Edexcel International A Level (IAL) Maths: Pure 4

Your notes

Vectors in 3D

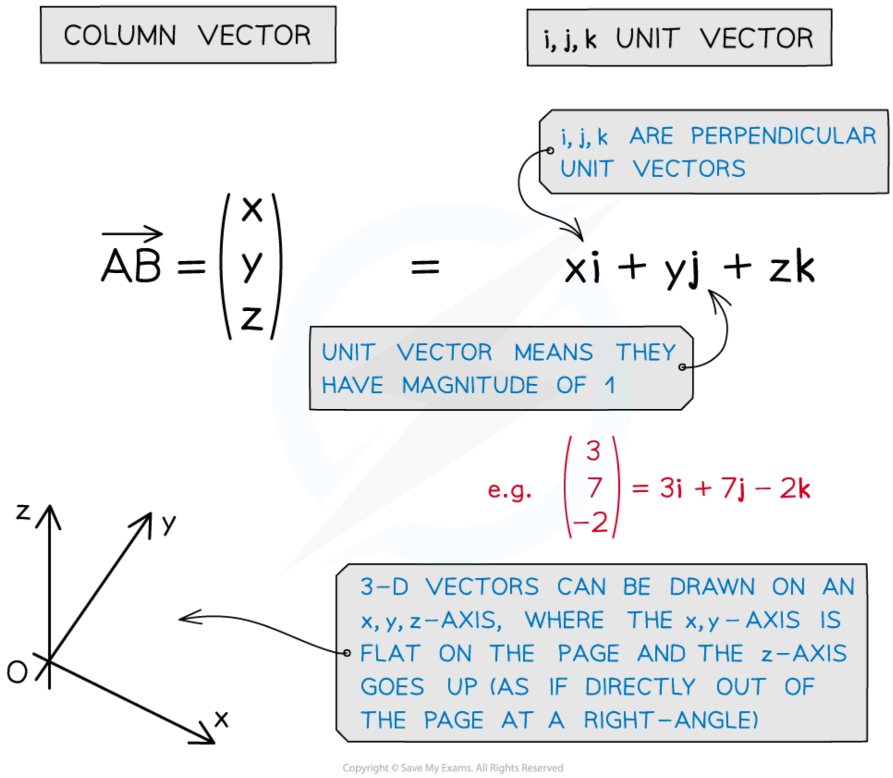
Contents

Vectors in 3 dimensions

What is a 3D vector?

- Vectors represent a movement of a certain **magnitude** (size) in a given **direction**

  You should have already come across (2D) **vectors** at AS (see Basic Vectors)
- 3D vectors describe the **position of a point** in a 3D space in relation to the origin
- They can be represented in different ways such as a **column vector** or in **i, j, k unit vector form**



Magnitude of a 3D vector

- The magnitude of a 3D vector is simply its size
- Like 2D vectors we can find the **magnitude** using Pythagoras' theorem (see Magnitude Direction)

Your notes

MAGNITUDE

$$|\mathbf{a}| = |x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \left|\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right| = \sqrt{x^2 + y^2 + z^2}$$

REMEMBER THERE ARE LOTS OF DIFFERENT WAYS TO REPRESENT THE SAME VECTOR

$$|\mathbf{a}| = |\overrightarrow{AB}| = \left|\begin{pmatrix} 3 \\ 7 \\ -2 \end{pmatrix}\right| = |3\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}|$$

A VECTOR'S MAGNITUDE IS SOMETIMES REFERRED TO AS ITS MODULUS

$$|\overrightarrow{AB}| = \sqrt{3^2 + 7^2 + 2^2} = \sqrt{62} = 7.874... = 7.9 \text{ (1 dp)}$$

YOU CAN IGNORE MINUS SIGN

- For 3D **position vectors** we can find the distance between two points
- By using the respective co-ordinates we can calculate the magnitude of the vector between them:

DISTANCE BETWEEN TWO POINTS FROM POSITION VECTORS

$$(x_1, y_1, z_1) \text{ AND } (x_2, y_2, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

WHERE x, y, z ARE FROM POSITION VECTORS $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

e.g. FIND $|\overrightarrow{AB}|$ FOR POSITION VECTORS A AND B, $5\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$ AND $3\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}$ RESPECTIVELY

A HAS CO-ORDINATES $(5, 3, -7)$

B HAS CO-ORDINATES $(3, -6, 5)$

$$|\overrightarrow{AB}| = \sqrt{(5-3)^2 + (3 - -6)^2 + (-7-5)^2} = \sqrt{4 + 81 + 144}$$

$$= \sqrt{229} = 15.1 \text{ UNITS (1 dp)}$$

3D vector addition, scalars, parallel vectors and unit vectors

Your notes

- 3D vectors work in the same way as 2D vectors, just in three dimensions rather than two
- **Vector addition** and subtraction and scalar multiplication can be carried out in exactly the same way, this time involving **i, j** and **k** or x, y and z
- 3D vectors are also **parallel** if one is a **multiple** of the other

3–D VECTORS

COLUMN VECTOR

$$\mathbf{a} = \begin{pmatrix} 2 \\ 6 \\ -4 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -3 \\ 7 \\ 4 \end{pmatrix}$$

$$2\mathbf{a} - 3\mathbf{b} = 2 \times \begin{pmatrix} 2 \\ 6 \\ -4 \end{pmatrix} - 3 \times \begin{pmatrix} -3 \\ 7 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 12 \\ -8 \end{pmatrix} - \begin{pmatrix} -9 \\ 21 \\ 12 \end{pmatrix} = \begin{pmatrix} 13 \\ -9 \\ -20 \end{pmatrix}$$

i, j, k NOTATION

$$\mathbf{a} = -3\mathbf{i} + 9\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{b} = 8\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$$

$$2\mathbf{a} + \frac{1}{2}\mathbf{b} = 2(-3\mathbf{i} + 9\mathbf{j} + 2\mathbf{k}) + \frac{1}{2}(8\mathbf{i} - 6\mathbf{j} + 4\mathbf{k})$$

$$= (-6\mathbf{i} + 18\mathbf{j} + 4\mathbf{k}) + (4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$$

$$= -2\mathbf{i} + 15\mathbf{j} + 6\mathbf{k}$$

PARALLEL VECTORS

$$\mathbf{a} = \begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix}$$

a IS PARALLEL TO **b** AS

$$\mathbf{b} = \frac{2}{3}\mathbf{a} \qquad \frac{2}{3}\begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix}$$

- **Unit vectors** in 3D are found in exactly the same way as in 2D

Your notes

TO FIND THE **UNIT VECTOR** IN THE SAME DIRECTION AS A GIVEN VECTOR, SIMPLY DIVIDE THE VECTOR BY ITS MAGNITUDE

- THE UNIT VECTOR IN THE SAME DIRECTION AS VECTOR $\overrightarrow{AB}$ IS $\dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$

- THE UNIT VECTOR IN THE SAME DIRECTION AS VECTOR $\mathbf{a}$ IS $\dfrac{\mathbf{a}}{|\mathbf{a}|}$

e.g. FIND THE UNIT VECTOR IN THE SAME DIRECTION AS THE VECTOR $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$

$$|\mathbf{a}| = \sqrt{3^2 + (-4)^2 + 5^2} = 5\sqrt{2}$$

$$\text{UNIT VECTOR} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}}{5\sqrt{2}} = \frac{1}{5\sqrt{2}}(3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$$

Worked Example

Your notes

Points A and B have position vectors $-\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $3\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$ respectively.

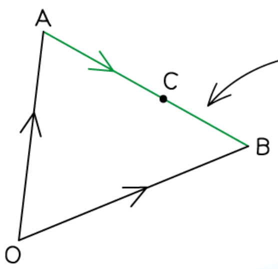
Point C divides the line AB in the ratio 3:2.

a) Find the position vector of C.

b) Calculate the distance of C from the origin.

Give your answer to 2 decimal places.

a)



START BY DRAWING A DIAGRAM TO HELP, DON'T WORRY ABOUT THE 3-D ASPECT, THE VECTORS WILL TAKE CARE OF THAT!

FIND VECTORS FOR ANY UNKNOWN PARTS OF THE DIAGRAM

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (3\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}) - (-\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})$$
$$= (3 - -1)\mathbf{i} + (7 - -2)\mathbf{j} + (-2 - 5)\mathbf{k}$$
$$= 4\mathbf{i} + 9\mathbf{j} - 7\mathbf{k}$$

$$\overrightarrow{AC} = \frac{3}{5}(4\mathbf{i} + 9\mathbf{j} - 7\mathbf{k})$$
$$= \frac{12}{5}\mathbf{i} + \frac{27}{5}\mathbf{j} - \frac{21}{5}\mathbf{k}$$

RATIO 3:2 MEANS C IS $\frac{3}{5}$ ALONG $\overrightarrow{AB}$

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = (-\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}) + \left(\frac{12}{5}\mathbf{i} + \frac{27}{5}\mathbf{j} - \frac{21}{5}\mathbf{k}\right)$$
$$= \frac{7}{5}\mathbf{i} + \frac{17}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}$$

POSITION VECTOR NEEDS TO START FROM ORIGIN

b) $$|\overrightarrow{OC}| = \left|\frac{7}{5}\mathbf{i} + \frac{17}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}\right|$$

DISTANCE = MAGNITUDE

$$|\overrightarrow{OC}| = \sqrt{\left(\frac{7}{5}\right)^2 + \left(\frac{17}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \frac{\sqrt{354}}{5}$$

$= 3.76$ UNITS (2 dp)

Your notes

Problem solving using 3D vectors

How are 3D vectors used for problem solving?

- 3D vector problems can be solved using the same principles as **2D vector problems** (see Problem Solving using Vectors)
- Vectors can be used to prove two lines are **parallel**, to show points are **collinear** (lie on the same straight line) or to find missing vertices of a given shape

Your notes

VECTORS ARE PARALLEL IF ONE IS A SCALAR **MULTIPLE** OF THE OTHER

PARALLEL VECTORS

$$\mathbf{a} = \begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix}$$

a IS PARALLEL TO **b** AS

$$\mathbf{b} = \frac{2}{3}\mathbf{a} \qquad \frac{2}{3}\begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix}$$

USE COORDINATES TO GIVE POSITION VECTORS THEN FIND AB AND AC TO PROVE THEY ARE COLLINEAR

THREE POINTS P, Q, R ARE **COLLINEAR** IF $\overrightarrow{PQ}$ AND $\overrightarrow{PR}$ ARE PARALLEL

SHOW THAT POINTS A(1, 2, 3) B(3, 8, 1) AND C(7, 20, −3) ARE COLLINEAR

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix}$$

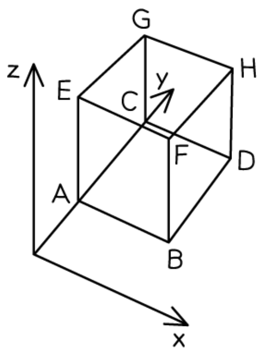
$$\overrightarrow{AC} = \begin{pmatrix} 7 \\ 20 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 18 \\ -6 \end{pmatrix}$$

$\overrightarrow{AC} = 3\overrightarrow{AB}$ SO $\overrightarrow{AC}$ IS PARALLEL TO $\overrightarrow{AB}$.

$\therefore$ ABC ARE COLLINEAR

3-D VECTOR PROBLEMS

PARALLEL SIDES OF A SHAPE SUCH AS A CUBOID HAVE THE **SAME VECTOR**



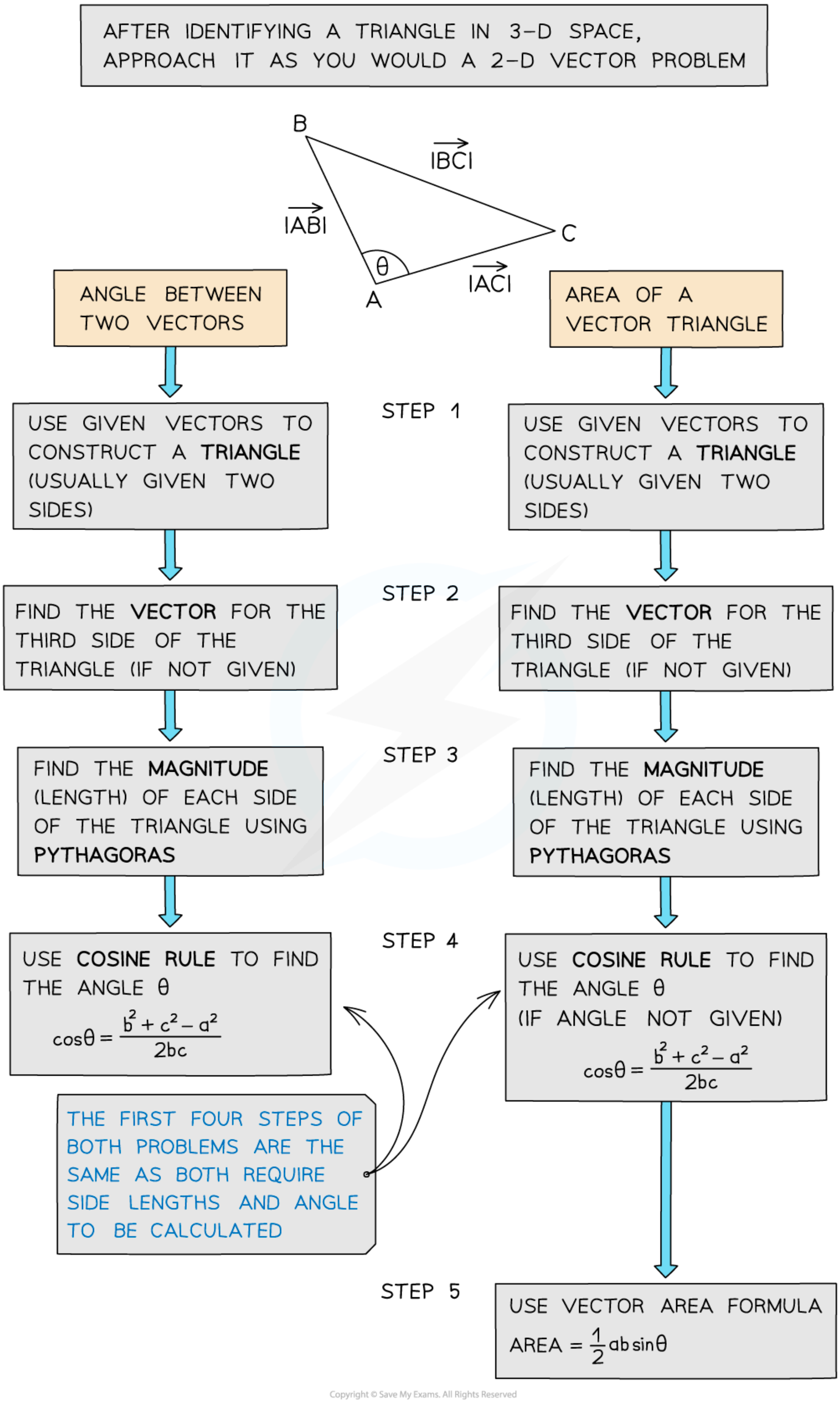
A CUBOID HAS POINTS B AND G SUCH THAT B HAS POSITION VECTOR $5\mathbf{i} + 3\mathbf{j}$ AND G HAS POSITION VECTOR $7\mathbf{j} + 6\mathbf{k}$.

FIND THE POSITION VECTOR OF POINT H IN UNIT VECTOR FORM

$$H = \begin{pmatrix} 5 \\ 7 \\ 6 \end{pmatrix} = 5\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$$

When do I use trigonometry in 3D vector problems?

- 3D vector problems can also involve using trigonometry to:
  - find the **angle between two vectors** using **Cosine Rule**
  - find the **area of a triangle** using a variation of **Area Formula**



- You can find the angle between a vector and any one of the coordinate axes by using the following formulae

Your notes

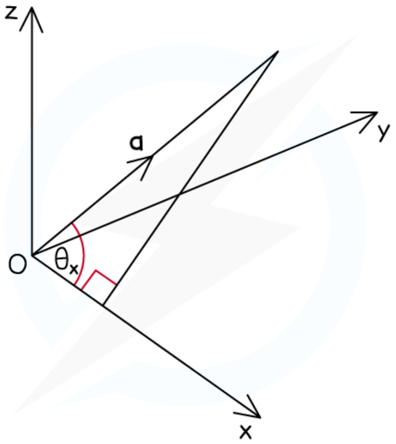
IF θ_x, θ_y AND θ_z ARE RESPECTIVELY THE ANGLES THAT THE VECTOR $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ MAKES WITH THE $x-$, $y-$ AND $z-$AXES, THEN

$$\cos\theta_x = \frac{x}{|\mathbf{a}|}$$

$$\cos\theta_y = \frac{y}{|\mathbf{a}|}$$

$$\cos\theta_z = \frac{z}{|\mathbf{a}|}$$

THESE FORMULAE COME FROM APPLYING SOHCAHTOA (SPECIFICALLY CAH) TO A RIGHT-TRIANGLE FORMED BY THE VECTOR AND ONE OF THE COORDINATE AXES.



IN THE DIAGRAM ABOVE FOR EXAMPLE, THE LENGTH OF THE SIDE ADJACENT TO ANGLE θ_x IS SIMPLY THE $\mathbf{x}$ COMPONENT OF THE VECTOR $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, WHILE THE LENGTH OF THE HYPOTENUSE IS $|\mathbf{a}|$.

- These formulae are not in the exam formulae book
- However, they may be derived using SOHCAHTOA as shown in the diagram
- The formulae also work for vectors in two dimensions

Examiner Tips and Tricks

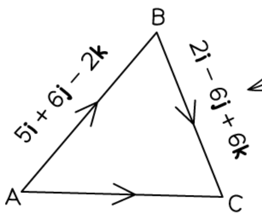
- If there is a diagram, labelling all known vectors and quantities will help, and don't worry about trying to make your diagram 3–D, as long as you label it well.

Your notes

Worked Example

The points A, B and C are such that $\overrightarrow{AB} = 5\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{BC} = 2\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}$.

Show that ABC is an isosceles triangle.



DRAW A DIAGRAM

TO SHOW ABC IS AN ISOSCELES TRIANGLE WE MUST SHOW TWO SIDE LENGHTS ARE EQUAL

FIND ANY MISSING VECTORS

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= (5\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}) + (2\mathbf{i} - 6\mathbf{j} + 6\mathbf{k})$

$= 7\mathbf{i} + 4\mathbf{k}$

$6\mathbf{j} - 6\mathbf{j} = 0\mathbf{j}$ SO DOES NOT APPEAR

FIND MAGNITUDE OF EACH SIDE

$|\overrightarrow{AB}| = \sqrt{5^2 + 6^2 + 2^2} = \sqrt{65}$

$|\overrightarrow{BC}| = \sqrt{2^2 + 6^2 + 6^2} = \sqrt{76}$

$|\overrightarrow{AC}| = \sqrt{7^2 + 4^2} = \sqrt{65}$

TWO SIDES ARE EQUAL LENGTH

$|\overrightarrow{AB}| = |\overrightarrow{AC}|$ THEREFORE TRIANGLE ABC IS ISOSCELES