Edexcel International A Level (IAL) Maths: Pure 4

Further Applications of Differentiation

Your notes

Contents

Connected rates of change

What are connected rates of change?

- In situations involving more than two variables you can use the **chain rule** to connect multiple rates of change into a single equation

e.g. THE VOLUME V ,OF, A CUBE IS INCREASING AT A CONSTANT RATE OF 3 cm^3 PER SECOND. ASSUMING THAT THE SHAPE REMAINS CUBICAL AT ALL TIMES, FIND THE RATE OF CHANGE OF THE SIDE LENGTH, S, IN cm PER SECOND AT THE MOMENT WHEN S = 5 cm.

$$\frac{dV}{dt} = 3 \text{ cm}^3/\text{sec}$$

WE KNOW THIS

WE WANT TO KNOW THIS

BY THE CHAIN RULE $\frac{dV}{dt} = \frac{dV}{ds} \times \frac{ds}{dt}$

USE THE CHAIN RULE TO CONNECT WHAT YOU KNOW AND WHAT YOU WANT TO KNOW

FOR A CUBE, $V = s^3$

SO $\frac{dV}{ds} = 3s^2$

USE DIFFERENTIATION TO GET THE 'MISSING' DERIVATIVE FROM THE CHAIN RULE EQUATION

SO $\frac{ds}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{ds}} = \frac{3}{3s^2} = \frac{1}{s^2}$

REARRANGE

WHEN $s = 5$,

$$\frac{ds}{dt} = \frac{1}{5^2} = \frac{1}{25} \text{ cm/sec}$$

- Equations involving derivatives (ie rates of changes) are known as **differential equations**
  - These can be solved using methods of integration (see Differential Equations)
  - However setting up the equation from the information given can involve the chain rule and connected rates of change

e.g. A RIGHT–ANGLED TRIANGLE HAS A BASE OF $3p$ cm AND A HEIGHT OF $2p$ cm, WHERE $p > 0$ IS A PARAMETER THAT IS INCREASING OVER TIME. THE RATE OF INCREASE OF p IS SUCH THAT THE RATE OF INCREASE OF THE THE AREA OF THE TRIANGLE, A, IS PROPORTIONAL TO THE SQUARE ROOT OF p. WRITE DOWN A DIFFERENTIAL EQUATION FOR THE RATE OF CHANGE OF p.

$\frac{dA}{dt} \propto \sqrt{p}$ SO $\frac{dA}{dt} = k\sqrt{p}$

A IS INCREASING SO $k > 0$

WHERE k IS A POSTITVE CONSTANT

WE KNOW THIS

BY THE CHAIN RULE, $\frac{dA}{dt} = \frac{dA}{dp} \times \frac{dp}{dt}$

USE THE CHAIN RULE TO CONNECT WHAT YOU KNOW AND WHAT YOU WANT TO KNOW

WE WANT TO KNOW THIS

THE AREA OF THE TRIANGLE IS

$A = \frac{1}{2} \times 3p \times 2p = 3p^2$

USE DIFFERENTIATION TO GET THE 'MISSING' DERIVATIVE FROM THE CHAIN RULE EQUATION

SO $\frac{dA}{dp} = 6p$

THEN $\frac{dA}{dt} = \frac{dA}{dp} \times \frac{dp}{dt}$

SUBSTITUTE AND REARRANGE

$k\sqrt{p} = 6p\left(\frac{dp}{dt}\right)$

AND $\frac{dp}{dt} = \frac{k\sqrt{p}}{6p}$

$\frac{dp}{dt} = \frac{k}{6\sqrt{p}}$ (WHERE k IS A POSITIVE CONSTANT)

- Note from the examples above that you will often need to differentiate a formula to get one of the rates of change you need

Examiner Tips and Tricks

- These problems can involve a lot of letters – be sure to keep track of what they all refer to.
- Be especially sure that you are clear about which letters are **variables** and which are **constants** – these behave very differently when differentiation is involved!

Your notes

Worked Example

In a manufacturing process, plastic spheres are produced in such a way that the volume, V, of a sphere increases at a constant rate of 10 cm³ per second. Find the rate of change of the surface area, A, of a sphere at the moment when the surface area is equal to 32π cm².

$\frac{dV}{dt} = 10 \text{ cm}^3/\text{sec}$

WE KNOW THIS

WE WANT TO KNOW THIS

BY THE CHAIN RULE $\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$

USE THE CHAIN RULE TO CONNECT WHAT YOU KNOW AND WHAT YOU WANT TO KNOW

$V = \frac{4}{3}\pi r^3$ (WHERE r IS THE RADIUS)

WRITE V IN TERMS OF A

$A = 4\pi r^2$

$A^{\frac{3}{2}} = (4\pi r^2)^{\frac{3}{2}} = 8\pi^{\frac{3}{2}} r^3 = 8\pi\sqrt{\pi}\, r^3$

$\frac{8\pi\sqrt{\pi}\, r^3}{6\sqrt{\pi}} = \frac{4}{3}\pi r^3$ SO $V = \frac{1}{6\sqrt{\pi}} A^{\frac{3}{2}}$

AND $\frac{dV}{dA} = \frac{3}{2} \times \frac{1}{6\sqrt{\pi}} A^{\frac{3}{2}-1} = \frac{1}{4\sqrt{\pi}} A^{\frac{1}{2}}$

DIFFERENTIATE TO FIND THE ‘MISSING’ DERIVATIVE

SO $\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$

SUBSTITUTE AND REARRANGE

$10 = \frac{1}{4\sqrt{\pi}} A^{\frac{1}{2}} \left(\frac{dA}{dt}\right)$

AND $\frac{dA}{dt} = \frac{40\sqrt{\pi}}{A^{\frac{1}{2}}} = 40\sqrt{\frac{\pi}{A}}$

WHEN $A = 32\pi$,

$\frac{dA}{dt} = 40\sqrt{\frac{\pi}{32\pi}} = \frac{40}{\sqrt{32}} = 5\sqrt{2} \text{ cm}^2/\text{sec}$