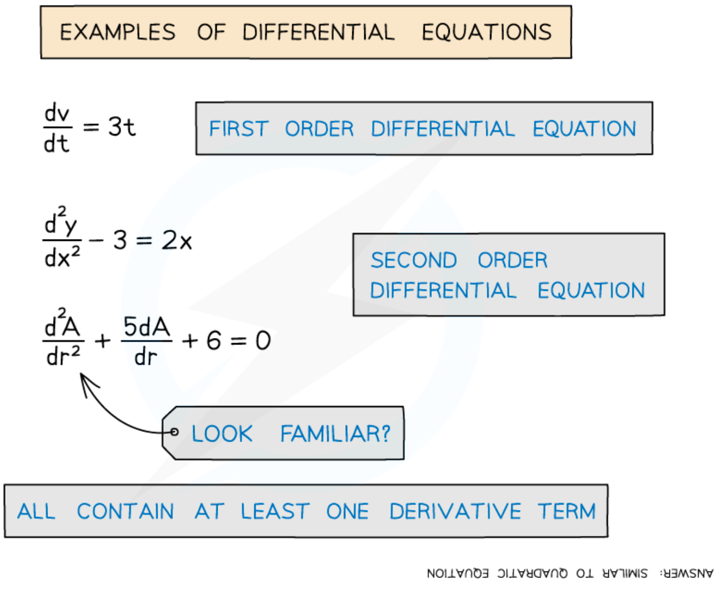
Differential Equations

Contents

Your notes

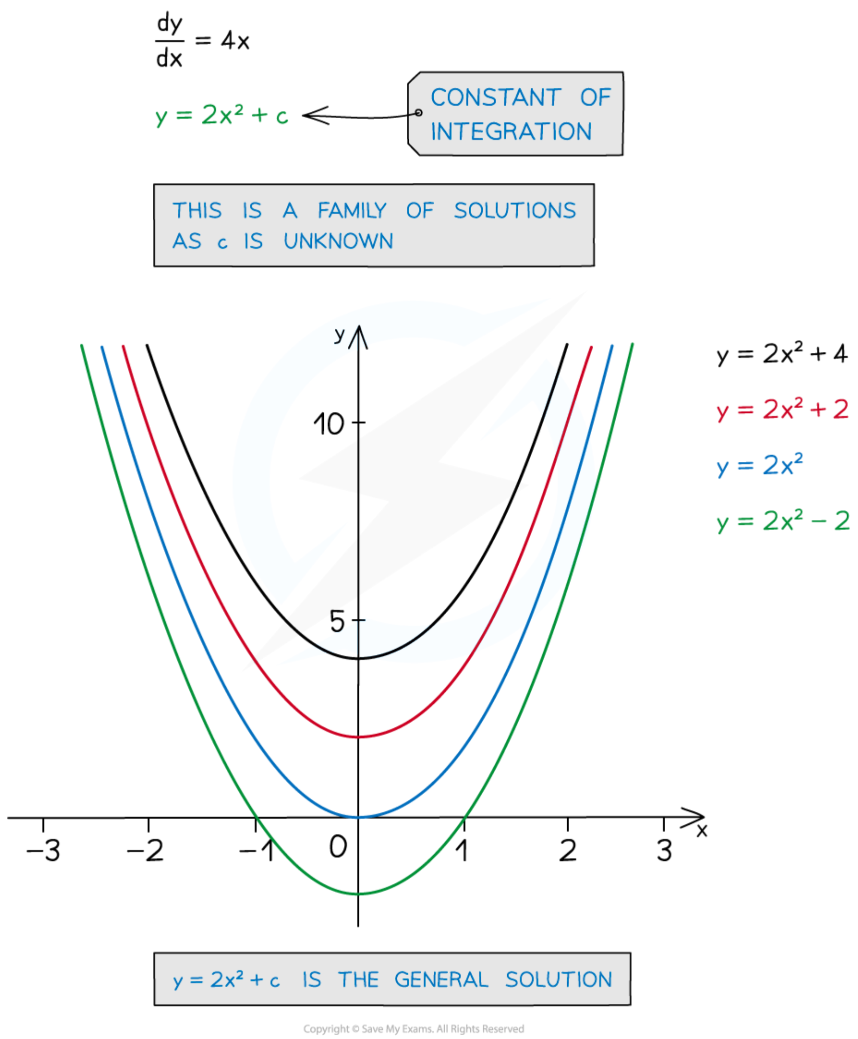
General solutions

What is a differential equation?



- Any equation, involving a **derivative term**, is a **differential equation**
- Equations involving only first derivative terms are called **first order differential equations**
- Equations involving second derivative terms are called **second order differential equations**

What is a general solution?



Your notes

- Integration will be involved in **solving** the differential equation
  - ie working back to "**y = f(x)**"
- A constant of integration, **c** is produced
- This gives an infinite number of solutions to the differential equation, each of the form **y = g(x) + c** (ie **y = f(x)** where **f(x) = g(x) + c**)
- These are often called a **family of solutions** ...
  - ... and the solution **y= g(x) + c** is called the **general solution**

Your notes

e.g. FIND THE GENERAL SOLUTION OF THE DIFFERENTIAL EQUATION

$$4 - \frac{dv}{dt} = 2$$

$$\frac{dv}{dt} = 2$$ REARRANGE

$$v = \int 2\,dt$$ THIS LINE IS NOT ESSENTIAL

$$v = 2t + c$$ INTEGRATE (BOTH SIDES)

GENERAL SOLUTION AS c UNKNOWN

Worked Example

Your notes

Find the general solution to the differential equation $\frac{d^2y}{dx^2} = 4$

$$\frac{d^2y}{dx^2} = 4$$

$$\frac{dy}{dx} = \int 4\,dx$$

INTEGRATE

$$\frac{dy}{dx} = 4x + c$$

... AND AGAIN

$$y = \int (4x + c)\,dx$$

NEITHER LINE ESSENTIAL

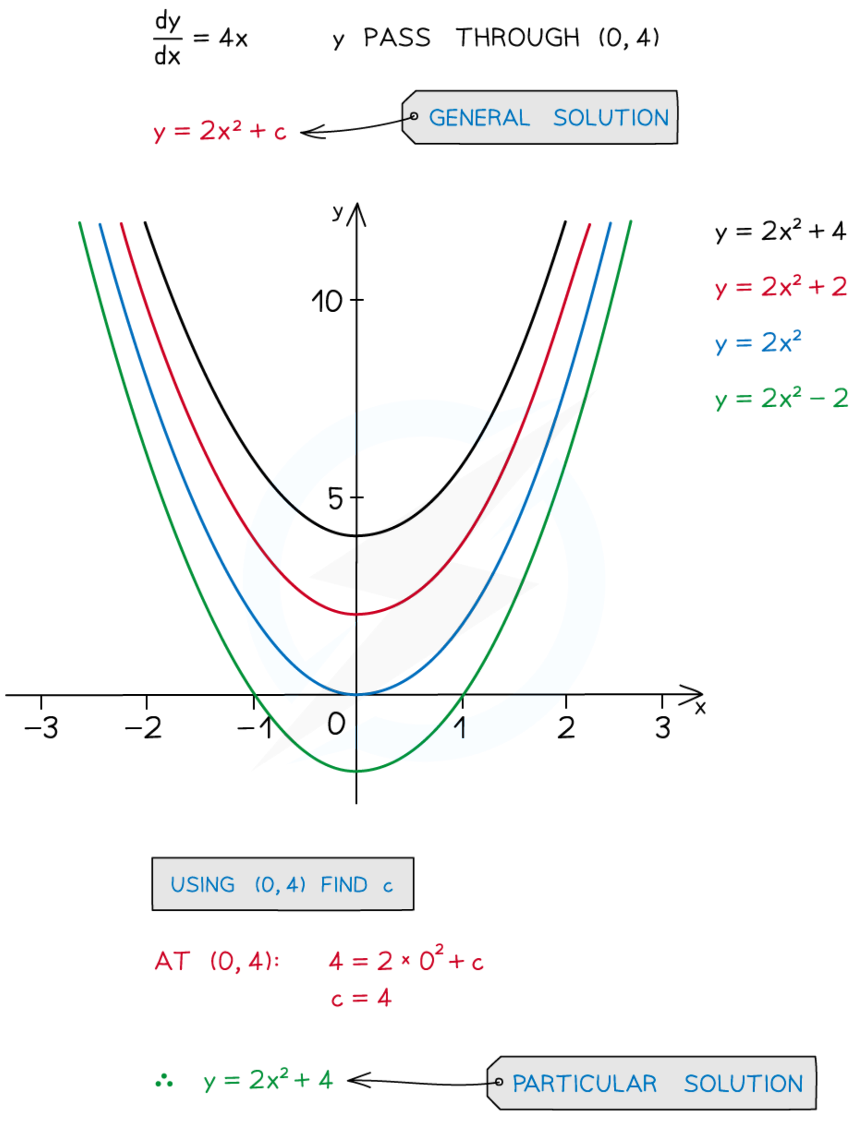
$$y = 2x^2 + cx + d$$

INTEGRATING TWICE WILL LEAD TO 2 UNKNOWN CONSTANTS OF INTEGRATION

Particular solutions

What is a particular solution?

- Ensure you are familiar with **General Solutions** first
- With extra information, the constant of integration, **c**, can be found
- This means the **particular solution** (from the family of solutions) can be found



What is a boundary condition and an initial condition?

Your notes

- A **boundary condition** is a piece of extra information that lets you find the particular solution
    - For example knowing **y = 4** when **x = 0** in the preceding example
    - In a model this could be a particle coming to rest after a certain time, ie **v = 0** at time **t**

e.g. FIND THE PARTICULAR SOLUTION OF THE DIFFERENTIAL EQUATION

$$4 - \frac{dv}{dt} = 2$$

GIVEN THAT $v = 0$ WHEN $t = 8$

$$\frac{dv}{dt} = 2$$

$$v = 2t + c$$

GENERAL SOLUTION

BOUNDARY CONDITION

AT $(8, 0)$: $0 = 2 \times 8 + c$

$c = -16$

$t = 0$ (NOT $v = 0$) WOULD BE AN INITIAL CONDITION

$\therefore$ PARTICULAR SOLUTION IS $v = 2t - 16$

- Differential equations are used in modelling, experiments and real-life situations
- A boundary condition is often called an **initial condition** when it gives the situation at the start of the model or experiment
    - This is often linked to time, so **t = 0**

Your notes

e.g. FIND THE PARTICULAR SOLUTION OF THE DIFFERENTIAL EQUATION

$$6 - \frac{dy}{dx} = 2x$$

GIVEN THAT AT $x = 0$, $y = 3$

$\frac{dy}{dx} = 6 - 2x$ ← REARRANGE

$y = \int (6 - 2x)\,dx$ ← (THIS LINE IS NOT ESSENTIAL)

$y = 6x - x^2 + c$ ← GENERAL SOLUTION

AT $(0,3)$: $3 = 6 \times 0 - 0^2 + c$

$c = 3$

INITIAL CONDITION

$\therefore$ PARTICULAR SOLUTION IS

$y = 6x - x^2 + 3$

- It is possible to have two boundary conditions
  - eg a particle initially at rest has velocity, $\mathbf{v = 0}$ and acceleration, $\mathbf{a = 0}$ at time, $\mathbf{t = 0}$
  - for a **second order** differential equation you need **two** boundary conditions to find the particular solution

Worked Example

Your notes

The velocity of a particle, initially at rest, is modelled by the differential equation

$$\frac{dv}{dt} = 2t - 2$$

where v is the velocity of the particle and t is time since the particle began moving.

(a) Find the velocity of the particle after 3 seconds

(b) According to this model when is the particle at rest (other than initially)?

a) $v = \int 2t - 2 \, dt$ INTEGRATE TO FIND v

$v = t^2 - 2t + c$ GENERAL SOLUTION

GIVEN IN WORDS "INITIALLY AT REST"

AT $t = 0$, $v = 0$, $\therefore c = 0$

$\therefore$ PARTICULAR SOLUTION IS $v = t^2 - 2t$

$\therefore$ AT $t = 3$, $v = 3^2 - 2 \times 3$

$v = 3 \text{ ms}^{-1}$

b) $v = 0$ GIVEN IN WORDS "AT REST"

$t^2 - 2t = 0$

$t(t-2) = 0$

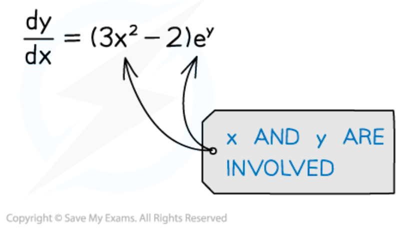
$\cancel{t = 0}$ INITIALLY $\quad t = 2$

THE PARTICLE IS AT REST AFTER 2 SECONDS

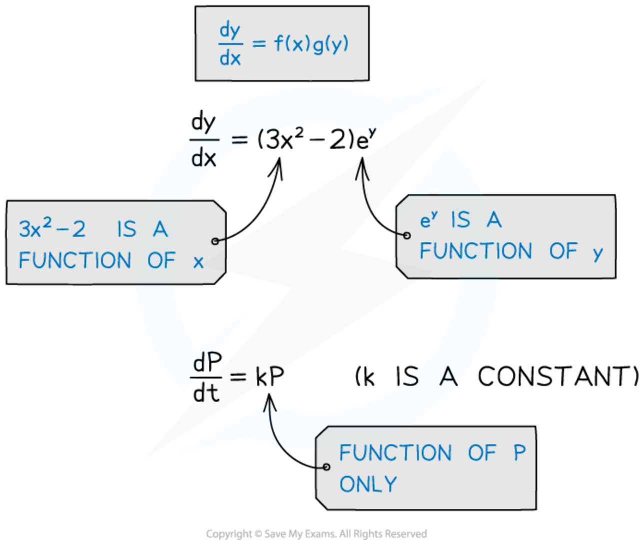
Your notes

Separation of variables

What does it mean for a differential equation to be separable?



- Many differential equations used in modelling have two variables involved (ie **x** and **y**)
- If there is a product of functions in different variables, the differential equation is separable
  - ie **dy/dx = f(x) × g(y)**



- Differential equations of the form **dy/dx= g(y)** should be thought of as **dy/dx= 1 × g(y)**
  - where f(x) = 1

How do I solve a differential equation using separation of variables?

SEPARATION OF VARIABLES

e.g. SOLVE $\frac{dy}{dx} = (3x^2 - 2)e^y$

GIVEN THAT $x = 1$ WHEN $y = 0$

STEP 1 — SEPARATE ALL y TERMS ON ONE SIDE AND ALL x TERMS ON THE OTHER

$\div e^y)$ $\quad \frac{1}{e^y}\frac{dy}{dx} = 3x^2 - 2$

$\times dx)$ $\quad e^{-y}dy = (3x^2 - 2)dx$

LHS HAS ONLY y TERMS

RHS HAS ONLY x TERMS

STEP 2 — INTEGRATE BOTH SIDES

$\int e^{-y}dy = \int (3x^2 - 2)dx$

$-e^{-y} = x^3 - 2x + c$

ONE "OVERALL" CONSTANT — STEP 3

STEP 4 — USE INITIAL/BOUNDARY CONDITION TO FIND THE PARTICULAR SOLUTION

AT $(1, 0)$ $\quad -e^0 = 1^3 - 2\times 1 + c$

$-1 = -1 + c$

$c = 0$

STEP 5 — WRITE THE PARTICULAR SOLUTION IN A SENSIBLE, OR REQUIRED, FORMAT

$-e^{-y} = x^3 - 2x$

$e^{-y} = 2x - x^3$

$y = -\ln(2x - x^3)$

- STEP 1: Separate all y terms on one side and all x terms on the other side
- STEP 2: Integrate both sides

- STEP 3: Include one “overall” constant of integration
- STEP 4: Use the initial or boundary condition to find the particular solution
- STEP 5: Write the particular solution in sensible, or required, format

Your notes

Worked Example

Your notes

Given that $A^2 = 20\pi + 4$ when $r = 2$ solve the differential equation $\dfrac{\mathrm{d}A}{\mathrm{d}r} = \dfrac{4\pi r + 2}{A}$

$$\frac{dA}{dr} = \frac{4\pi r + 2}{A}$$

g(r)

f(A)

STEP 1 SEPARATE VARIABLES (A AND r)

$A\,dA = (4\pi r + 2)\,dr$

STEP 2 INTEGRATE BOTH SIDES

$\int A\,dA = \int (4\pi r + 2)\,dr$

$\frac{A^2}{2} = 2\pi r^2 + 2r + c$

GENERAL SOLUTION

STEP 3 ONE "OVERALL" CONSTANT

STEP 4 USE THE BOUNDARY CONDITION

$20\pi + 4 = 4\pi \times 2^2 + 4 \times 2 + c$

$20\pi + 4 = 16\pi + 8 + c$

$c = 4\pi - 4$

STEP 5 WRITE THE PARTICULAR SOLUTION IN A SENSIBLE, OR REQUIRED, FORMAT

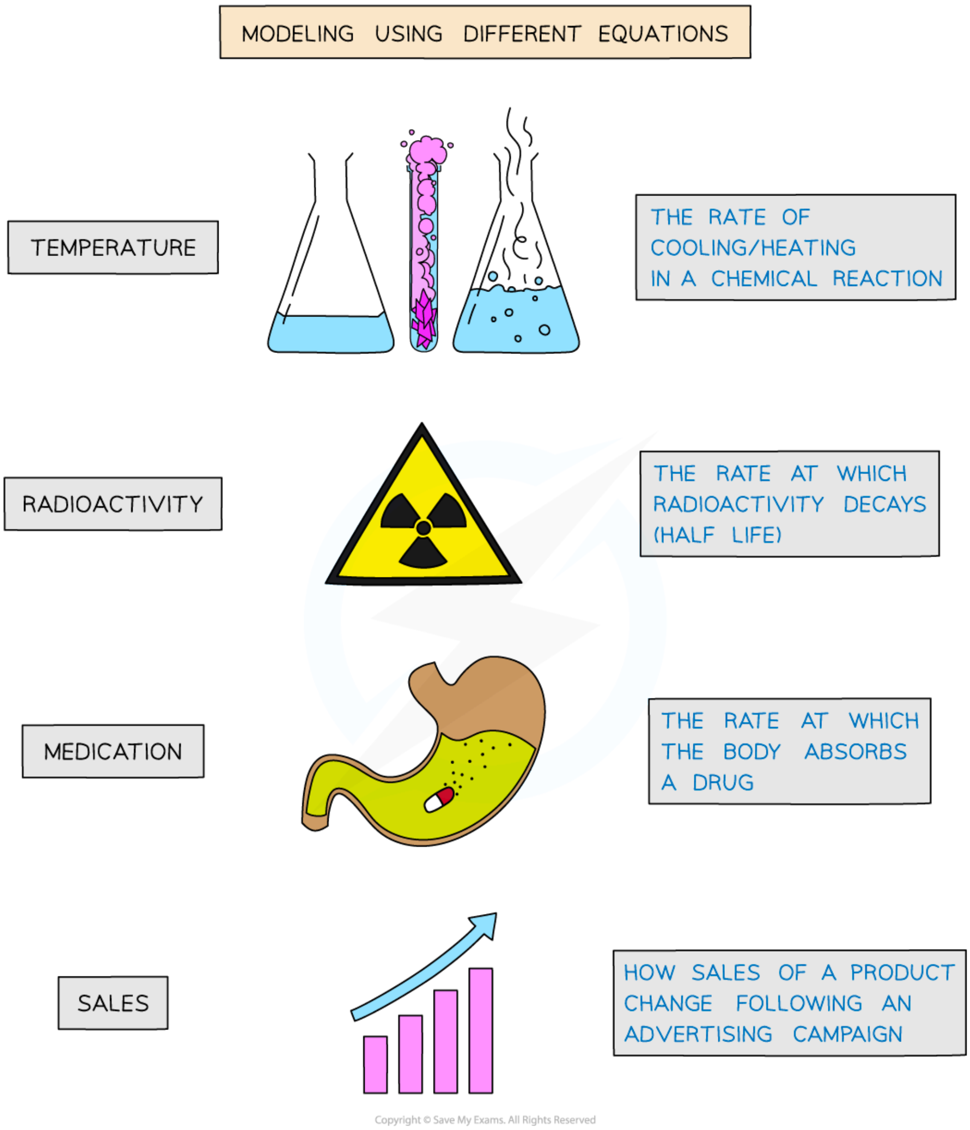
$A^2 = 4\pi r^2 + 4r + 4\pi - 4$

$A^2 = 4\pi(r^2 + 1) + 4(r - 1)$

Modelling with differential equations

What can be modelled with differential equations?

- Derivative terms like $\frac{dy}{dx}$ are "rates of change"
- There are many situations that involve "change"
- Temperature
- Radioactivity
- Medication
- Sales



How do I set up a model with differential equations?

- The first task is to set up a differential equation from a description in words:

Your notes

e.g. THE RADIUS OF A SPHERICAL BUBBLE IS INCREASING AT A RATE PROPORTIONAL TO THE SQUARE OF ITS RADIUS. FORMULATE A DIFFERENTIAL EQUATION, DEFINING ANY VARIABLES YOU SEE.

IT IS OFTEN EASIER TO WRITE THE EQUATION FIRST, USING "OBVIOUS" OR "NATURAL" LETTERS, AND DEFINE THEM AFTER

RADIUS CHANGES WITH TIME

$$\frac{dr}{dt} \propto r^2$$

r IS THE RADIUS OF THE BUBBLE
t IS TIME

$$\frac{dr}{dt} = kr^2 \quad \text{WHERE } k \text{ IS A CONSTANT}$$

HOW WOULD YOUR ANSWER DIFFER IF THE RADIUS WAS DECREASING?

ANSWER: $\frac{dr}{dt} = -kr^2$ (KEEPS k POSITIVE)

- Important phrases here are ...
  - ... "**rate of change**" ... reference to a derivative term like $\frac{dy}{dx}$
  - ... **directly/inversely proportional** to ... $y = kx$, $y = \frac{k}{x}$
  - ... **formulate** ... means to write as an equation
    - you may need to choose and define letters for variables
    - **V** for volume, **h** for height (of a cylinder, say)

e.g. NEWTON'S LAW OF COOLING STATES THAT THE RATE OF CHANGE OF THE TEMPERATURE OF AN OBJECT IS PROPORTIONAL TO THE DIFFERENCE BETWEEN ITS OWN TEMPERATURE AND THE TEMPERATURE OF ITS SURROUNDINGS (AMBIENT TEMPERATURE).
USING T AS THE TEMPERATURE OF A CUP OF TEA, t FOR TIME AND T_0 AS THE TEMPERATURE AROUND THE CUP WRITE DOWN A DIFFERENTIAL EQUATION FOR NEWTON'S LAW OF COOLING.

- "RATE OF" – DEPENDENT ON TIME t
- PROPORTIONAL TO... $\propto$
- T IS "CURRENT" TEMPERATURE DIFFERENCE BETWEEN T AND T_0

YOU MAY HAVE TO READ THE QUESTION SEVERAL TIMES.

$$\frac{dT}{dt} \propto T - T_0$$

IF COOLING, $T_0 \leqslant T$

$$\frac{dT}{dt} = -k(T - T_0)$$

WHERE k IS A CONSTANT

COOLING SO EXPECTING A NEGATIVE IMPLIES $k>0$

- Some differential equations may involve Connected Rates of Change

Your notes

e.g. A RIGHT–ANGLED TRIANGLE HAS A BASE OF **3p** cm AND A HEIGHT OF **2p** cm, WHERE **p > 0** IS A PARAMETER THAT IS INCREASING OVER TIME. THE RATE OF INCREASE OF **p** IS SUCH THAT THE RATE OF INCREASE OF THE THE AREA OF THE TRIANGLE, **A**, IS PROPORTIONAL TO THE SQUARE ROOT OF **p**. WRITE DOWN A DIFFERENTIAL EQUATION FOR THE RATE OF CHANGE OF **p**.

$\frac{dA}{dt} \propto \sqrt{p}$ SO $\frac{dA}{dt} = k\sqrt{p}$

A IS INCREASING SO $k > 0$

WHERE k IS A POSTITVE CONSTANT

WE KNOW THIS

BY THE CHAIN RULE, $\frac{dA}{dt} = \frac{dA}{dp} \times \frac{dp}{dt}$

USE THE CHAIN RULE TO CONNECT WHAT YOU KNOW AND WHAT YOU WANT TO KNOW

WE WANT TO KNOW THIS

THE AREA OF THE TRIANGLE IS

$A = \frac{1}{2} \times 3p \times 2p = 3p^2$

USE DIFFERENTIATION TO GET THE 'MISSING' DERIVATIVE FROM THE CHAIN RULE EQUATION

SO $\frac{dA}{dp} = 6p$

THEN $\frac{dA}{dt} = \frac{dA}{dp} \times \frac{dp}{dt}$

$k\sqrt{p} = 6p\left(\frac{dp}{dt}\right)$

SUBSTITUTE AND REARRANGE

AND $\frac{dp}{dt} = \frac{k\sqrt{p}}{6p}$

$\frac{dp}{dt} = \frac{k}{6\sqrt{p}}$ (WHERE k IS A POSITIVE CONSTANT)

Examiner Tips and Tricks

- Use a highlighter (or underline) to pick out important words/phrases
- Read and re-read the question several times

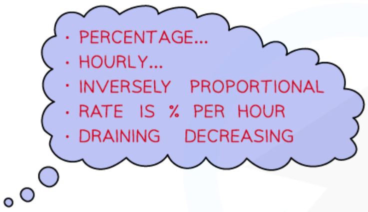
- Jot down bits and pieces as you go; do not expect to go straight from reading to writing down a differential equation.

Your notes

Worked Example

A battery's charge, as a percentage of its full capacity, is draining at an hourly rate inversely proportional to the cube root of its charge.

Formulate a differential equation to model the percentage charge of the battery.



LET p% BE THE BATTERY'S CHARGE AT TIME t HOURS

$$\frac{dp}{dt} \propto \frac{1}{\sqrt[3]{p}}$$

$$\frac{1}{\sqrt[3]{p}} = p^{-\frac{1}{3}}$$

$$\frac{dp}{dt} = -kp^{-\frac{1}{3}}$$ WHERE k IS A CONSTANT

DRAINING

Solving & interpreting differential equations

How do I solve a differential equation from a given model?

- Solving differential equations from a given model using the same techniques as before, for example separation of variables
- The precise integration method will depend on the type of question (see Decision Making)
- **Particular solutions** are usually required to Differential Equations
    - An **initial/boundary condition** is needed

e.g. SOLVE THE DIFFERENTIAL EQUATION $\frac{dP}{dt} = -0.1P$ GIVEN THAT THE INITIAL VALUE OF P IS 2

$$\frac{1}{P}\,dP = -0.1\,dt$$

SEPARATE VARIABLES

$$\int \frac{1}{P}\,dP = \int -0.1\,dt$$

INTEGRATE

$$\ln|P| = -0.1t + c$$

AT $t = 0,\ P = 2$

$$\therefore\ c = \ln 2$$

USE THE INITIAL CONDITION TO FIND c

$$P = e^{-0.1t + \ln 2}$$

$$P = e^{-0.1t} \times e^{\ln 2}$$

$$P = 2e^{-0.1t}$$

- Solutions can be rewritten in a format relevant to the model

IN THE LAST EXAMPLE, AFTER INTEGRATING THE SOLUTION WAS

$\ln |P| = -kt + c$

GENERAL SOLUTION

$e^{\ln|P|} = e^{-kt+c}$

$P = e^{-kt} \times e^{c}$

$e^{c} = A$

$\therefore \; P = Ae^{-kt}$

AT $t=0$, $P=A$, SO A IS THE INITIAL VALUE OF P

RECOGNISING THIS TYPE OF SOLUTION CAN SAVE TIME

$P = Ae^{-kt}$

INITIAL VALUE

NEGATIVE EXPONENTIAL – DECAY

THE EQUATION IS IN A FORM THAT RELATES TO THE MODEL

- The solution can be used to make predictions at other times
  - Temperature after four minutes
  - Volume of sales after another three months

How can I use the solution of a differential equation to answer modelling questions?

- Questions may ask you to interpret your solutions in the context of the problem

Your notes

e.g. THE AMOUNT OF DRUG, N mg, IN A PATIENT'S BODY t HOURS AFTER INJECTION, IS MODELLED BY THE DIFFERENT EQUATION

$$\frac{dN}{dt} = -\frac{1}{8}N(t+1)$$

a) GIVEN THE INITIAL DOSE OF THE DRUG IS 30 mg FIND THE AMOUNT OF DRUG IN A PATIENT'S BODY AFTER 8 HOURS.

b) IT IS SAFE FOR A PATIENT TO HAVE ANOTHER DOSE OF THE DRUG ONCE THE AMOUNT LEFT IN THE BODY FALLS BELOW 0.1 mg. FIND, TO THE NEAREST MINUTE, HOW LONG A PATIENT WOULD HAVE TO WAIT FOR A SECOND INJECTION OF THE DRUG.

Your notes

a) $\frac{1}{N}dN = -\frac{1}{8}(t+1)dt$

SEPARATE VARIABLES

$\int \frac{1}{N}dN = \int\left(-\frac{1}{8}t - \frac{1}{8}\right)dt$

INTEGRATE

$\ln|N| = -\frac{1}{16}t^2 - \frac{1}{8}t + c$

$N = e^{-\frac{1}{16}t^2 - \frac{1}{8}t + c}$

$N = Ae^{-\frac{1}{16}t(t+2)}$

RECOGNISE THIS FORM

AT $t=0$, $N=A$, $\therefore A = 30$

$N = 30e^{-\frac{1}{16}t(t+2)}$

A IS INITIAL VALUE

AT $t=8$, $N = 30e^{-\frac{1}{2}\times 10}$

SUBSTITUTE VALUE OF t IN

$N = 0.202$ mg (3 sf)

b) REQUIRE $N < 0.1$

$30e^{-\frac{1}{16}t(t+2)} < 0.1$

$e^{-\frac{1}{16}t(t+2)} < \frac{1}{300}$

$-\frac{1}{16}t(t+2) < -\ln 300$

• ln BOTH SIDES

• $\ln\frac{1}{300} = \ln 300^{-1}$
$= -\ln 300$

$t^2 + 2t > 16\ln 300$

$\times -16$

$t^2 + 2t - 16\ln 300 > 0$

$\cancel{t < -10.605...}$ $\quad t > 8.605...$

USE CALCULATOR TO SOLVE

$t \geq 0$

IMPLIED AS TIME

8.605 HOURS IS
8 HOURS (60×0.605) MINUTES

$\therefore$ PATIENT NEEDS TO WAIT (AT LEAST)
8 HOURS 37 MINUTES
FOR THEIR SECOND DOSE

CALCULATOR SAYS
36.3 MINUTES USING
0.605 BUT t NEEDS

TO BE GREATER
THAN THIS

- There could be links to other areas of A level maths – such as mechanics

THE ACCELERATION, a ms^{-2}, OF A PARTICLE TRAVELLING IN A VACUUM IS PROPORTIONAL TO HALF OF ITS VELOCITY, v ms^{-1}, AT TIME t SECONDS. GIVEN THE INITIAL VELOCITY OF THE PARTICLE IS 3 ms^{-1} AND IT TAKES 4 SECONDS FOR ITS SPEED TO DOUBLE, FIND v IN TERMS OF t AND FIND THE VELOCITY AFTER 12 SECONDS.

Your notes

Your notes

- ACCELERATION IS "RATE OF CHANGE" OF VELOCITY
- $a = \frac{dv}{dt}$ FROM MECHANICS
- AT $t = 0$, $v = 3$ AT $t = 4$, $v = 6$

USING A MAY HIDE THE FACT THIS IS A DIFFERENTIAL EQUATION

$$a = \frac{dv}{dt} \propto \frac{1}{2}v$$

$$\frac{dv}{dt} = \frac{1}{2}kv \quad \text{WHERE } k \text{ IS A CONSTANT}$$

$$\int \frac{1}{v}\,dv = \frac{1}{2}k\,dt$$

SEPARATE VARIABLES AND INTEGRATE

$$\ln|v| = \frac{1}{2}kt + c$$

$$v = Ae^{\frac{1}{2}kt}$$

AT $t = 0$, $v = A = 3$

$$\therefore \ v = 3e^{\frac{1}{2}kt}$$

USE GIVEN CONDITIONS TO FIND UNKNOWNS

AT $t = 4$, $v = 6$

$$\therefore \ e^{2k} = 2$$

$$2k = \ln 2$$

$$k = \frac{1}{2}\ln 2$$

$$\boxed{k = \ln\sqrt{2}}$$

$$\therefore \ v = 3e^{\frac{1}{4}t\ln 2}\ \text{ms}^{-1} \quad (*)$$

AFTER 12 SECONDS

$$v = 3e^{3\ln 2}$$

$$v = 3e^{\ln 8}$$

$$v = 3 \times 8$$

$$v = 24\ \text{ms}^{-1}$$

$(*)$ $v = 3e^{\ln 2^{\frac{1}{4}t}}$

$v = 3 \times 2^{\frac{1}{4}t}$

THIS MODEL IS REALLY BASED ON POWERS OF 2

- Sometimes **multiple** rates of change may be involved in a model or problem

- See Connected Rates of Change

Your notes

e.g. AN AIR BUBBLE'S VOLUME IS INCREASING AT THE RATE OF $0.5\ m^3s^{-1}$. THE INITIAL RADIUS OF THE BUBBLE IS 0.1m. ASSUMING THE BUBBLE IS SPHERICAL, FIND THE RATE AT WHICH THE RADIUS OF THE BUBBLE IS INCREASING AFTER 12 SECONDS.

$\frac{dV}{dt} = 0.5$

AT $t = 0,\ r = 0.1$

$V = \frac{4}{3}\pi r^3$

$\frac{dr}{dt}$

CONVERT THE WORDED QUESTIONS INTO MATHEMATICAL STATEMENTS

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

0.5

CONNECTED RATES OF CHANGE

TRYING TO FIND

CAN FIND $\frac{dV}{dr}$

$\frac{dV}{dr} = 4\pi r^2$

$$\frac{dV}{dr} = 4\pi r^2$$

USING $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

$$\therefore \frac{dr}{dV} = \frac{1}{4\pi r^2}$$

$$\therefore \frac{dr}{dt} = \frac{1}{4\pi r^2} \times 0.5$$

$$\frac{dr}{dt} = \frac{1}{8\pi r^2}$$

WE NEED TO FIND r AT $t = 12$ AND SUBSTITUTE THIS VALUE OF r INTO $\frac{dr}{dt}$

$$8\pi r^2 \, dr = dt$$

SEPARATE VARIABLES

$$\int 8\pi r^2 \, dr = \int dt$$

$$\frac{8\pi r^3}{3} = t + c$$

INTEGRATE

AT $t = 0$, $r = 0.1$, $c = \frac{\pi}{375}$

AT $t = 12$, $\frac{8\pi r^3}{3} = 12 + \frac{\pi}{375}$

$r = 1.13$ (3 sf)

$$\therefore \frac{dr}{dt} = \frac{1}{8\pi(1.13)^2}$$

$$\frac{dr}{dt} = 0.0313 \text{ ms}^{-1} \text{ (3 sf)}$$

THIS IS ABOUT 3cm PER SECOND

How can I interpret a model using the solution to the differential equation?

- Models may not always be realistic in the long term
  - A population will not grow indefinitely – it will reach a natural limit
  - You will be expected to interpret and comment on the model

Your notes

Your notes

e.g. SCIENTISTS ARE MODELLING THE POPULATION, P, OF APES IN A PARTICULAR AREA USING THE DIFFERENTIAL EQUATION

$$\frac{dP}{dt} = \frac{1}{10}P$$

a) IF THE INITIAL POPULATION IS 600 APES, USE THE MODEL TO ESTIMATE THE POPULATION OF APES 5 YEARS LATER.

b) WHAT IS ONE LIMITATION OF THE MODEL IN THE LONG TERM? HOW CAN THE MODEL BE IMPROVED?

a) $$\int \frac{1}{P}\,dP = \int \frac{1}{10}\,dt$$

$$\ln|P| = \frac{1}{10}t + c$$

$$P = Ae^{\frac{1}{10}t}$$

AT $t = 0$, $P = A = 600$

$$P = 600e^{\frac{1}{10}t}$$

AT $t = 5$, $P = 600e^{0.5}$

$P = 989$ (3 sf)

b) THE MODEL SUGGESTS THE POPULATION OF APES WILL INCREASE INDEFINITELY FOREVER – THERE IS NO UPPER LIMIT.

AN IMPROVEMENT TO THE MODEL WOULD BE TO INTRODUCE AN UPPER LIMIT BASED ON THE SIZE/RESOURCES OF THE AREA BEING STUDIED.

Worked Example

Your notes

A large tank has a small hole from which water is leaking at a rate directly proportional to the square root of the volume, V m^3, of water in the tank at time, t seconds.

Initially the tank holds 400 m^3.

(a) Write a differential equation in terms of V and t

After 2 minutes, the tank contains 324 m^3 of water

(b) Find the particular solution for your differential equation

(c) Explain a potential problem with this model for large values of t

- "...LEAKING..." – NEGATIVE $\frac{dV}{dt}$
- AT $t = 0$, $V = 400$

a) $\frac{dV}{dt} \propto \sqrt{V}$

$\frac{dV}{dt} = -kV^{\frac{1}{2}}$

b) $\int V^{-\frac{1}{2}} dV = \int -k \, dt$

SEPARATE VARIABLES & INTEGRATE

$2V^{\frac{1}{2}} = -kt + c$

AT $t = 0$, $V = 400$

INITIAL CONDITION

$\therefore 2 \times 20 = c \quad c = 40$

2 MINUTES, t IN SECONDS

AT $t = 120$, $V = 324$

$2 \times 18 = -120k + 40$

$-120k = -4$

$k = \frac{1}{30}$

$\therefore 2\sqrt{V} = -\frac{1}{30}t + 40$

Your notes

$\sqrt{V} = -\frac{1}{60}t + 20$

$\therefore$ PARTICULAR SOLUTION IS

$V = \left(20 - \frac{t}{60}\right)^2$

c) FOR LARGE VALUES OF t THE MODEL WILL PREDICT THAT THE VOLUME OF WATER IN THE TANK WILL INCREASE.

SINCE $V = (\ldots)^2$ IT IS ALWAYS ≥ 0.
QUADRATIC – A POSITIVE QUADRATIC. ∪
SO THE MODEL WILL HAVE A MINIMUM POINT OF 0 WHEN $t = 20 \times 60 = 1200$.
THIS ALSO MEANS TANK TAKES 20 MINUTES TO EMPTY.