Edexcel International A Level (IAL) Maths: Pure 4

Your notes

Further Integration

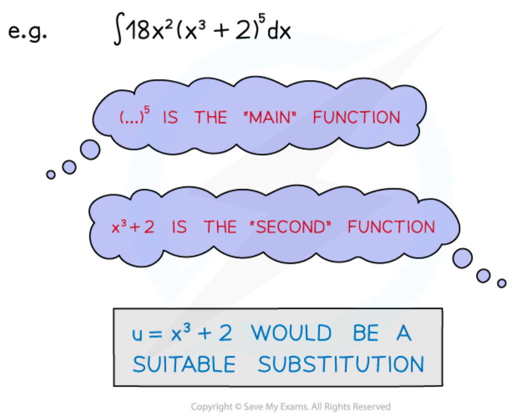
Contents

Your notes

Substitution (Reverse Chain Rule)

What is integration by substitution?

- Make sure you are familiar with Chain Rule and Reverse Chain Rule
- In more awkward problems it is harder to spot the reverse chain rule
- It is possible to use a substitution
- These kinds of substitutions usually will **not** be given
  - The substitution is deemed 'obvious'



How do I integrate when a substitution is not given?

- Look to substitute the 'second' (rather than the 'main') function
- STEP 1: Determine the substitution
  - What is the 'main' function? 'second' function?
- STEP 2: Differentiate the substitution and rearrange
  - **du/dx** here can be treated like a fraction (eg × **dx** to get rid of fractions)
- STEP 3: Replace all parts of the integral
  - all **x** terms should be replaced with equivalent **u** terms, including **dx**
  - if a definite integral change the limits from **x** to **u** too
- STEP 4: Integrate and either ...
  - ...substitute **x** back in

or

- ... evaluate the definite integral using the **u** limits

- STEP 5: Find **c**, the constant of integration, if needed

$I = \int 6u^5\,du$

THE RESULT SHOULD BE REALLY EASY TO INTEGRATE

STEP 4: INTEGRATE, SUBSTITUTE x BACK IN

$I = u^6 + c$

$I = (x^3 + 2)^6 + c$

FIND $\int_0^1 15x^2\cos(5x^3)\,dx$ GIVING YOUR ANSWER TO 3 SIGNIFICANT FIGURES.

$I = \int_0^1 15x^2\cos(5x^3)\,dx$

STEP 1: DETERMINE THE SUBSTITUTION

LET $u = 5x^3$

STEP 2: DIFFERENTIATE AND REARRANGE

$\frac{du}{dx} = 15x^2$

$du = 15x^2\,dx$

STEP 3: REPLACE ALL PARTS OF THE INTEGRAL

$x = 0, \quad u = 0$

$x = 1, \quad u = 5$

CHANGE LIMITS TOO!

Your notes

$$I = \int_0^5 \cos u \, du$$

$\cos(5x^3) = \cos u$

$du = 15x^2 dx$

STEP 4: INTEGRATE AND EVALUATE

$$I = [\sin u]_0^5$$

$$I = \sin 5 - \sin 0$$

RADIANS FOR CALCULUS!

$$I = -0.95892...$$

$$I = -0.959 \quad (3 \text{ sf})$$

Your notes

Examiner Tips and Tricks

- A lot of the "work" in these problems happens in your head :
  - what is the 'main' function?
  - what should the substitution be?
  - are there any numbers to 'adjust' and 'compensate' for?
- Be sure that what you write down is clear and easy to follow, and remember that you can check your final answer by differentiating it.

Worked Example

Your notes

Given that y passes through the point $(1, 2e^3)$, and that

$$y = \int 5x^2 e^{x^3+2}\,dx$$

find y in terms of x.

$y = \int 5x^2 e^{x^3+2}\,dx$

STEP 1: DETERMINE THE SUBSTITUTION

$e^{\cdots}$ IS THE MAIN FUNCTION

LET $u = x^3 + 2$

STEP 2: DIFFERENTIATE AND REARRANGE

$\frac{du}{dx} = 3x^2$

$du = 3x^2 dx$

STEP 3: REPLACE ALL PARTS OF THE INTEGRAL

$y = 5 \times \frac{1}{3}\int 3x^2 e^u\,dx$

CONSTANT

"ADJUST" AND "COMPENSATE"

$$y = \frac{5}{3}\int e^u \, du$$

DEAD EASY!

STEP 4: INTEGRATE, SUBSTITUTE x BACK IN

$$y = \frac{5}{3}e^u + c$$

$$y = \frac{5}{3}e^{x^3+2} + c$$

STEP 5: FIND c

AT $(1, 2e^3)$ $\quad 2e^3 = \frac{5}{3}e^3 + c$

$$c = \frac{1}{3}e^3$$

$$\therefore \quad y = \frac{5}{3}e^{x^3+2} + \frac{1}{3}e^3$$

Your notes

Your notes

Harder substitution

What is integration by substitution?

- Make sure you are familiar with Reverse Chain Rule and Substitution (RCR)
- In more difficult questions the substitution will be given to you

e.g. USE THE SUBSTITUTION $u = x + 3$

TO FIND $\int \left(\frac{x}{x+3}\right)^3 dx.$

How do I integrate using a given substitution?

- The process is very much the same as before but expect the algebra to be trickier

e.g. USE THE SUBSTITUTION $u = x + 3$

TO FIND $\int \left(\frac{x}{x+3}\right)^3 dx.$

STEP 1: DIFFERENTIATE AND REARRANGE

$u = x + 3$

$\frac{du}{dx} = 1$

$du = dx$

$x = u - 3$

SPOT YOU MAY NEED THIS FOR THE NUMERATOR

STEP 2: REPLACE ALL PARTS OF THE INTEGRAL

$\int \left(\frac{x}{x+3}\right)^3 dx = \int \left(\frac{u-3}{u}\right)^3 du$

ALGEBRA MORE AWKWARD!

$= \int (1 - 3u^{-1})^3 du$

Your notes

USE BINOMIAL EXPANSION

$$= \int \left(1 + 3(-3u^{-1}) + 3(-3u^{-1})^2 + (-3u^{-1})^3\right) du$$

$$= \int (1 - 9u^{-1} + 27u^{-2} - 27u^{-3})\, du$$

STEP 4: INTEGRATE, SUBSTITUTE x BACK IN

$$= u - 9\ln|u| + \frac{27u^{-1}}{-1} - \frac{27u^{-2}}{-2} + c$$

$$= x + 3 - 9\ln|x+3| - \frac{27}{x+3} + \frac{27}{2(x+3)^2} + c$$

- STEP 1: Differentiate the substitution and rearrange
  - $\frac{du}{dx}$ here can be treated like a fraction (eg × **dx** to get rid of fractions)
- STEP 2: Replace all parts of the integral
  - all **x** terms should be replaced with equivalent **u** terms, including **dx**
  - if a definite integral change the limits from **x** to **u** too
- STEP 3: Integrate and either
  - substitute **x** back in or
  - evaluate the definite integral using the **u** limits
- STEP 4: Find **c**, the constant of integration, if needed

Worked Example

Your notes

Use the substitution $u^2 = x^3 + 1$ to find the exact value of

$$\int_1^2 \frac{x^2}{\sqrt{1+x^3}}\,dx$$

$$I = \int_1^2 \frac{x^2}{\sqrt{1+x^3}}\,dx$$

$$u^2 = x^3 + 1$$

STEP 1: DIFFERENTIATE AND REARRANGE

$$2u\frac{du}{dx} = 3x^2$$

$$2u\,du = 3x^2\,dx$$

DEAL WITH THIS LATER

$$x^3 = u^2 - 1$$

STEP 2: REPLACE ALL PARTS OF THE INTEGRAL

$x = 1, \quad u^2 = 1^3 + 1 = 2, \quad u = \sqrt{2}$

$x = 2, \quad u^2 = 2^3 + 1 = 9, \quad u = 3$

EXACT VALUE REQUIRED AT END

"ADJUST" AND "COMPENSATE"

$$I = \frac{1}{3}\int_{1}^{2} \frac{3x^2}{\sqrt{1+x^3}}\,dx$$

$$I = \frac{1}{3}\int_{\sqrt{2}}^{3} \frac{2u\,du}{\sqrt{1+(u^2-1)}}$$

$$I = \frac{2}{3}\int_{\sqrt{2}}^{3} \frac{u}{\sqrt{u^2}}\,du$$

$$I = \frac{2}{3}\int_{\sqrt{2}}^{3} du$$

$$\int_{\sqrt{2}}^{3} 1\,du$$

STEP 3: INTEGRATE AND EVALUATE

$$I = \frac{2}{3}\left[u\right]_{\sqrt{2}}^{3}$$

$$I = \frac{6-2\sqrt{2}}{3}$$

Your notes

Your notes

Integration by parts

What is integration by parts?

INTEGRATION BY PARTS

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

- For integrating the product of two functions - reverse product rule
- Crucially the product is made from **u** and **dv/dx** (rather than **u** and **v**)
- Alternative notation may be used

ALTERNATIVE NOTATION

FUNCTION NOTATION

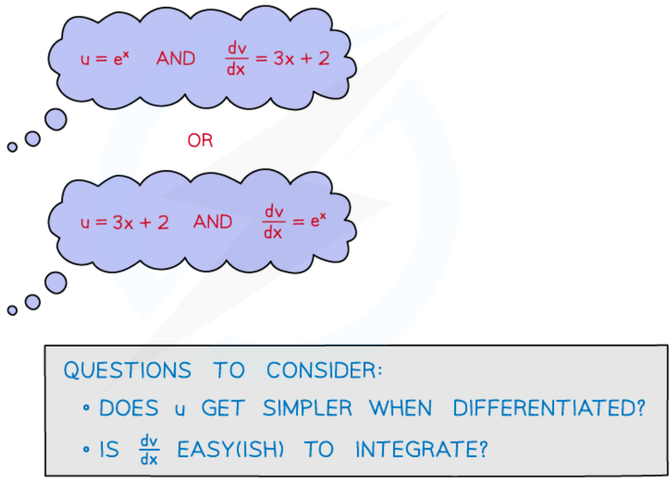
$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

DASH NOTATION

$$\int uv'dx = uv - \int vu'dx$$

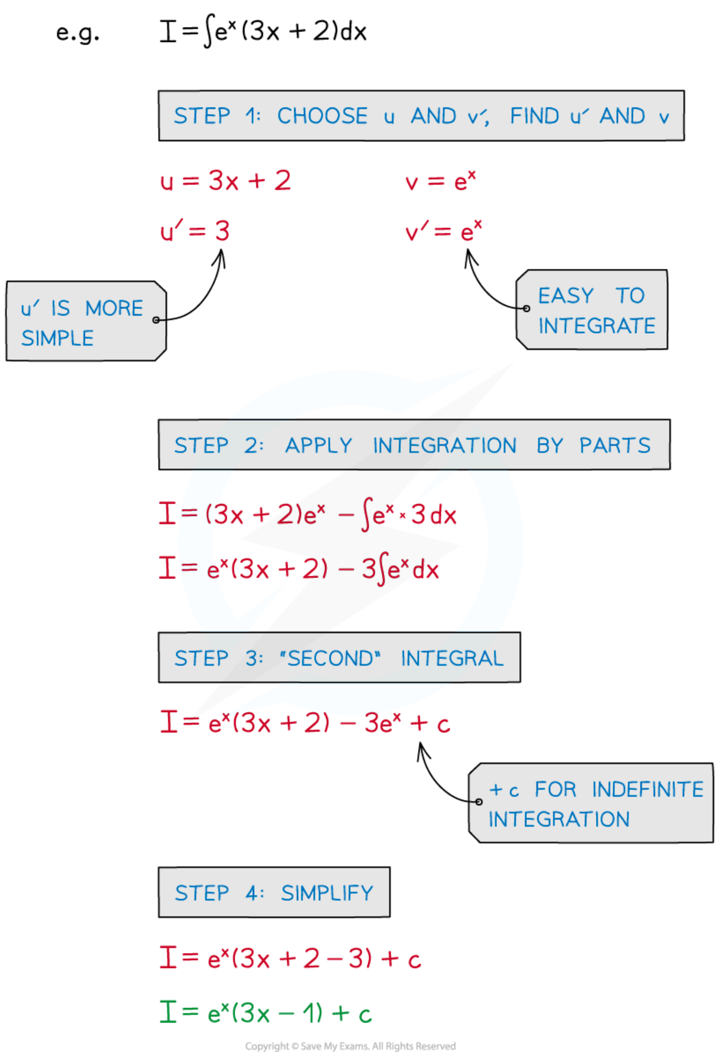
How do I use integration by parts?

e.g. $\int e^x(3x+2)\,dx$



- The hardest part is choosing **u** and **dv/dx** as there is no method for doing so
- **u**, ideally, becomes **simpler** when differentiated but this is not always possible
- **dv/dx** should be a function that can be integrated fairly easily
- Be wary of functions that 'cycle'/'repeat' when differentiated/integrated
  - $e^x \rightarrow e^x$
  - $\sin x \rightarrow \cos x \rightarrow -\sin x \rightarrow -\cos x \rightarrow \sin x$

Your notes



- **STEP 1**
  Choose u and v', find u' and v
- **STEP 2**
  Apply Integration by Parts
  - Simplify anything straightforward
- **STEP 3**
  Do the 'second' integral
  - If an **indefinite** integral remember "**+c**", the **constant of integration**
- **STEP 4**
  Simplify and/or apply limits

Can I use integration by parts twice?

- It is possible that integration by parts may need to be applied more than once e.g. $I = \int x^2 \cos x \, dx$

STEP 1: CHOOSE u AND v′, FIND u′ AND v

$u = x^2 \qquad v = \sin x$

$u' = 2x \qquad v' = \cos x$

SIMPLER AFTER DIFFERENTIATING

"CYCLES" BETWEEN cos AND sin

STEP 2: APPLY INTEGRATION BY PARTS

$I = x^2 \sin x - \int 2x \sin x \, dx$

STEP 3: REPEAT STEP 1 AND STEP 2 FOR THE SECOND INTEGRAL

$u = 2x \qquad v = -\cos x$

$u' = 2 \qquad v' = \sin x$

$I = x^2 \sin x - \left[-2x \cos x - \int -2 \cos x \, dx \right]$

STEP 4: INTEGRATE AND SIMPLIFY

$I = x^2 \sin x + 2x \cos x - 2 \sin x + c$

$I = (x^2 - 2)\sin x + 2x \cos x + c$

"+c"!!

How do I integrate ln x?

Your notes

$$\boxed{\int \ln x \, dx}$$

$$\int \ln x \, dx = \int 1 \times \ln x \, dx$$

$u = \ln x \qquad v = x$

$u' = \frac{1}{x} \qquad v' = 1$

$$= x \ln x - \int x \times \frac{1}{x} \, dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + c$$

$$\boxed{\int \ln x \, dx = x \ln x - x + c}$$

- A classic 'set piece' in almost every A level maths textbook ever written!
- In general, rewriting **f(x)** as **1×f(x)** can be a powerful problem-solving technique
- This could be a question in the exam

Can I find definite integrals using integration by parts?

- You can find the value of a definite integral using integration by parts
  - Use the layout shown in the example below

Your notes

e.g. FIND THE VALUE OF $I = \int_0^1 3xe^{2x}\,dx$

STEP 1: CHOOSE u AND v′, FIND u′ AND v

$u = 3x \qquad v = \frac{1}{2}e^{2x}$ ← REVERSE CHAIN RULE

$u' = 3 \qquad v' = e^{2x}$

STEP 2: APPLY INTEGRATION BY PARTS

$$I = \left[\frac{3}{2}xe^{2x} - \int \frac{3}{2}e^{2x}\,dx\right]_0^1$$

$$I = \left[\frac{3}{2}xe^{2x} - \frac{3}{2} \times \frac{1}{2}\int 2e^{2x}\,dx\right]_0^1$$

STEP 3: DO THE "SECOND" INTEGRAL

$$I = \left[\frac{3}{2}xe^{2x} - \frac{3}{4}e^{2x}\right]_0^1$$

$$I = \left[\frac{3}{4}e^{2x}(2x - 1)\right]_0^1$$

STEP 4: APPLY LIMITS AND SIMPLIFY

$$I = \left(\frac{3}{4}e^{2}\right) - \left(\frac{3}{4}e^{0} \times -1\right)$$

$$I = \frac{3}{4}e^{2} + \frac{3}{4}$$

$$I = \frac{3}{4}(e^{2} + 1)$$

Examiner Tips and Tricks

- Always think about what an elegant, slick, professional maths solution looks like – solutions normally get more complicated at first but quickly get simpler.
- If your work is continuing to get more complicated, stop and check for an error.
- Try to develop a sense of 'having gone too far down the wrong path'.
- This general advice is useful to remember:
- Is the second integral harder than the first?

- Try swapping your choice of **u** and **dv/dx**
- It is rare to have to apply integration by parts more than twice

Your notes

Worked Example

Find $\int x^2 \ln x \, dx$

LET $I = \int x^2 \ln x \, dx$

STEP 1: CHOOSE u AND v', FIND u' AND v

$u = \ln x \qquad v = \frac{1}{3}x^3$

$u' = \frac{1}{x} \qquad v' = x^2$

NO CHOICE AS CANNOT INTEGRATE ln x EASY

STEP 2: APPLY INTEGRATION BY PARTS

$I = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \times \frac{1}{x} dx$

$I = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$

STEP 3: DO THE "SECOND" INTEGRAL

$I = \frac{1}{3}x^3 \ln x - \frac{1}{3}\left(\frac{1}{3}x^3\right) + c$

STEP 4: SIMPLIFY

$I = \frac{1}{9}x^3(3\ln x - 1) + c$

Your notes

Integration using partial fractions

What are partial fractions?

- This is the reverse process to adding (or subtracting) fractions
  - The common polynomial denominator is split into factors
- Make sure you are familiar with the notes on Partial Fractions

$$\frac{1}{x^2+3x+2} = \frac{1}{(x+2)(x+1)}$$

$$\frac{1}{(x+2)(x+1)} \equiv \frac{A}{(x+2)} + \frac{B}{(x+1)}$$

$$1 \equiv A(x+1) + B(x+2)$$

$$\text{LET } x=-1, \quad 1=B, \quad B=1$$
$$\text{LET } x=-2, \quad 1=-A, \quad A=-1$$

$$\therefore \frac{1}{x^2+3x+2} = \frac{1}{x+1} - \frac{1}{x+2}$$

How do I integrate using partial fractions?

- Fractions with linear denominators can be integrated (See Integrating Other Functions)

$$\int \frac{1}{x+3}\,dx = \ln|x+3| + c$$

- A fraction with a polynomial denominator (degree 2+) can be integrated by ...
  - ... splitting using partial fractions
  - ... integrating each partial fraction

Your notes

e.g. $\displaystyle\int \frac{3}{x^2+5x+4}\,dx$

STEP 1: FACTORISE DENOMINATOR, IF NEEDED

$$\frac{3}{x^2+5x+4} = \frac{3}{(x+4)(x+1)}$$

STEP 2: EXPRESS AS PARTIAL FRACTIONS

$$\frac{3}{(x+4)(x+1)} \equiv \frac{A}{x+4} + \frac{B}{x+1}$$

$$3 \equiv A(x+1) + B(x+4)$$

LET $x = -1$, $3 = 3B$, $B = 1$

LET $x = -4$, $3 = -3A$, $A = -1$

$$\therefore \frac{3}{x^2+5x+4} = \frac{1}{x+1} - \frac{1}{x+4}$$

STEP 3: INTEGRATE EACH TERM

$$\int \frac{3}{x^2+5x+4}\,dx = \int\left(\frac{1}{x+1} - \frac{1}{x+4}\right)dx$$

$$= \ln|x+1| - \ln|x+4| + c$$

REMEMBER MODULUS WHEN INTEGRATING TO ln

- STEP 1: Factorise denominator, if needed
- STEP 2: Express as partial fractions
- STEP 3: 'Adjust' and 'compensate' each term so it can be integrated (See Reverse Chain Rule)
- STEP 4: Integrate each term (usually involves **ln**) and simplify

Examiner Tips and Tricks

- When there are several parts, read the whole question:
- An early part may be a "show that" involving partial fractions
- A later part may be to integrate the original fraction
- If you can't do, or get stuck on, the partial fractions bit of the question you can still use the "show that" result to help with the integration.

Your notes

Worked Example

(a) Fully factorise $4x^3 - 4x^2 + x$

(b) Express $\dfrac{2}{4x^3 - 4x^2 + x}$ in partial fractions

(c) Find $\displaystyle\int \frac{2}{4x^3 - 4x^2 + x}\,dx$

a) $4x^3 - 4x^2 + x = x(4x^2 - 4x + 1)$

$= x(2x-1)^2$

IF x IS NOT A FACTOR, FACTOR THEOREM AND / OR ALGEBRAIC DIVISION WOULD BE NEEDED TO FACTORISE A CUBIC

LOOKING AHEAD, THIS IS STEP 1 FOR PART c)

Your notes

b) $\dfrac{2}{4x^3 - 4x^2 + x} = \dfrac{2}{x(2x-1)^2}$ USING a)

$\equiv \dfrac{A}{x} + \dfrac{B}{2x-1} + \dfrac{C}{(2x-1)^2}$

SQUARED LINEAR DENOMINATOR

$2 \equiv A(2x-1)^2 + Bx(2x-1) + Cx$

LET $x = 0$, $2 = A$, $A = 2$

LET $x = \frac{1}{2}$, $2 = \frac{1}{2}C$, $C = 4$

LET $x = 1$, $2 = A + B + C$

$2 = 2 + B + 4$, $B = -4$

NO VALUES OF x WILL LEAD TO AN EQUATION IN B ONLY

$\dfrac{2}{4x^3 - 4x^2 + x} \equiv \dfrac{2}{x} - \dfrac{4}{2x-1} + \dfrac{4}{(2x-1)^2}$

LOOKING AHEAD, THIS IS STEP 2 FOR PART c)

Your notes

c) $\int \frac{2}{4x^3 - 4x^2 + x}\,dx = \int\left(\frac{2}{x} - \frac{4}{2x-1} + \frac{4}{(2x-1)^2}\right)dx$

STEP 1: FACTORISE DENOMINATOR

PART a)

STEP 2: EXPRESS AS PARTIAL FRACTIONS

PART b)

STEP 3: "ADJUST" AND "COMPENSATE"

4 SPLIT INTO 2×2
2 FROM REVERSE CHAIN RULE

4 SPLIT INTO 2×2
2 FROM REVERSE CHAIN RULE
−1 FROM POWERS

$$= \int\left(\frac{2}{x} - \frac{2\times 2}{2x-1} + \frac{2\times 2\times -1}{-1}(2x-1)^{-2}\right)dx$$

STEP 4: INTEGRATE EACH TERM AND SIMPLIFY

$$= 2\ln|x| - 2\ln|2x-1| - 2(2x-1)^{-1} + c$$

"EXTRA" 2 ON TOP

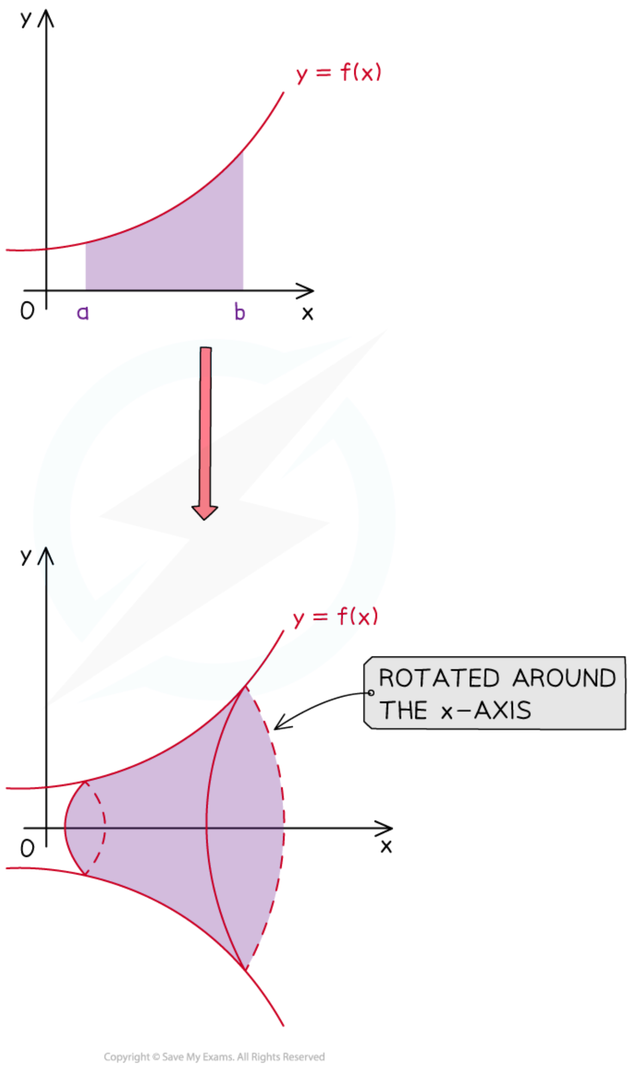
$\frac{2}{-1} = -2$

$$= 2\ln|x| - 2\ln|2x-1| - \frac{2}{2x-1} + c$$

Volumes of revolution around the x-axis

What is a volume of revolution around the x-axis?

- A **solid** of **revolution** is formed when an **area** bounded by a function $y = \mathrm{f}(x)$ (and other boundary equations) is rotated 360° around the x-axis
- A **volume** of **revolution** is the volume of this solid formed



Example of a solid of revolution that is formed by rotating the area bounded by the function $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis $360°$ about the x--axis

How do I find the volume of revolution around the x-axis?

- To find the **volume** of **revolution** created when the area bounded by the function $y = \mathrm{f}(x)$, the lines $x = a$ and $x = b$, and the *x*-axis is rotated 360° about the *x*-axis use the formula

$$V = \pi \int_a^b y^2 \mathrm{d}x$$

- The formula may look complicated or confusing at first due to the *y* and d*x*
  - remember that *y* is a function of *x*
  - once the expression for *y* is substituted in, everything will be in terms of *x*
- π is a constant so you may see this written either inside or outside the integral
- This is **not** given in the formulae booklet
  - The formulae booklet does list the volume formulae for some common 3D solids – it may be possible to use these depending on what information about the solid is available

How do I solve problems involving volumes of revolution around the x-axis?

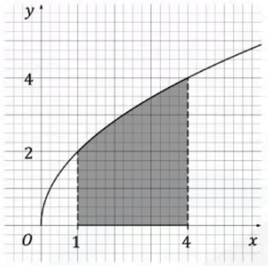
- **Visualising** the solid created is helpful
  - Try sketching some functions and their solids of revolution to help
- **STEP 1** Square *y*
  - Do this first without worrying about π or the integration and limits
- **STEP 2** Identify the limits *a* and *b* (which could come from a graph)
- **STEP 3** Use the formula by evaluating the integral and multiplying by π
  - The answer may be required in **exact form** (leave in terms of π)
    - If not, round to **three significant figures** (unless told otherwise)
- Trickier questions may give you the volume and ask for the value of an unknown constant elsewhere in the problem

Worked Example

Your notes

Your notes

The area bounded by the graph with equation $y = 2\sqrt{x}$, the lines $x = 1$ and $x = 4$ and the x-axis is shown in the diagram below.



Find the volume of the solid of revolution formed when the area is rotated 360° around the x-axis.

STEP 1 SQUARE y

$y^2 = (2\sqrt{x})^2 = 4x$

STEP 2 IDENTIFY THE LIMITS

$x = 1$ AND $x = 4$

STEP 3 EVALUATE THE INTEGRAL

$V = \pi \int_1^4 4x \, dx$

$V = \pi \left[2x^2\right]_1^4$

$V = \pi(32 - 2) = 30\pi$

∴ VOLUME OF REVOLUTION IS 30π CUBIC UNITS

Examiner Tips and Tricks

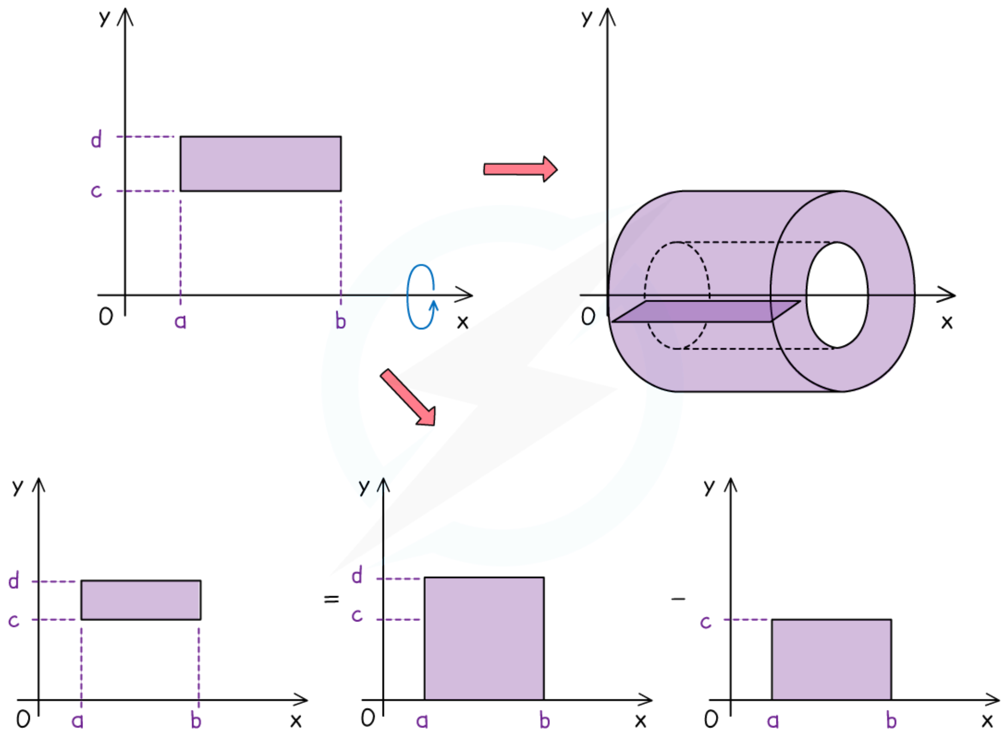
- To help remember the formula note that it is only y^2 - volume is 3D so you may have expected a cubic expression
  - If rotating a **single point** around the x-axis a **circle** of radius would be formed
    - The **area** of that circle would then be πy^2
    - Integration then adds up the areas of all circles between a and b creating the third dimension and **volume**
      (In 2D, integration creates area by adding up lots of 1D lines)

Your notes

Adding and Subtracting Volumes

Why might I need to add or subtract volumes of revolution?

- As with the **area between a curve and a line** or the **area between 2 curves**, a required volume may be created by two functions
  - Make sure you are familiar with the methods in **Volumes of Revolution**
- The volumes created here can be created from areas that do not have the x-axis as one its boundaries
  - A **cylinder** is created by rotating a rectangle that borders the x-axis around the x-axis by 360°
  - An **annular prism** (a cylinder with a whole through it – like a toilet roll) is created by rotating a rectangle that does **not** have a boundary with the x-axis around the x-axis by 360°
- A rectangle would be defined by two vertical and two horizontal lines

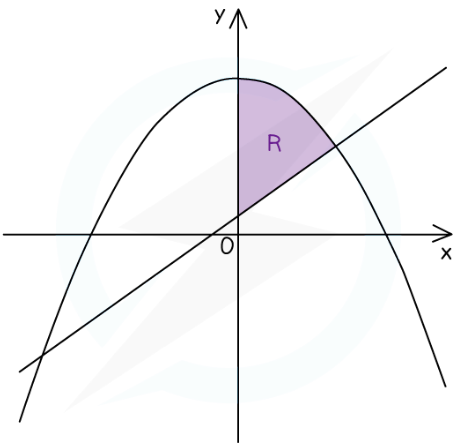


- $x = a,\ x = b,\ y = c,\ y = d$
  - Where a, b, c & d are all positive and a < b and c < d

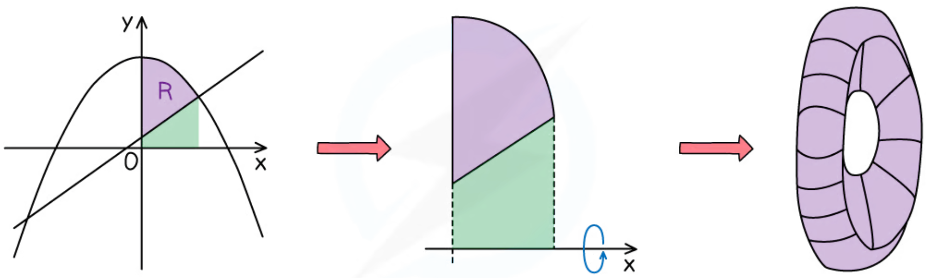
- The volume of revolution of this rectangle would be
  - $V = \pi \int_a^b d^2 \mathrm{d}x - \pi \int_a^b c^2 \mathrm{d}x$

How do I know whether to add or subtract volumes of revolution?

- When the **area** to be **rotated** around an axis has more than one function (and an axis) defining its boundary it can be trickier to tell whether to **add** or **subtract volumes** of **revolution**
  - It will depend on the **nature** of the **functions** and their **points** of **intersection**
  - Whether **rotation** is around the x**-axis** or the y**-axis**
- Consider the region ***R***, bounded by a curve, a line and the y-axis, in the diagram below



- If ***R*** is **rotated** around the x**-axis** the **solid** of **revolution** formed will have a 'hole' in its centre



- Think in 2D and area

- "region under the curve"
  SUBTRACT
  "region under the line"

How do I solve problems involving adding or subtracting volumes of revolution?

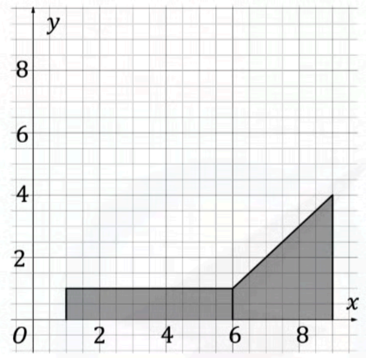
- Visualising the solid created becomes increasingly useful (but also trickier) for shapes generated by separate volumes of revolution
  - Continue trying to sketch the functions and their solids of revolution to help
- **STEP 1:** Identify the functions $(y_1, y_2, \ldots)$ involved in generating the volume
  - Determine whether these will need to be added or subtracted
- **STEP 2:** Square *y* for all functions
  - Do this first without worrying about π or the integration and limits
- **STEP 3:** Identify the limits for each volume involved and form the integrals required
  - The limits could come from a graph
- **STEP 4:** Evaluate the integral for each function and add or subtract as necessary
  - The answer may be required in **exact form**
  - If not, round to **three significant figures** (unless told otherwise)

Your notes

Worked Example

Your notes

The area shown below is bounded by the function $y = x - 5$,
the horizontal line $y = 1$ and the vertical lines $x = 1$ and $x = 9$.
The area is rotated 360° around the x-axis.



(a) Find the exact volume of the solid of revolution formed.

(b) Sketch and briefly describe the shape of the solid of revolution.

a) STEP 1 IDENTIFY FUNCTIONS AND DETERMINE IF ADDING OR SUBTRACTING

$y_1 = 1$

$y_2 = x - 5$

ADDITION

STEP 2 SQUARE ALL FUNCTIONS

$y_1 = 1^2 = 1$

$y_2 = (x-5)^2 = x^2 - 10x + 25$

STEP 3 IDENTIFY LIMITS AND DETERMINE INTEGRALS

$$\pi \int_1^6 1 \, dx$$

$$\pi \int_6^9 (x^2 - 10x + 25) \, dx$$

STEP 4 EVALUATE AND ADD

$$V = \pi [x]_1^6 + \pi \left[\frac{1}{3}x^3 - 5x^2 + 25x\right]_6^9$$

$$V = \pi(6-1) + \pi(63-42)$$

$$V = 26\pi$$

$\therefore$ VOLUME OF THE SOLID OF REVOLUTION IS 26π CUBIC UNITS

b) THE SHAPE CREATED WOULD BE LIKE A 'FUNNEL' 

Your notes

Examiner Tips and Tricks

- It is possible, in subtraction questions, to combine the separate integrals into one
  - This is possible when the limits for each function are often the same in subtraction questions

$$V = \pi \int_a^b {y_1}^2 \mathrm{d}x - \pi \int_a^b {y_2}^2 \mathrm{d}x = \pi \int_a^b \left({y_1}^2 - {y_2}^2\right) \mathrm{d}x$$

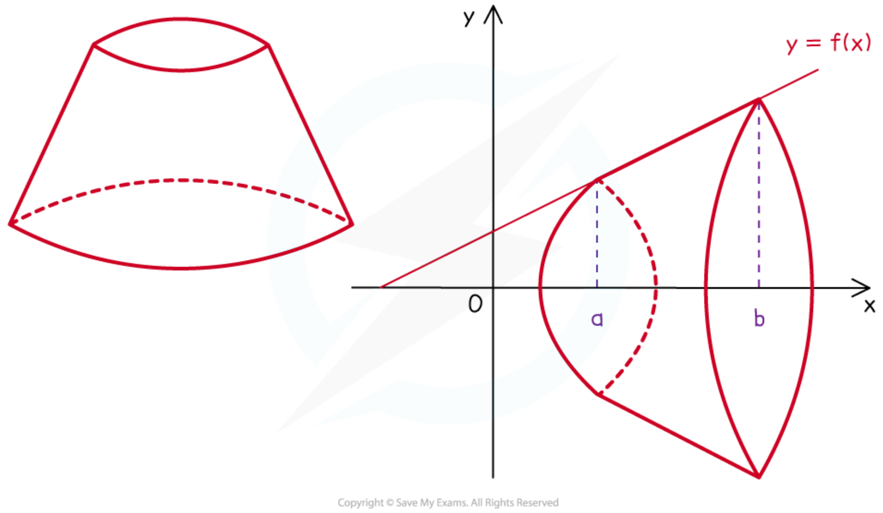
- This doesn't really apply to addition questions as if the limits are the same, you would be adding some of the same volume twice
- If in any doubt avoid this approach as **accuracy** is far more important

Your notes

Modelling with Volumes of Revolution

What is meant by modelling with volumes of revolution?

- Many every day objects such as buckets, beakers, vases and lamp shades can be modelled as a **solid of revolution**
- This can then be used to find the **volume** of the **solid** (**volume of revolution**) and/or other information about the solid that could be useful before an object is manufactured
- An object that would usually stand **upright** may have to be **modelled horizontally** so its **volume** of **revolution** can be found



An upright lamp shade modelled horizontally as a straight-line segment ($y = \mathrm{f}(x)$ between $x = a$ and $x = b$) rotated $360°$ around the x -axis

What modelling assumptions are there with volumes of revolution?

- The solids formed are usually the **main shape** of the **body** of the **object**
  - For example, the handles on a vase would **not** be included
  - The lip on the top edge of a bucket would **not** be considered
- A common question or **assumption** concerns the **thickness** of a container
  - The thickness is generally ignored as it is relatively small compared to the size of the object
    - thickness will depend on the purpose of the object and the material it is made from

- Some questions may refer to the solid formed being the 'inside' of an object or refer to the 'internal' dimensions
- If the thickness of the material is significant it would involve two related solids of revolution (**Adding & Subtracting Volumes**)

Your notes

How do I solve modelling problems with volumes of revolution?

- **Visualising** and sketching the solid formed can help with starting problems
- Familiarity with applying the volume of revolution formulae for rotations around both the x-axis

$$x\text{-axis}\quad V = \pi \int_a^b y^2 \, dx$$

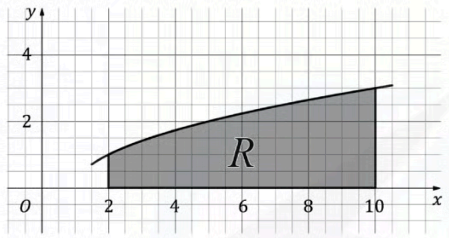
- The volume of a solid may involve adding or subtracting different volumes of revolution
  - Subtraction would need to be used for solids formed from areas that do not have a boundary with the axis of rotation
- Questions may go on to ask related questions in context so do take notice of the context
  - A question about a **bucket** being formed may ask about its **capacity**
- This would be measured in **litres** so there may also be a mix of **units** that will need conversion (e.g. 1000cm^3 = 1 litre)

Worked Example

Your notes

The diagram below shows the region R, which is bounded by the function $y = \sqrt{x-1}$, the lines $x = 2$ and $x = 10$, and the x-axis.

Dimensions are in centimetres.



A mathematical model for a miniature vase is produced by rotating the region R through 360° about the x-axis.

Find the volume of the miniature vase, giving your answer in litres to three significant figures.

STEP 1 SQUARE y

$y^2 = (\sqrt{x-1})^2 = x - 1$

STEP 2 IDENTIFY LIMITS

$a = 2$

$b = 10$

STEP 3 EVALUATE THE INTEGRAL

$$V = \pi \int_2^{10} (x-1)\,dx = \pi [0.5x^2 - x]_2^{10}$$

$$= \pi [(50-10) - (2-2)]$$

$$= 40\pi$$

NOW WE NEED TO INTERPRET THIS IN THE CONTEXT OF THE MINIATURE VASE

$V = 40\pi \text{ cm}^3$

$V = \frac{40\pi}{1000}$ LITRES $\quad$ $1000 \text{ cm}^3 = 1$ LITRE

$V = 0.125\,663...$

VOLUME OF THE MINIATURE VASE IS 0.126 LITRES (3.s.f.)

Examiner Tips and Tricks

- Consider the context of the question to gauge whether your final answer is realistic
  - a vase holding just 0.126 litres of water will not hold many flowers, but the question did state it was a *miniature* vase

Integration decision making

What is integration decision making?

- The hardest part of integration is deciding which technique to use
- There are lots of them!

How do I know which integration method to use?

Your notes

EXPERIENCE

PRACTISE AS MANY QUESTIONS AS POSSIBLE.

e.g. $\int 3xe^{x^2}\,dx$ ← HAVE A GO NOW!

FAMILIARITY

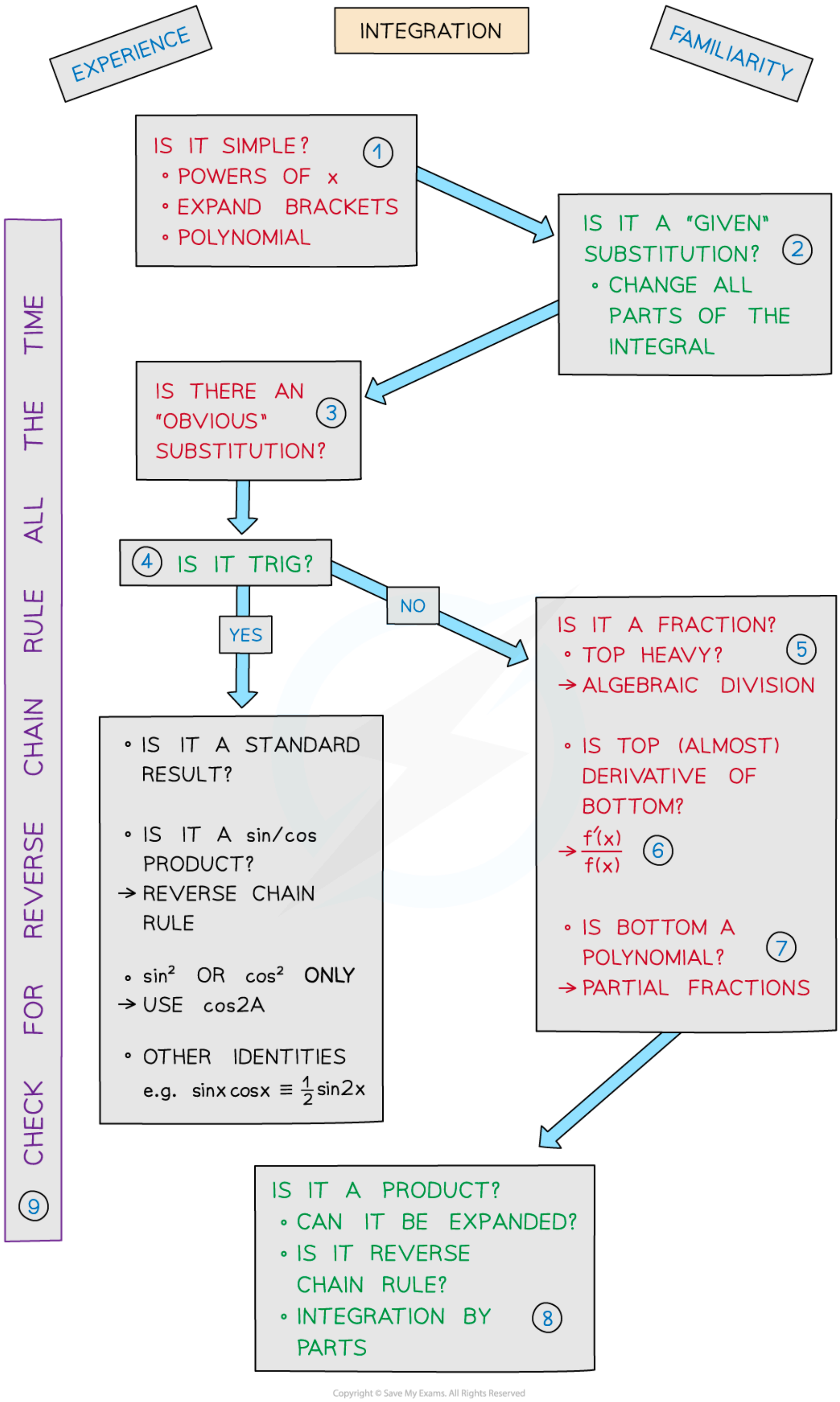
SOME, BUT NOT ALL, STANDARD RESULTS ARE IN THE FORMULAE BOOKLET.

- MEMORISE THOSE YOU HAVE TO
- BE FAMILIAR, BY EXPERIENCE, WITH THOSE IN THE FORMULA BOOKLET

e.g. $\int 2\cot 3x$ ← HAVE A GO NOW!

ANSWERS: $\frac{3}{2}e^{x^2}+c$, $\frac{2}{3}\ln|\sin 3x|+c$

- The main two requirements for success with integration are
  - **Experience**
  - **Familiarity**
- Always check for Reverse Chain Rule ...
  - ... particularly after changing part of an integral ...
  - ... by using a trig identity for example



Your notes

Key to diagram above:

(1) Powers of X

(2) Substitution

(3) Obvious substitution

(4) Trig

(5) Algebraic Division

(6) $\dfrac{f'(x)}{f(x)}$

(7) Partial fractions

(8) Integration by Parts

(9) Reverse Chain Rule

Examiner Tips and Tricks

- If in doubt, try something ... You may gain marks for attempting a suitable substitution.
- You may gain marks from using a trig identity.
- You may gain marks from attempting to simplify or rewrite a fraction.

Worked Example

Your notes

Your notes

Find the following

(a) $\int sin\theta cos\theta \; d\theta$

(b) $\int \frac{3x^2}{2+x^3} \, dx$

(c) $\int 3x\sqrt{x^2+1} \, dx$

a) $\int \sin\theta \cos\theta \, d\theta$

CLEAR TRIG?

$\sin 2A \equiv 2\sin A\cos A$

$= \int \frac{1}{2}\sin 2\theta \, d\theta$

$= \frac{1}{2} \times -\frac{1}{2}\int -2\sin 2\theta \, d\theta$

"ADJUST" & "COMPENSATE"

$= -\frac{1}{4}\cos 2\theta + c$

b) $\int \frac{3x^2}{2+x^3} dx$

$\frac{f'(x)}{f(x)}$

TOP IS DERIVATIVE OF BOTTOM

$= \ln|2+x^3| + c$

Your notes

c) $\int 3x\sqrt{x^2+1}\,dx$

THIS IS, IF YOU SPOT IT, REVERSE CHAIN RULE.
LET'S ASSUME THIS WAS NOT SPOTTED...

CHOOSE A SENSIBLE SUBSTITUTION

LET $u = x^2 + 1$

$x = \sqrt{u-1}$ $\quad \frac{du}{dx} = 2x$

$\frac{1}{2}du = x\,dx$

$$\int 3x\sqrt{x^2+1}\,dx = \int 3u^{\frac{1}{2}} \times \frac{1}{2}du$$

$$= \int \frac{3}{2}u^{\frac{1}{2}}$$

$$= \frac{\frac{3}{2}u^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= u^{\frac{3}{2}} + c$$

$$= (x^2+1)^{\frac{3}{2}} + c$$