Edexcel International A Level (IAL) Maths: Pure 4

General Binomial Expansion

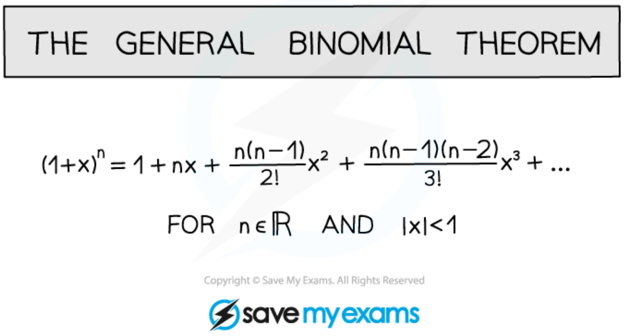
Your notes

Contents

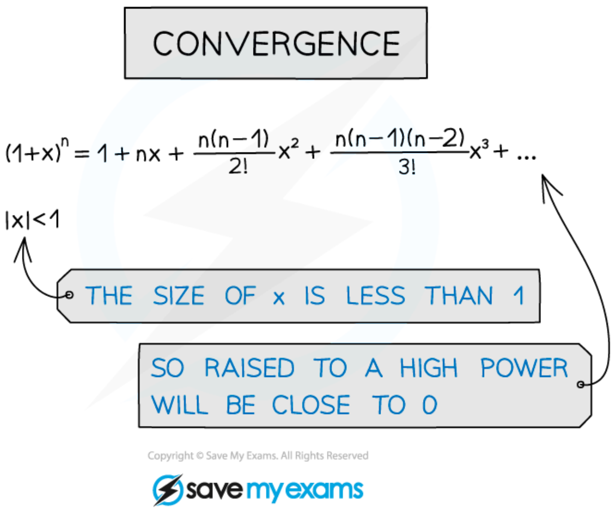
General binomial expansion

What is the general binomial expansion?

- The binomial expansion applies for positive integers, $n \in \mathbb{N}$
- The **general binomial expansion** applies to other types of powers too



- The **general binomial expansion** applies for all real numbers, $n \in \mathbb{R}$
- Usually fractional and/or negative values of **n** are used
- It is derived from $(\mathbf{a} + \mathbf{b})^n$, with $\mathbf{a = 1}$ and $\mathbf{b = x}$
- $\mathbf{a = 1}$ is the main reason the expansion can be reduced so much



- Unless $n \in \mathbb{N}$, the expansion is infinitely long
- It is only valid for $|x| < 1$

Your notes

- This is another way of writing $-1 < x < 1$
- This is often called the **validity statement**
- The restriction $|x| < 1$ means the **series** will **converge**
- Higher powers of x can be ignored (as $r \to \infty$, $x^r \to 0$)
- Only the first few terms of an expansion are needed

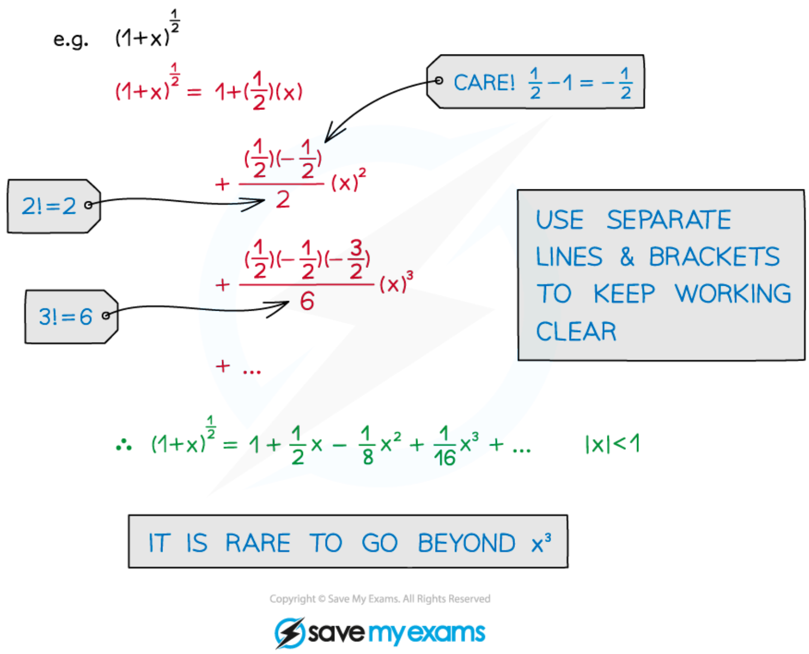
How do I apply the general binomial expansion?

STEP 1 Write the expression in the form $(1 + x)^n$
STEP 2 Expand and simplify

Use a line for each term to make things easier to read and follow

Use brackets - fractions and negatives get ugly!

STEP 3 If required, check and state the validity statement



How do I use the general binomial expansion to expand $(1+bx)^n$?

STEP 1 Write the expression in the form $(1 + bx)^n$
STEP 2 Replace "**x**" by "**bx**" in the expansion

Check carefully to see if **b** is negative

STEP 3 Expand and simplify

Use a line for each term to make things easier to read and follow

Use brackets

STEP 4 If required, check and state the validity statement

The validity statement changes

Replace "**x**" with "**bx**" so now **|bx| < 1**

e.g. FIND THE FIRST THREE TERMS IN THE EXPANSION OF $\frac{1}{\sqrt{1-x}}$

WRITE IN THE FORM $(1+bx)^n$

STEP 1: $\frac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}}$

REPLACE "x" WITH "−x"

STEP 2: $= 1 + (-\frac{1}{2})(-x)$

$+ \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(-x)^2 + \ldots$

STEP 3:

EXPAND AND SIMPLIFY

$\therefore \frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \ldots$

e.g. FOR WHICH VALUES OF x IS THE EXPANSION OF $\frac{1}{\sqrt{1-x}}$ VALID?

STEP 4: $\frac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}}$

VALID WHEN $|-x|<1$

REPLACE "x" WITH "−x"

$\therefore$ VALID WHEN $|x|<1$

Worked Example

Your notes

Your notes

(a) Find, up to the term in x^2, in ascending powers of x, the expansion of $\dfrac{3}{(1-\frac{2}{3}x)^2}$

(b) Write down the values of x for which the expansion in part (a) is valid.

a) STEP 1: $\dfrac{3}{(1-\frac{2}{3}x)^2} = 3(1-\frac{2}{3}x)^{-2}$

WRITE IN THE FORM $(1+bx)^n$

STEP 2:

REPLACE "x" WITH "$-\frac{2}{3}x$"

$= 3[1+(-2)(-\frac{2}{3}x) + \frac{(-2)(-3)}{2}(-\frac{2}{3}x)^2 + \ldots]$

STEP 3: $= 3(1+\frac{4}{3}x+\frac{4}{3}x^2+\ldots)$

EXPAND AND SIMPLIFY

$\therefore \dfrac{3}{(1-\frac{2}{3}x)^2} \approx 3+4x+4x^2$

b) STEP 4: $|-\frac{2}{3}x| < 1$

REPLACE "x" WITH "$-\frac{2}{3}x$"

$\therefore |x| < \frac{3}{2}$

Subtleties of the general binomial expansion

What is the general binomial expansion?

- The binomial expansion applies for positive integers, $\mathbf{n} \in \mathbb{N}$
- The **general binomial expansion** applies to other types of powers too

THE GENERAL BINOMIAL THEOREM

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

FOR $n \in \mathbb{R}$ AND $|x|<1$

- The **general binomial expansion** applies for all **real** numbers, $\mathbf{n} \in \mathbb{R}$
- Usually fractional and/or negative values of **n** are used
- It is derived from $(\mathbf{a} + \mathbf{b})^n$, with $\mathbf{a = 1}$ and $\mathbf{b = x}$
- Even when $\mathbf{a \neq 1}$ the general binomial expansion can still be used

How do I use the general binomial expansion to expand $(a+kx)^n$?

The general binomial expansion can be applied to expanding $(\mathbf{a} + \mathbf{kx})^n$

STEP 1 Rewrite the question into $(\mathbf{1} + \mathbf{bx})^n$ form

- Roots are fractional powers
- Denominators are negative powers
- A factor may be needed to make "**a = 1**"

STEP 2 Replace "x" by "bx" in the expansion

Check carefully to see if **b** is negative

STEP 3 Expand and simplify

Use a line for each term to make things easier to read and follow

Use brackets

STEP 4 If required, check and state the validity statement, **$|bx| < 1$**

Your notes

e.g. FIND THE FIRST 3 TERMS IN THE EXPANSION OF $(3+2x)^{-4}$ AND STATE THE VALUES OF x FOR WHICH IT IS VALID.

STEP 1: $(3+2x)^{-4} = [3(1+\frac{2}{3}x)]^{-4}$

$= 3^{-4}(1+\frac{2}{3}x)^{-4}$ ← $(1+bx)^n$ FORM

STEP 2: $= \frac{1}{81}[1 + (-4)(\frac{2}{3}x) + \frac{(-4)(-5)}{2}(\frac{2}{3}x)^2 + \ldots]$

REPLACE "x" WITH "$-\frac{2}{3}x$"

STEP 3: $= \frac{1}{81}(1 - \frac{8}{3}x + \frac{40}{9}x^2 + \ldots)$

EXPAND AND SIMPLIFY

$\therefore \quad (3+2x)^{-4} \approx \frac{1}{81} - \frac{8}{243}x + \frac{40}{729}x^2$

STEP 4: $|\frac{2}{3}x| < 1$

$|x| < \frac{3}{2}$

"$|bx| < 1$"

Worked Example

Your notes

(a) Find, up to the term in x^2, in ascending powers of x, the expansion of $\dfrac{1}{\sqrt[3]{8-3x}}$

(b) Write down the values of x for which the expansion in part (a) is valid.

a) STEP 1: $\dfrac{1}{\sqrt[3]{8-3x}} = (8-3x)^{-\frac{1}{3}}$

$= [8(1-\frac{3}{8}x)]^{-\frac{1}{3}}$

$= 8^{-\frac{1}{3}}(1-\frac{3}{8}x)^{-\frac{1}{3}}$

REWRITE IN $(1+bx)^n$ FORM

REPLACE "x" WITH "$-\frac{3}{8}x$"

CARE! $-\frac{1}{3}-1=-\frac{4}{3}$

STEP 2: $= \dfrac{1}{\sqrt[3]{8}}\left[1+(-\frac{1}{3})(-\frac{3}{8}x)+\dfrac{(-\frac{1}{3})(-\frac{4}{3})}{2}(-\frac{3}{8}x)^2+\ldots\right]$

STEP 3: $= \frac{1}{2}(1+\frac{1}{8}x+\frac{1}{32}x^2+\ldots)$

EXPAND AND SIMPLIFY

$\therefore \dfrac{1}{\sqrt[3]{8-3x}} \approx \frac{1}{2}+\frac{1}{16}x+\frac{1}{64}x^2$

b) STEP 4: $|-\frac{3}{8}x|<1$

"$|bx|<1$"

$\therefore |x|<\frac{8}{3}$

Multiple general binomial expansions

What is the general binomial expansion?

- The **general binomial expansion**, as given in the formula booklet, is

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{1\times 2\times \dots \times r}x^r + \dots \qquad (|x|<1,\ n\in\mathbb{R})$$

- If $n \in \mathbb{N}$ then the expansion is **finite** (see Binomial Expansion)
- Otherwise the expansion is infinitely long
  - It is only valid for $|x| < 1$ $(-1 < x < 1)$
  - Only the first few terms of an expansion are usually needed

What are multiple general binomial expansions?

- More than one part of an expression can be a binomial expansion
- These may sometimes be called **compound expressions**
- The expansion will only be valid for the lowest $|x|$ boundary from all the expansions used

How do I expand multiple binomials?

STEP 1 Break the expression down into binomial expansions

STEP 2 Expand each binomial individually, up to a suitable number of terms

- Be careful with negatives and fractions
- Use brackets as appropriate

STEP 3 Collect the expansions together and simplify

- This could be expanding brackets, collecting like terms, etc
- Ignore any terms of degree higher than required

STEP 4 Check the validity of each binomial expansion

- The overall validity is the intersection (∩)

COMPOUND EXPRESSIONS

e.g. FIND THE BINOMIAL EXPANSION OF $\frac{\sqrt{1+x}}{3+2x}$ UP TO AND INCLUDING THE TERM IN x^3. STATE THE VALUES OF x FOR WHICH THE EXPANSION IS VALID.

$$\frac{\sqrt{1+x}}{3+2x} = (1+x)^{\frac{1}{2}}(3+2x)^{-1}$$

STEP 1: BREAK THE EXPRESSION DOWN INTO BINOMIAL EXPANSIONS

$$(1+x)^{\frac{1}{2}} = 1 + (\tfrac{1}{2})x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{6}x^3 + \ldots$$

USE BRACKETS

$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + \ldots$$

STEP 2: EXPAND EACH BINOMIAL INDIVIDUALLY

$$(3+2x)^{-1} = 3^{-1}(1+\tfrac{2}{3}x)^{-1}$$

$$= \tfrac{1}{3}\left[1 + (-1)(\tfrac{2}{3}x) + \frac{(-1)(-2)}{2}(\tfrac{2}{3}x)^2 + \frac{(-1)(-2)(-3)}{6}(\tfrac{2}{3}x)^3 + \ldots\right]$$

$$= \tfrac{1}{3}\left[1 - \tfrac{2}{3}x + \tfrac{4}{9}x^2 - \tfrac{8}{27}x^3 + \ldots\right]$$

$$= \tfrac{1}{3} - \tfrac{2}{9}x + \tfrac{4}{27}x^2 - \tfrac{8}{81}x^3 + \ldots$$

$$\frac{\sqrt{1+x}}{3+2x} = (1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + \ldots)(\tfrac{1}{3} - \tfrac{2}{9}x + \tfrac{4}{27}x^2 - \tfrac{8}{81}x^3 + \ldots)$$

$$\approx \tfrac{1}{3} - \tfrac{2}{9}x + \tfrac{4}{27}x^2 - \tfrac{8}{81}x^3$$
$$+ \tfrac{1}{6}x - \tfrac{1}{9}x^2 + \tfrac{2}{27}x^3$$
$$- \tfrac{1}{24}x^2 + \tfrac{1}{36}x^3$$
$$+ \tfrac{1}{48}x^3$$

x^4 AND HIGHER IGNORED

STEP 3: EXPAND BRACKETS IN THE USUAL WAY

SEPARATE LINES KEEP IT TIDY

$$= \tfrac{1}{3} - \tfrac{1}{18}x - \tfrac{1}{216}x^2 + \tfrac{31}{1296}x^3$$

Your notes

$\sqrt{1+x} = (1+x)^{\frac{1}{2}} \quad |x| < 1$

$(3+2x)^{-1} = \frac{1}{3}(1+\frac{2}{3}x)^{-1} \quad |\frac{2}{3}x| < 1 \quad |x| < \frac{3}{2}$

STEP 4: CHECK THE VALIDITY

$\therefore$ EXPANSION IS ONLY VALID FOR $|x|<1$

Your notes

How do I use partial fractions to form multiple general binomial expansions?

- Partial fractions allow rational expressions to be written in a form where the general binomial expansion can then be applied e.g. FIND THE BINOMIAL EXPANSION OF $\frac{9x+10}{(x+4)(3x-1)}$

UP TO AND INCLUDING THE TERM IN x^2

$$\frac{9x+10}{(x+4)(3x-1)} = \frac{2}{x+4} + \frac{3}{3x-1}$$

WRITE AS PARTIAL FRACTIONS

$$= 2(x+4)^{-1} + 3(3x-1)^{-1}$$

STEP 1: BREAK EXPRESSION DOWN INTO BINOMIAL EXPANSIONS

$$2(x+4)^{-1} = 2(4+x)^{-1}$$

$$= 2[4^{-1}(1+\tfrac{1}{4}x)^{-1}]$$

$$= \tfrac{1}{2}(4+x)^{-1} = \tfrac{1}{2}[1 + (-1)(\tfrac{x}{4}) + \frac{(-1)(-2)}{2}(\tfrac{x}{4})^2 + \ldots]$$

$$= \tfrac{1}{2} - \tfrac{1}{8}x + \tfrac{1}{32}x^2 - \ldots$$

STEP 2: EXPAND EACH BINOMIAL INDIVIDUALLY

$$3(3x-1)^{-1} = 3(-1+3x)^{-1}$$

$$= 3[(-1)^{-1}(1-3x)^{-1}]$$

$$= -3(1-3x)^{-1} = -3[1+(-1)(-3x) + \frac{(-1)(-2)}{2}(-3x)^2 + \ldots]$$

$$= -3 - 9x - 27x^2 - \ldots$$

$$\therefore \frac{9x+10}{(x+4)(3x-1)} = (\tfrac{1}{2} - \tfrac{1}{8}x + \tfrac{1}{32}x^2 - \ldots)$$

$$+(-3 - 9x - 27x^2 - \ldots)$$

STEP 3: COLLECT EXPANSIONS AND SIMPLIFY

LINE UP "LIKE TERMS" MAKES SIMPLIFYING EASIER

- Validity is an important part of the general binomial expansion

Your notes

Your notes

e.g. FOR WHICH VALUES OF x IS THE EXPANSION OF $\frac{9x+10}{(x+4)(3x-1)}$ VALID?

EXPANSIONS USED ARE $(1+\frac{1}{4}x)^{-1}$ AND $(1-3x)^{-1}$

STEP 4: CONSIDER VALIDITY OF EXPANSIONS USED SEPARATELY

$(1+\frac{1}{4}x)^{-1}$ IS VALID FOR $|\frac{1}{4}x|<1$ $\quad |x|<4$

$(1-3x)^{-1}$ IS VALID FOR $|-3x|<1$ $\quad |x|<\frac{1}{3}$

$\therefore$ EXPANSION OF $\frac{9x+10}{(x+4)(3x-1)}$ IS VALID FOR $|x|<\frac{1}{3}$

STEP 4: THE INTERSECTION ($\cap$) (OVERLAP) OF THE TWO INEQUALITIES

Worked Example

Your notes

(a) Find, up to the term in x^2, in ascending powers of x, the expansion of $\left(\frac{1-x}{1+2x}\right)^{\frac{1}{3}}$

(b) Write down the values of x for which the expansion in part (a) is valid.

STEP 1: WRITE AS BINOMIALS

a) $\left(\frac{1-x}{1+2x}\right)^{\frac{1}{3}} = (1-x)^{\frac{1}{3}}(1+2x)^{-\frac{1}{3}}$

$\frac{1}{3}-1=-\frac{2}{3}$

STEP 2: EXPAND INDIVIDUALLY

$(1-x)^{\frac{1}{3}} = 1+\frac{1}{3}(-x)+\frac{(\frac{1}{3})(-\frac{2}{3})}{2}(-x)^2+\ldots$

$= 1-\frac{1}{3}x-\frac{1}{9}x^2+\ldots$

$(1+2x)^{-\frac{1}{3}} = 1+(-\frac{1}{3})(2x)+\frac{(-\frac{1}{3})(-\frac{4}{3})}{2}(2x)^2+\ldots$

$= 1-\frac{2}{3}x+\frac{8}{9}x^2+\ldots$

STEP 3: COLLECT EXPANSION AND SIMPLIFY

$\therefore \left(\frac{1-x}{1+2x}\right)^{\frac{1}{3}} = (1-\frac{1}{3}x-\frac{1}{9}x^2+\ldots)(1-\frac{2}{3}x+\frac{8}{9}x^2+\ldots)$

$\approx 1-\frac{2}{3}x+\frac{8}{9}x^2$

$-\frac{1}{3}x+\frac{2}{9}x^2$

$-\frac{1}{9}x^2$

$\left(\frac{1-x}{1+2x}\right)^{\frac{1}{3}} \approx 1-x+x^2$

Your notes

b) $(1-x)^{\frac{1}{3}}$ IS VALID FOR $|-x|<1$ $\quad |x|<1$

$(1+2x)^{-\frac{1}{3}}$ IS VALID FOR $|2x|<1$ $\quad |x|<\frac{1}{2}$

$\therefore$ EXPANSION IN PART a) IS VALID FOR $|x|<\frac{1}{2}$

STEP 4: CHECK THE INDIVIDUAL VALIDITIES
OVERALL IS THE INTERSECTION $(\cap)$

Approximating values using the binomial expansion

What is the general binomial expansion?

- The **general binomial expansion**, as given in the formula booklet, is

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1\times 2\times \ldots \times r}x^r + \ldots \qquad (|x|<1, \ n\in\mathbb{R})$$

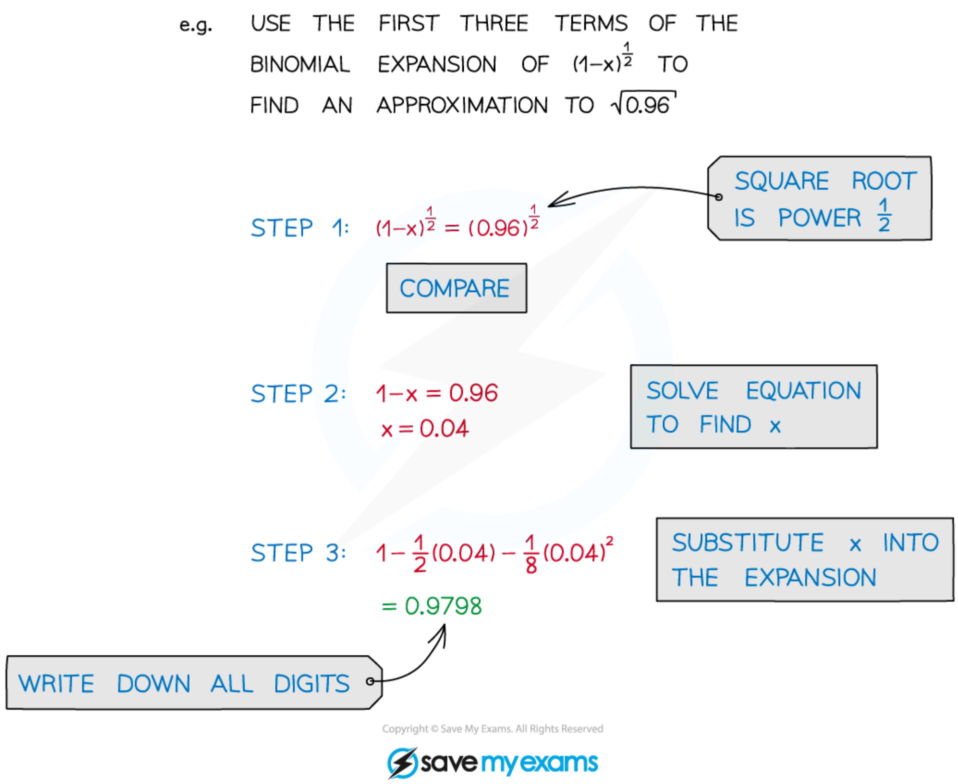
- If $n \in \mathbb{N}$ then the expansion is **finite** (see Binomial Expansion)
- Otherwise the expansion is infinitely long
    - It is only valid for **|x| < 1** (-1 < x < 1)
    - Only the first few terms of an expansion are usually needed

How do I use a binomial expansion to approximate a value?

$$\text{e.g.} \quad (1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \ldots \qquad |x|<1$$

SINCE $|x|<1$ x^4, x^5 etc WILL BE VERY SMALL

- Ignoring higher powers of **x** leads to an approximation
- The more terms the closer the approximation is to the true value
- For most purposes, squared or cubed terms are accurate enough



STEP 1 Compare the value you are approximating to $(\mathbf{a + bx})^n$
STEP 2 Solve the appropriate equation to find the value of **x**
STEP 3 Substitute this value of **x** into the expansion to find the approximation

Examiner Tips and Tricks

- You can get a good idea if your approximation is correct by working out the "real" answer using your calculator.
- Sometimes it helps to factorise out a number before approximating
  - $\sqrt{710} = 10\sqrt{7.1}$

Worked Example

Your notes

Your notes

(a) Find, up to the term including x^2, the binomial expansion of $\sqrt[4]{81-9x}$

(b) Use your expansion from part (a) to approximate $\sqrt[4]{85}$

(c) Explain why your expansion would not be suitable for approximating $\sqrt[4]{171}$

a)

$$\sqrt[4]{81-9x} = (81-9x)^{\frac{1}{4}}$$

$$= 81^{\frac{1}{4}}\left(1-\tfrac{1}{9}x\right)^{\frac{1}{4}}$$

$$= 3\left[1+\left(\tfrac{1}{4}\right)\left(-\tfrac{1}{9}x\right) + \frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2}\left(-\tfrac{1}{9}x\right)^2+\ldots\right]$$

$$\approx 3\left(1-\tfrac{1}{36}x-\tfrac{1}{864}x^2\right)$$

$$\therefore \sqrt[4]{81-9x} \approx 3-\tfrac{1}{12}x-\tfrac{1}{288}x^2$$

b) STEP 1: $\sqrt[4]{81-9x} = \sqrt[4]{85}$ COMPARE

STEP 2:

$$81-9x = 85$$

$$9x = -4$$

$$x = -\tfrac{4}{9}$$

SOLVE TO FIND x

STEP 3:

$$\sqrt[4]{81-9x} \approx 3-\tfrac{1}{12}\left(-\tfrac{4}{9}\right)-\tfrac{1}{288}\left(-\tfrac{4}{9}\right)^2$$

$$=3.036351166$$

SUBSTITUTE INTO EXPANSION

Your notes

c) $\sqrt[4]{81-9x} = \sqrt[4]{171}$

$81-9x = 171$

$9x = -90$

$x = -10$

THIS IS A WELL HIDDEN QUESTION ABOUT VALIDITY!

EXPANSION IS ONLY VALID FOR $|-\frac{1}{9}x|<1$

$|x|<9$

$\therefore$ EXPANSION IS NOT VALID WHEN $x=-10$ AND SO $\sqrt[4]{171}$ CANNOT BE APPROXIMATED