Edexcel International A Level (IAL) Maths: Pure 3

Your notes

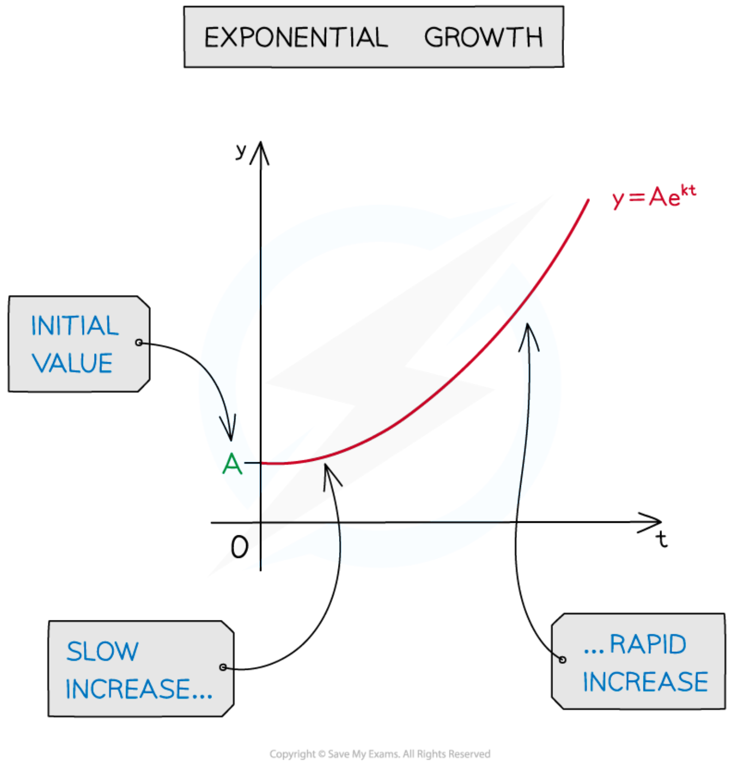
Modelling with Exponentials & Logarithms

Contents

Exponential growth & decay

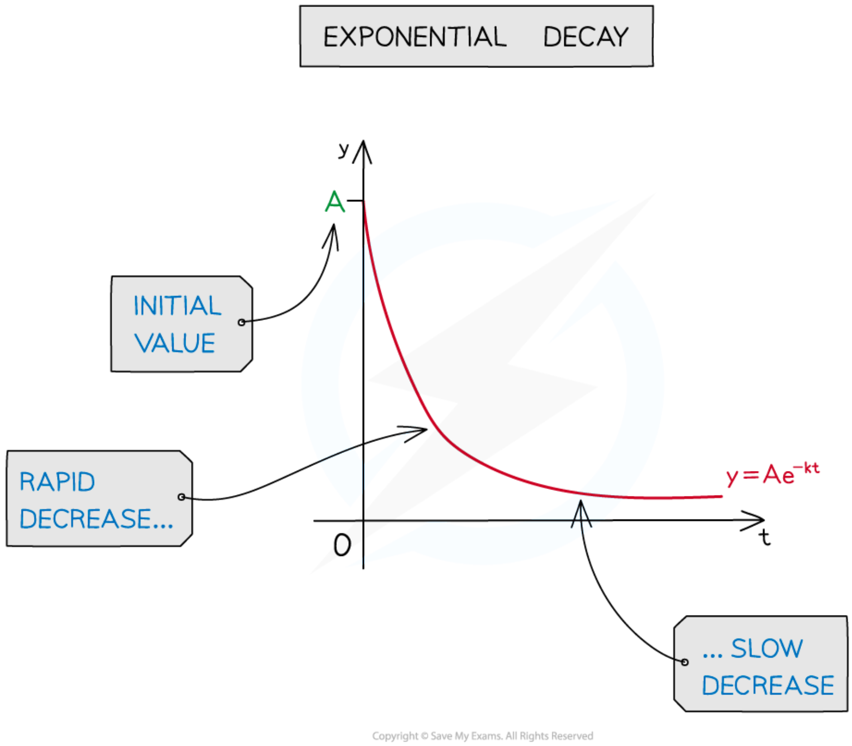
What is exponential growth?

- $y = Ae^{kt}$ is exponential **growth**



What is exponential decay?

- $y = Ae^{-kt}$ is exponential **decay**



- **k > 0** so plus or minus sign in equation immediately indicates growth or decay
- **A** is the **starting** value or **initial** value, when **t = 0**
- **t** is often used, rather than **x**, as many models are time-dependent

How do I write a^x in terms of e^x?

- Using **e** makes the maths easier
- With the proper choice of **k**, all exponential functions of the form a^x can be written in the form e^{kx}
  - $a^x = e^{kx}$

Your notes

e.g. WRITE $\left(\frac{1}{5}\right)^x$ IN THE FORM e^{kx}

$$\left(\frac{1}{5}\right)^x = e^{-kx}$$

$$(5^{-1})^x = e^{-kx} \qquad \boxed{\text{LAWS OF INDICES}}$$

$$5^{-x} = e^{-kx}$$

$$\ln 5^{-x} = \ln e^{-kx} \qquad \boxed{\text{LAWS OF LOGARITHMS}}$$

$$-\cancel{x}\ln 5 = -k\cancel{x}\overset{1}{\cancel{\ln e}} \qquad \boxed{\ln e = 1}$$

$$k = \ln 5$$

$$\therefore \left(\frac{1}{5}\right)^x = e^{-x\ln 5}$$

NOTE: THIS WOULD BE EXPONENTIAL DECAY

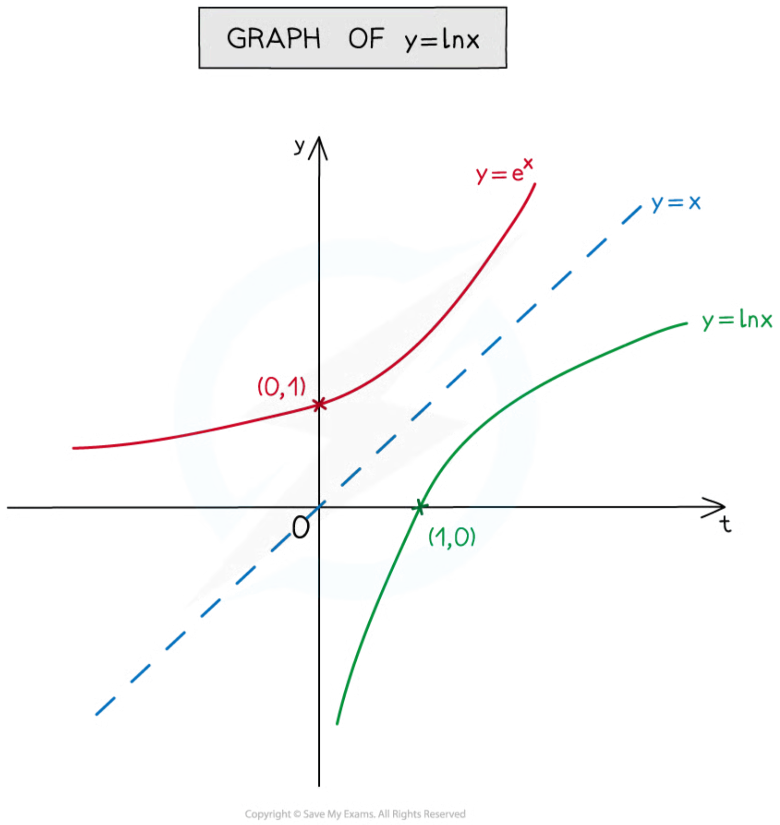
How are e^x and $\ln x$ inverses of each other?

$$x \xrightarrow{e} e^x$$

$$x \xleftarrow{\ln} e^x$$

$$\boxed{\ln x \equiv \log_e x}$$

- **ln** is the inverse of **e**



- The graph of **y = ln x** is the reflection of the graph of **y = e^x** in the line **y = x**

Worked Example

Your notes

(a) Write $\left(\frac{2}{5}\right)^{2t}$ in the form e^{-kt}, where k is a positive constant.

Give your value of k to 3 significant figures.

(b) State whether $\left(\frac{2}{5}\right)^{2t}$ would represent exponential growth or exponential decay.

(c) Sketch the graph of $y = \left(\frac{2}{5}\right)^{2t} - 3$

a) $\left(\frac{2}{5}\right)^{2t} = e^{-kt}$

$\ln\left(\frac{2}{5}\right)^{2t} = \ln e^{-kt}$ — LOGS OF BOTH SIDES

$2\cancel{t}\ln\left(\frac{2}{5}\right) = -k\cancel{t}\cancel{\ln e}^{\,1}$ — $\ln e = 1$

$k = -2\ln\left(\frac{2}{5}\right)$

$k = 1.83258\ldots$

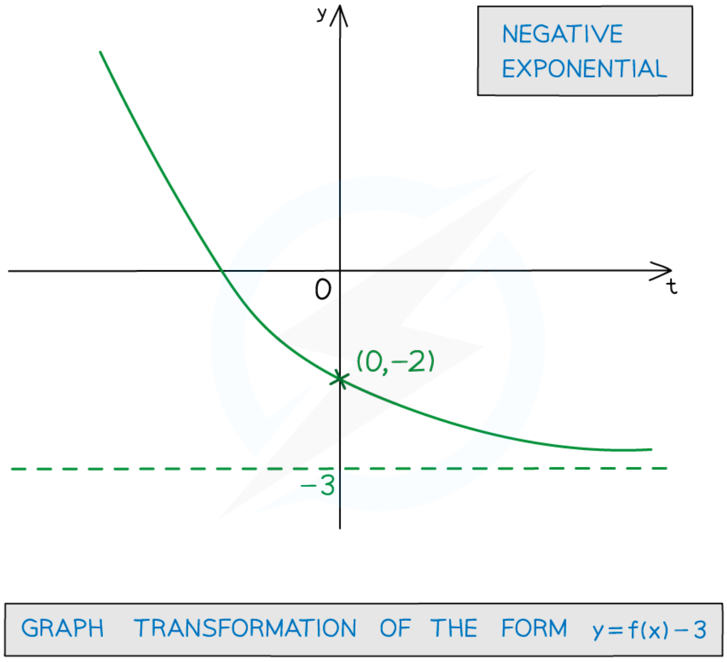
$k = 1.83$ (3 sf)

b) AS THE POWER OF e IS NEGATIVE, THIS IS A NEGATIVE EXPONENTIAL AND SO IS EXPONENTIAL DECAY.

k IS POSITIVE AND WE CAN ASSUME t IS TIME SO ALSO POSITIE, SO THE POWER OVERALL WILL BE NEGATIVE

Your notes

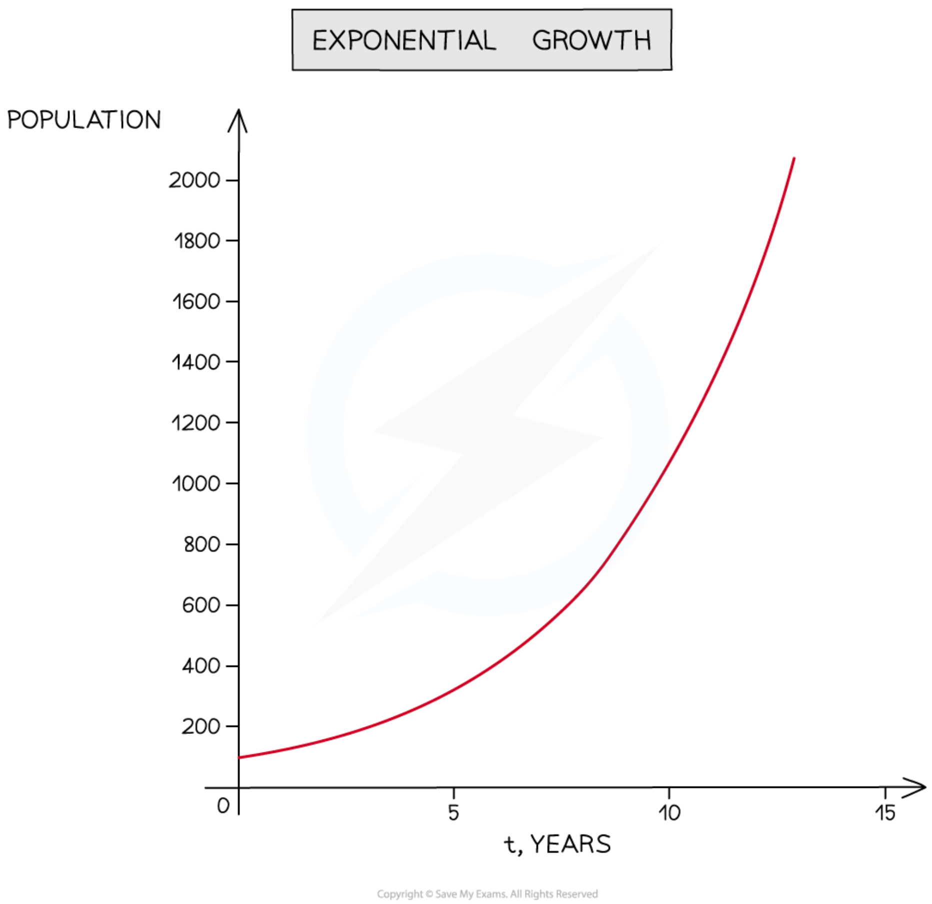
c) $y = \left(\frac{2}{5}\right)^{2t} - 3$



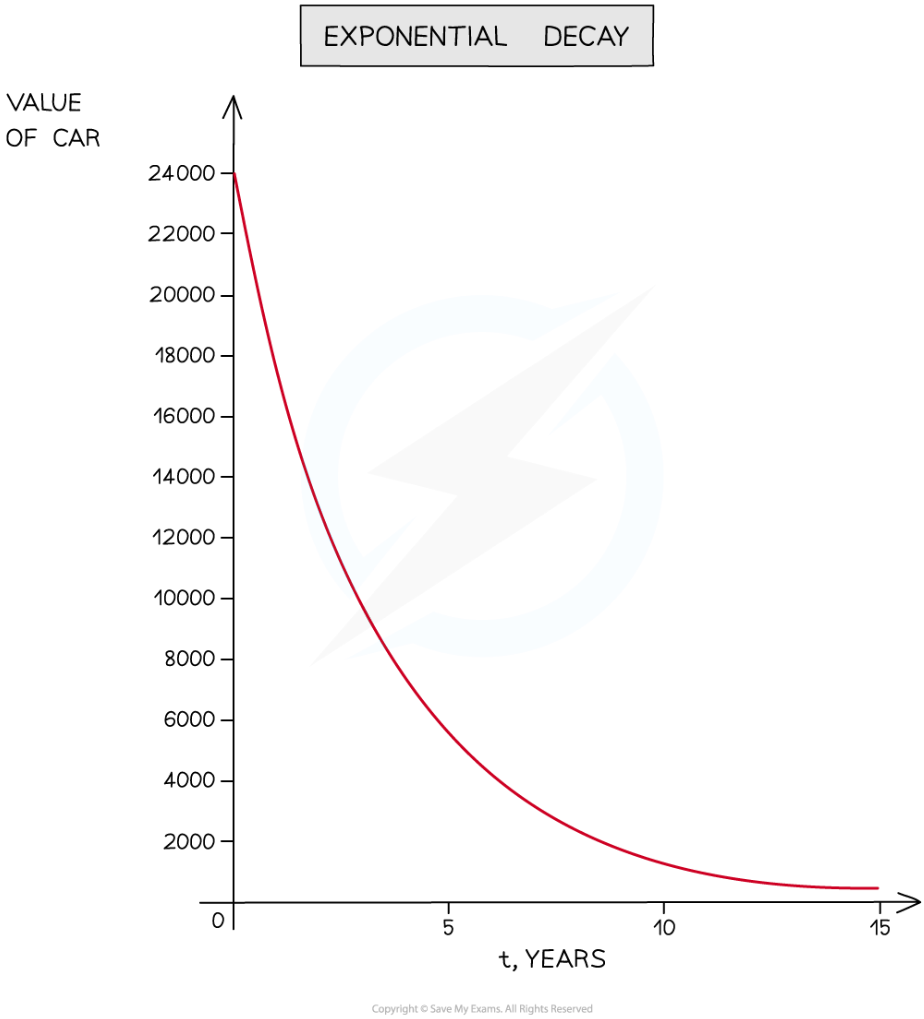
Using exponentials in modelling

What is exponential modelling?

- Exponential modelling is where real-life situations are modelled using exponential growth and decay
- Exponential growth is typically used for …
  - … population (animals and humans) increase
  - … the value of an investment under compound interest



- Exponential decay is typically used for …
  - … levels of radioactivity
  - … quantity of a drug in a person's bloodstream
  - … depreciation (eg value of a standard car)



- It is important to consider if a model is only relevant for a certain period of time
- Can a population increase forever?
  - Will a very old car still be worth a small amount of money?
- For example: Consider a substance that initially contains 2000 radioactive atoms where the number of radioactive atoms, N, at time, t hours, is modelled by $N = 2000e^{-kt}$
  - **y** is **N**
  - **A = 2000** – the initial number of radioactive atoms
  - **k > 0** so the negative indicates this must be exponential **decay**
  - **t** is used (rather than **x**) as the situation is time-dependent

How do I use exponential models?

- Recognise the form of an exponential model
  - Growth $y = Ae^{kt}$

Your notes

- Decay $\mathbf{y = Ae^{-kt}}$
- **A** (constant) is the starting or initial value
  - When $\mathbf{t = 0}$,

    $\mathbf{e^{kt} = e^0 = 1}$

    so $\mathbf{y = A}$
- **k** (constant) determines the **rate** of growth/decay
- Many modelling problems are about **rates of change** –
  - If $\mathbf{y = Ae^{kt}}$

    then $\mathbf{dy/dt = Ake^{kt}}$
  - If $\mathbf{y = Ae^{-kt}}$

    then $\mathbf{dy/dt = -Ake^{-kt}}$

e.g. A SUBSTANCE INITIALLY CONTAINS 2000 RADIOACTIVE ATOMS. THE NUMBER OF RADIOACTIVE ATOMS, N, AT TIME t HOURS, IS MODELLED BY $N = 2000e^{-kt}$.

GIVEN THAT AFTER 2 HOURS THERE WERE 1200 RADIOACTIVE ATOMS,

a) FIND THE VALUE OF k,

b) FIND THE NUMBER OF RADIOACTIVE ATOMS AFTER SIX HOURS

a) AT $t = 2$, $N = 1200$,

$1200 = 2000e^{-2k}$

$e^{-2k} = 0.6$

$-2k = \ln 0.6$

$k = -\frac{1}{2}\ln 0.6 = 0.255$ (3 sf)

b) $N = 2000e^{-2k}$ $\quad k = -\frac{1}{2}\ln 0.6 = 0.255$

AT $t=6$,

$N = 2000e^{-0.255\times 6}$

$N = 432$

USE EXACT VALUE ON CALCULATOR USING $\boxed{\text{ANS}}$ OR MEMORY

Your notes

Worked Example

The population of an endangered animal in a conservation area is modelled by $P = P_0e^{kt}$, where P is the population t years after the population was first recorded.

(a) Write down the meaning of the value P_0

When first recorded the population was 560.

3 years later there were 1980 animals.

(b) Write down the value of P_0

(c) During which year will the population first exceed 4000?

(d) How quickly was the population growing after 5 years?

(e) State one limitation of using this model

a) P_0 WILL BE THE POPULATION OF ANIMALS WHEN IT WAS FIRST RECORDED

Your notes

b) $P_0 = 560$ INITIAL POPULATION

c) $P = 560e^{kt}$

AT $t = 3$, $P = 1980$ FIRST, FIND k

$1980 = 560e^{3k}$ $\div 560$

$e^{3k} = \frac{99}{28}$ ln BOTH SIDES

$3k = \ln\left(\frac{99}{28}\right)$

$k = \frac{1}{3}\ln\left(\frac{99}{28}\right) = 0.421$ (3 sf) SET UP INEQUALITY

$560e^{0.421t} > 4000$

$e^{0.421t} > \frac{50}{7}$ USE EXACT VALUE ON CALCULATOR USING $\boxed{\text{ANS}}$ OR MEMORY

$0.421t > \ln\left(\frac{50}{7}\right)$

$t > 4.6704...$ BETWEEN YEAR 4 AND YEAR 5

$\therefore$ POPULATION FIRST EXCEEDS 4000 DURING THE 5^{th} YEAR

Your notes

d) $\frac{dp}{dt} = P_0 k e^{kt}$

"RATE OF GROWTH" IS GRADIENT

$P_0 = 560 \qquad k = \frac{1}{3}\ln\left(\frac{99}{28}\right) = 0.421 \ (3 \text{ sf})$

AT $t = 5$,

$\frac{dp}{dt} = 560 \times 0.421 \times e^{0.421 \times 5}$

$= 1934.5039...$

THE POPULATION WAS GROWING BY 1930 ANIMALS (3 sf) PER YEAR AFTER 5 YEARS

CHOOSE SENSIBLY!

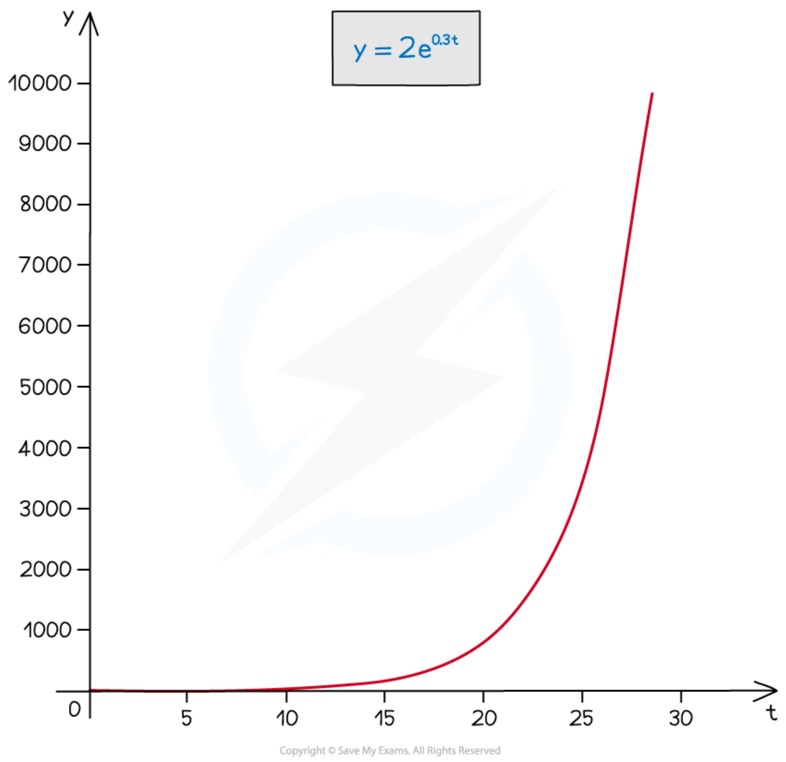
e) THE MODEL SUGGESTS THE POPULATION OF ANIMALS WILL INCREASE INDEFINITELY

POPULATION WILL HAVE A LIMIT – SPACE, RESOURCES, FOOD, etc

Transforming graphs using logs

What are logarithmic axes?

- For graphs that increase or decrease exponentially
  - it can be very difficult to read specific values from the scales on the axes



- Taking **ln** of both sides allows the equation to be rearranged into the form "**y = mx + c**"

$$y = 2e^{0.3t}$$

TAKE LOGS OF BOTH SIDES

$$\ln y = \ln(2e^{0.3t})$$

$$\ln y = \ln 2 + \ln e^{0.3t}$$

$\ln xy = \ln x + \ln y$

$$\ln y = \ln 2 + 0.3t$$

$$\ln y = 0.3t + \ln 2$$

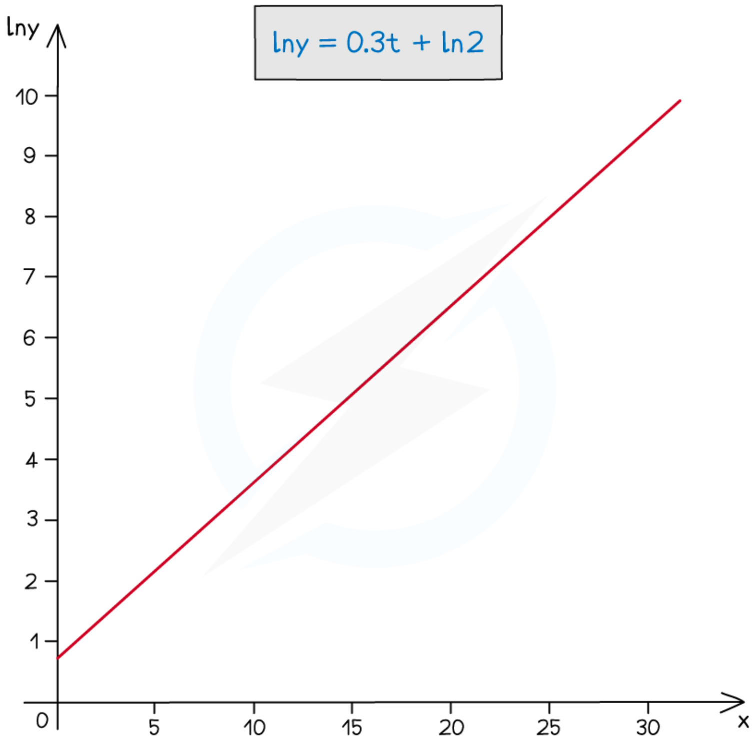
THIS IS OF THE FORM "$y=mx+c$"

"y" $= \ln y$

"m" $= 0.3$

"c" $= \ln 2$

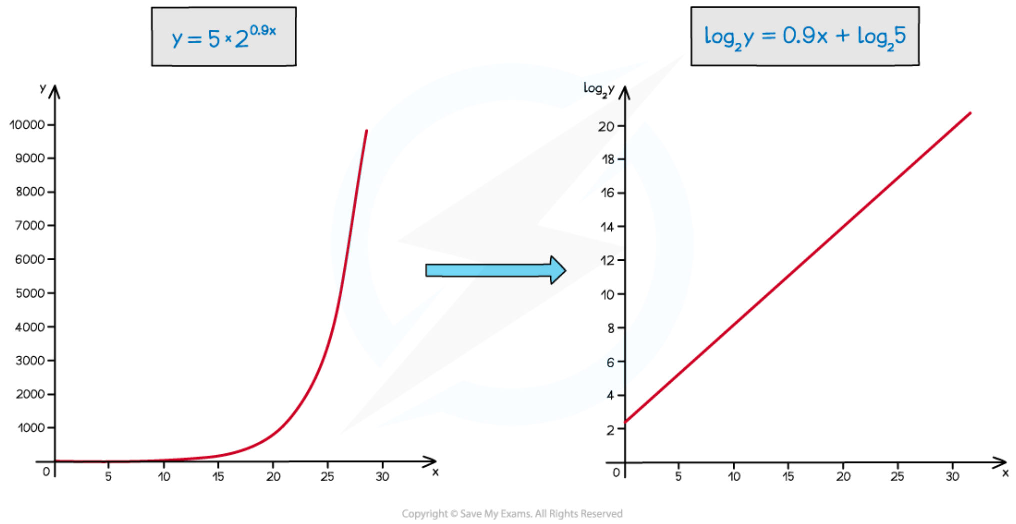
- Plotting **ln y** against **t** produces a straight line
  - The y-axis is a **logarithmic axis**



- Note the second graph has **ln y** on the **y**-axis

Your notes

    - Log graphs have at least one logarithmic axis
- Reading a value for **ln y** at **t = 20** is easier than reading the value for ***y***
- Logarithmic axes are used where a wide range of numbers can occur
    - it makes numbers smaller and easier to deal with
    - a curve can be turned into a straight line



How do I transform graphs into straight lines using logs?

- Exponential models are of the form
    - **$y=Ab^{kx}$ (growth)**
    - **$y=Ab^{-kx}$ (decay)**
- To use a model to make predictions the values of **A**, **b** and **k** are needed
- **A**, **b** and **k** will usually be estimated from observed data values
    - **A** may be known exactly, as it is a starting/initial value
- Estimating from a straight line graph is easier than from a curve

TYPE 1

$y = Ab^{kx}$

BASE b (WHERE b>0 IS A CONSTANT)

$\log_b y = \log_b Ab^{kx}$

$\log_b y = \log_b A + \log_b b^{kx}$

$\log_b y = \log_b A + kx \log_b b$

$\log_b y = kx + \log_b A$

"$y = mx + c$"

TYPE 2

$y = Ae^{kx}$

BASE e

$\ln y = \ln Ae^{kx}$

$\ln y = \ln A + \ln e^{kx}$

$\ln y = \ln A + kx \overset{1}{\cancel{\ln e}}$

$\ln y = kx + \ln A$

"$y = mx + c$"

THIS IS A SPECIAL CASE OF TYPE 1

TYPE 3

$y = Ax^b$

x IS THE BASE

$\log y = \log Ax^b$

$\log y = \log A + \log x^b$

$\log y = \log A + b\log x$

$\log y = b\log x + \log A$

"$y = mx + c$"

Your notes

e.g. GIVEN $y = Ae^{kx}$ AND THAT $\ln y = 3.2x + \ln 12$
FIND THE VALUES OF A AND k.

$\boxed{\text{TYPE 2} \quad y = Ae^{kx}}$

$3.2x = kx$

$k = 3.2$

$\ln A = \ln 12$

$A = 12$

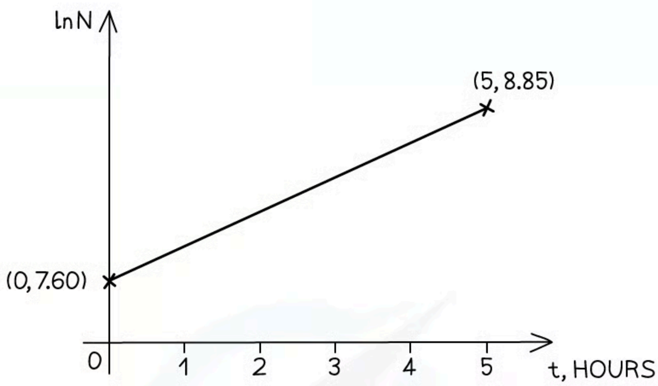
Worked Example

Your notes

A scientist is studying the growth of bacteria in a laboratory experiment.

The number of bacteria, N, is recorded every hour from the start of the experiment for 5 hours.

The graph of $\ln N$ against time, t hours, is shown below.



The scientist suggests that a suitable model for the number of bacteria, N, after t hours is of the form $N = Ae^{kt}$, where A and k are constants.

(a) Find the values of A and k, giving your answers correct to 3 significant figures.

(b) Use the model and your answers from part (a) to predict the number of bacteria after 9 hours.

a) $\ln N = kt + \ln A$

TYPE 2 "$y = Ae^{kx}$"

$\ln A = 7.60$

AT $t=0$, $\ln N = \ln A$

$A = e^{7.60}$

$A = 2000$ (3 sf)

$k = \dfrac{8.85 - 7.60}{5 - 0}$

k WILL BE THE GRADIENT OF THE LINE

$k = 0.25$

b) MODEL IS $N = 2000e^{0.25t}$

AT $t = 9$, $N = 2000\,e^{0.25 \times 9}$

$N = 18\,975.47...$

$\therefore$ AFTER 9 HOURS THERE WILL BE APPROXIMATELY 19 000 BACTERIA

Your notes