Edexcel International A Level (IAL) Maths: Pure 4

Your notes

Implicit Differentiation

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Implicit differentiation

What is implicit differentiation?

- An equation connecting x and y is not always easy to write **explicitly** in the form $y = f(x)$ or $x = f(y)$
- However you can still differentiate such an equation **implicitly** using the **chain rule**:

$$1. \quad \frac{d}{dx} f(y) = f'(y)\frac{dy}{dx}$$

- $\frac{d}{dx}$ MEANS "THE DERIVATIVE WITH RESPECT TO x (OF WHAT FOLLOWS)". SO $\frac{d}{dx}f(y)$ HERE IS DERIVATIVE OF $f(y)$ **WITH RESPECT TO x.**
- $f'(y)$ IS THE DERIVATIVE OF $f(y)$ **WITH RESPECT TO y.**
- IT IS IMPORTANT TO KEEP THIS STRAIGHT! (SEE THE EXAMPLE BELOW).

e.g. FIND THE DERIVATIVE OF $\sin y$ WITH RESPECT TO x.

$$\frac{d}{dx}(\sin y) = (\cos y)\frac{dy}{dx}$$

$f(y) = \sin y$

$f'(y) = \cos y$

NOTE THAT WE DON'T HAVE AN EXPLICIT EXPRESSION FOR $\frac{dy}{dx}$ HERE

- Combining this with the **product rule** gives us:

2. $$\frac{d}{dx}(f(x)g(y)) = f'(x)g(y) + f(x)g'(y)\frac{dy}{dx}$$

- $f'(x)$ IS THE DERIVATIVE OF $f(x)$ WITH RESPECT TO x.
- $g'(y)$ IS THE DERIVATIVE OF $g(y)$ WITH RESPECT TO y.

e.g. FIND THE DERIVATIVE OF $3x^2e^{-4y}$ WITH RESPECT TO x.

$f(x) = 3x^2$
$g(y) = e^{-4y}$

$$\frac{d}{dx}(3x^2e^{-4y}) = \underbrace{6x}_{f'(x)}\,\underbrace{e^{-4y}}_{g(y)} + \underbrace{3x^2}_{f(x)}\underbrace{(-4e^{-4y})}_{g'(y)}\frac{dy}{dx}$$

$$= 6xe^{-4y} - 12x^2e^{-4y}\frac{dy}{dx}$$

AGAIN, WE DO NOT HAVE AN EXPLICIT EXPRESSION FOR $\frac{dy}{dx}$

- These two special cases are especially useful:

3. $$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$$

SPECIAL CASE OF 1. WITH $f(y) = y^n$

4. $$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$$

SPECIAL CASE OF 2. WITH $f(x) = x$ AND $g(y) = y$

Your notes

- When x and y are connected in an **equation** you can differentiate both sides with respect to x and rearrange to find a formula (usually in terms of ***x* and *y***) for **dy/dx**
  - Note that **dy/dx** is a single algebraic object
  - When rearranging do not treat **dy/dx** as a fraction
  - Especially do not try to separate **dy** and **dx** and treat them as algebraic objects on their own!

e.g. A CURVE IS DESCRIBED BY THE EQUATION

$$\sin y = 3x^2e^{-4y}$$

USE IMPLICIT DIFFERENTIATION TO FIND AN EXPRESSION FOR $\frac{dy}{dx}$.

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(3x^2e^{-4y})$$

DIFFERENTIATE BOTH SIDES WITH RESPECT TO x

$$\cos y\frac{dy}{dx} = 6xe^{-4y} - 12x^2e^{-4y}\frac{dy}{dx}$$

WE ALREADY WORKED OUT THESE DERIVATIVES ABOVE!

$$(12x^2e^{-4y} + \cos y)\frac{dy}{dx} = 6xe^{-4y}$$

REARRANGE TO MAKE $\frac{dy}{dx}$ THE SUBJECT

$$\frac{dy}{dx} = \frac{6xe^{-4y}}{12x^2e^{-4y} + \cos y}$$

Examiner Tips and Tricks

- When using implicit differentiation you will not always be able to write **dy/dx** simply as a function of x.
- However, this does not stop you from answering questions involving the derivative.

Worked Example

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A curve C has the equation

$$e^{3x} = y^2 - 4xy$$

a) Use implicit differentiation to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

b) Hence find the gradient of C at the point $(0, \ -1)$.

a) $\frac{d}{dx}(e^{3x}) = \frac{d}{dx}(y^2 - 4xy)$

DIFFERENTIATE BOTH SIDES WITH RESPECT TO x

$$3e^{3x} = \underbrace{2y\frac{dy}{dx}}_{\text{FORMULA 3}} \underbrace{- 4y - 4x\frac{dy}{dx}}_{\text{FORMULA 4}}$$

$$3e^{3x} + 4y = (2y - 4x)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3e^{3x} + 4y}{2y - 4x}$$

REARRANGE TO MAKE $\frac{dy}{dx}$ THE SUBJECT

b) AT $(0, -1)$ $x = 0$ AND $y = -1$

SO $\frac{dy}{dx} = \frac{3e^{3(0)} + 4(-1)}{2(-1) - 4(0)} = \frac{1}{2}$

SUBSTITUTE x- AND y-COORDINATES INTO THE EQUATION FOR $\frac{dy}{dx}$