Edexcel International A Level (IAL) Maths: Pure 3

Your notes

Exponential & Logarithms

Contents

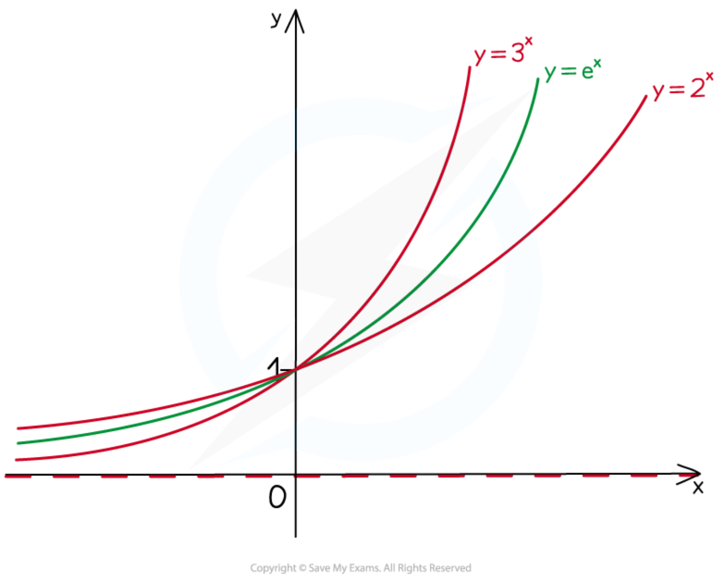
Your notes

"e"

What is the number "e" in maths?

- **e** is an irrational number, sometimes called Euler's number
  - **e** $\approx$ **2.718**

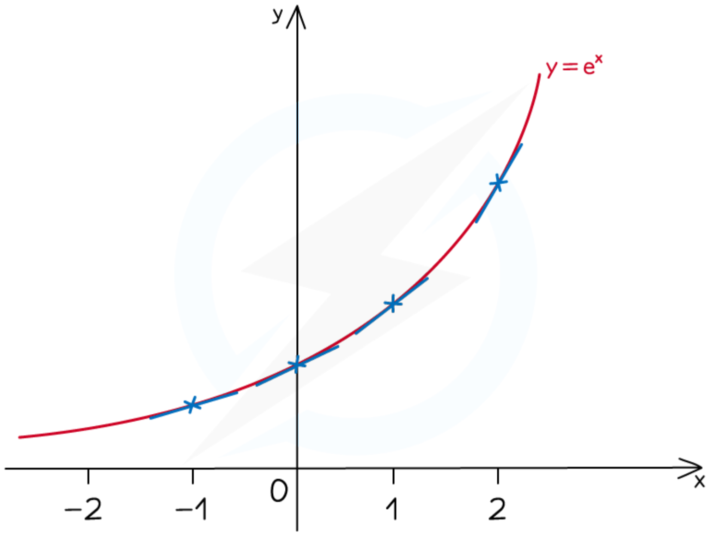
How do I sketch y = e^x?



- As other exponential graphs do, $\mathbf{y = e^x}$
  - passes through (0, 1)
  - has the x-axis as an asymptote

What is the special property of "e"?

Your notes

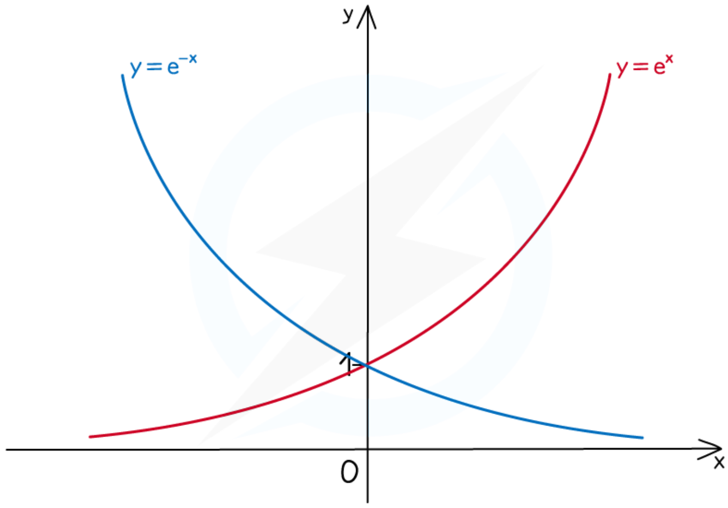


x	y	dy / dx
−1	0.3678...	0.3678...
0	1	1
1	2.7182...	2.7182...
2	7.3890...	7.3890...

- $\mathbf{y = e^x}$ has the particular property
  - $\mathbf{dy/dx = e^x}$
  - ie for *every* real number $\mathbf{x}$, the gradient of $\mathbf{y = e^x}$ is also equal to $\mathbf{e^x}$

    (see Derivatives of Exponential Functions)

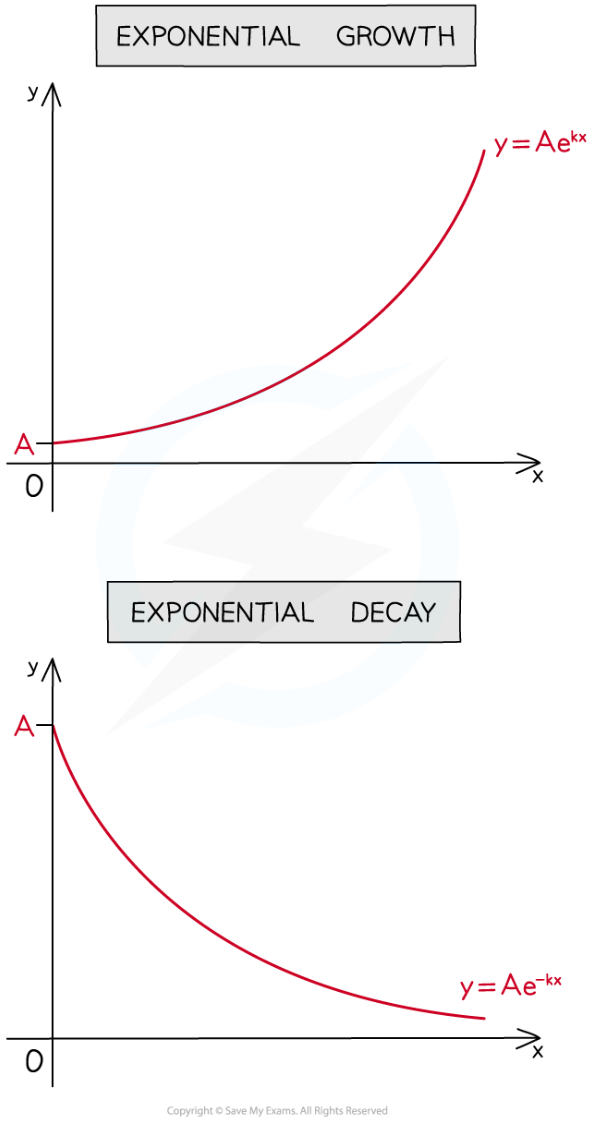
How do I sketch $y = e^{-x}$?



- $\mathbf{y = e^{-x}}$ is a reflection in the **y**-axis of $\mathbf{y = e^{x}}$
  - They are of the form **y = f(x)** and **y = f(−x)**

  (see Transformations of Functions - Reflections)

What is exponential growth and decay?



- **$y = Ae^{kx}$** (**$k > 0$**) is exponential growth
- **$y = Ae^{-kx}$** (**$k > 0$**) is exponential decay
- **A** is the initial value
- **k** is a (usually positive) constant
- "–" is used in the equation making clear whether it is growth or decay

Worked Example

Your notes

(a) On the same diagram sketch the graphs of

(i) $y = e^x$

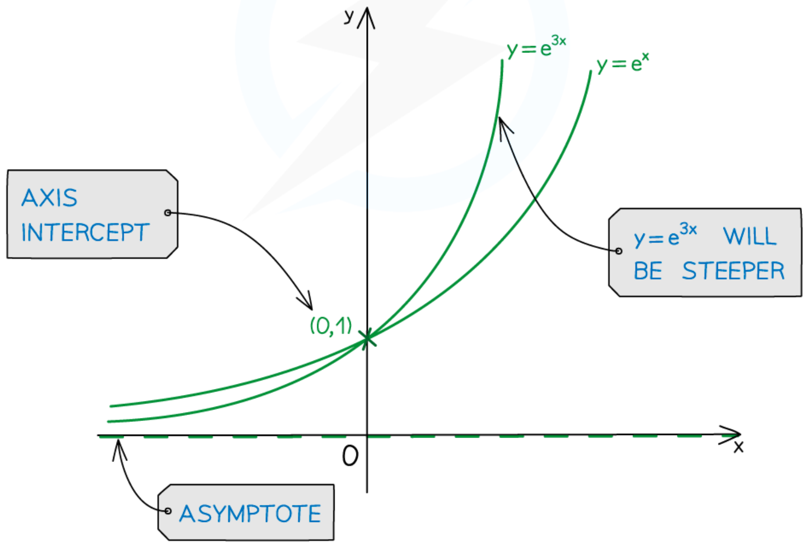
(ii) $y = e^{3x}$

(b) On the same diagram sketch the graphs of

(i) $y = e^{-x}$

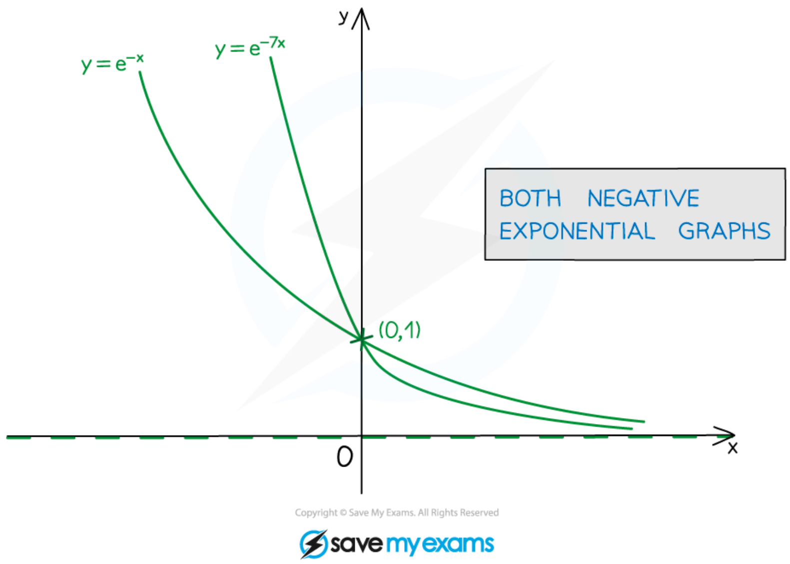
(ii) $y = e^{-7x}$

a) (i) & (ii)



Your notes

b) (i) & (ii)



"ln"

What is ln x?

- **ln** is a function that stands for **natural logarithm**
- It is a logarithm where the **base** is the constant "**e**"
  - $\ln x \equiv \log_e x$
  - It is important to remember that **ln** is a function and **not a number**

What properties of ln x do I need to know?

- Using the **definition** of a **logarithm** you can see
  - $\ln 1 = 0$
  - $\ln e = 1$
  - $\ln e^x = x$
  - $\ln x$ is only defined for positive *x*
- As ln is a logarithm you can use the **laws of logarithms**
  - $\ln a + \ln b = \ln(ab)$
  - $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$
  - $n \ln a = \ln(a^n)$

- Any logarithm can be written in terms of the natural logarithm using the **change of base** formula
  - $\log_a b = \frac{\ln b}{\ln a}$

How do I solve equations involving e^x & ln x?

- The functions e^x and $\ln x$ are **inverses** of each other
  - If $e^{f(x)} = g(x)$ then $f(x) = \ln g(x)$
  - If $\ln f(x) = g(x)$ then $f(x) = e^{g(x)}$
- If your equation involves "e" then try to get all the "e" terms on one side
  - If "e" terms are multiplied, you can add the powers
    - $e^x \times e^y = e^{x+y}$
    - You can then apply ln to both sides of the equation
  - If "e" terms are added, try transforming the equation with a substitution
    - For example: If $y = e^x$ then $e^{4x} = y^4$
    - You can then solve the resulting equation (usually a quadratic)
    - Once you solve for *y* then solve for *x* using the substitution formula
- If your equation involves "ln", try to combine all "ln" terms together
  - Use the laws of logarithms to combine terms into a single term
  - If you have $\ln f(x) = \ln g(x)$ then solve $f(x) = g(x)$
  - If you have $\ln f(x) = k$ then solve $f(x) = e^k$

Worked Example

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Solve the following equations. Give your answers in exact form.

(a) $e^{3x+2} = 5e^{x-3}$.

(b) $\ln(8x) - \ln(x+4) = 2$.

a) COLLECT THE "e" TERMS ON ONE SIDE: $\frac{e^{3x+2}}{e^{x-3}} = 5$

SIMPLIFY USING INDEX LAWS: $e^{2x+5} = 5$ $\quad \boxed{(3x+2)-(x-3)}$

APPLY NATURAL log TO BOTH SIDES: $\cancel{\ln(e}^{2x+5}) = \ln 5$

USE $\ln(e^{f(x)}) = f(x)$ $\quad 2x + 5 = \ln 5$

REARRANGE FOR x $\quad x = \frac{1}{2}(-5 + \ln 5)$

$\boxed{\text{WRITE } \frac{\ln a}{k} \text{ AS } \frac{1}{k}\ln a}$ $\quad \boxed{\text{WRITE } \ln a + k \text{ as } k + \ln a}$

b) COMBINE "ln" USING LAWS OF LOGARITHMS: $\ln\frac{8x}{x+4} = 2$

THE INVERSE OF $\ln(x)$ IS e^x: $\frac{8x}{x+4} = e^2$

REARRANGE FOR x: $8x = e^2x + 4e^2$ $\quad \boxed{\text{COLLECT x TERMS ON ONE SIDE}}$

$\boxed{\text{FACTORISE OUT THE x}}$ $\quad 8x - e^2x = 4e^2$

$(8 - e^2)x = 4e^2$

$x = \frac{4e^2}{8 - e^2}$

Examiner Tips and Tricks

- Always simplify your answer if you can
  - for example, $\frac{1}{2}\ln 25 = \ln\sqrt{25} = \ln 5$
  - you wouldn't leave your final answer as $\sqrt{25}$ so don't leave your final answer as $\frac{1}{2}\ln 25$

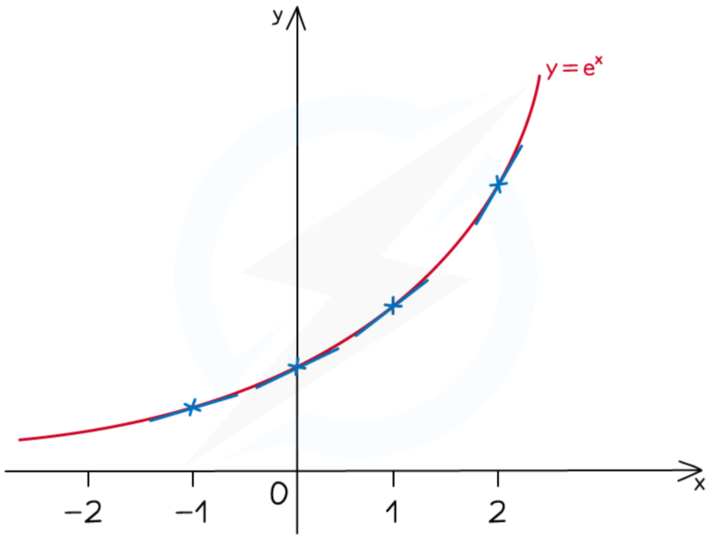
Derivatives of exponential functions

What is the derivative of e^x?

- **$y = e^x$** has the particular property that

$$\frac{dy}{dx} = e^x$$

- i.e. for *every* real number **x**, the gradient of **$y = e^x$** is also equal to **e^x** (see Derivatives of Exponential Functions)



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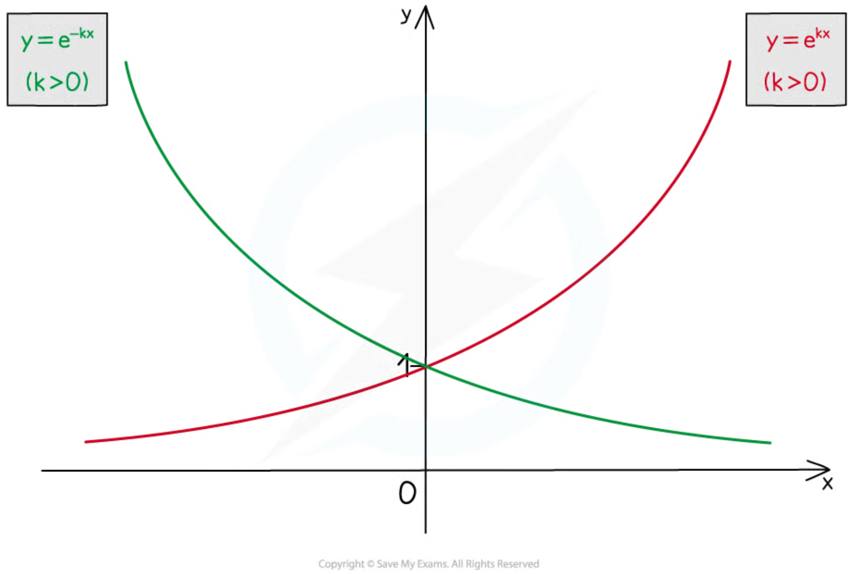
- **$e \approx 2.718$** (see "e")

Your notes

- Recall that the derivative is the gradient function for a curve (see First Principles Differentiation)

What is the derivative of $y = e^{kx}$ and $y=e^{-kx}$?



- The derivative of **y = e^kx** is
  - $$\frac{dy}{dx} = ke^{kx}$$
- The derivative of **y = e^-kx** is
  - $$\frac{dy}{dx} = -ke^{-kx}$$

Examiner Tips and Tricks

- Remember that (like **π**) **e** is a **number**.
- Exam questions can ask for answers to be given as exact values in terms of **e** (see the Worked Example below).

Worked Example

Your notes

Find, in terms of e, the gradient at $x = 3$ for each of the following functions ...

(a) $y = e^x$

(b) $y = e^{5x}$

(c) $y = e^{-x}$

(d) $y = 3e^{2x}$

(e) $y = 5e^{-4x}$

a) $y = e^x$ $\quad \frac{dy}{dx} = e^x$

$\frac{dy}{dx} = y$

AT $x=3$, $\quad \frac{dy}{dx} = e^3$

$\frac{dy}{dx} = e^3$

FINAL ANSWER IN TERMS OF e

b) $y = e^{5x}$ $\quad \frac{dy}{dx} = 5e^{5x}$

$\frac{dy}{dx} = ke^{kx}$

AT $x=3$, $\quad \frac{dy}{dx} = 5e^{5 \times 3}$

$\frac{dy}{dx} = 5e^{15}$

Your notes

c) $y = e^{-x}$ $\qquad \frac{dy}{dx} = -e^{-x}$

$\frac{dy}{dx} = -ke^{-kx}$

$k = 1$

AT $x = 3$, $\quad \frac{dy}{dx} = -e^{-3}$

$\frac{dy}{dx} = -e^{-3} = -\frac{1}{e^3}$ $\quad$ NOT ESSENTIAL

d) $y = 3e^{2x}$ $\qquad \frac{dy}{dx} = 6e^{2x}$

$\frac{dy}{dx} = ke^{kx}$

3 IS A CONSTANT SO IS MULTIPLIED BY k

AT $x = 3$, $\quad \frac{dy}{dx} = 6e^{2 \times 3}$

$\frac{dy}{dx} = 6e^{6}$

e) $y = 5e^{-4x}$ $\qquad \frac{dy}{dx} = -20e^{-4x}$

$\frac{dy}{dx} = -ke^{-kx}$

5 IS A CONSTANT

AT $x = 3$, $\quad \frac{dy}{dx} = -20e^{-4 \times 3}$

$\frac{dy}{dx} = -20e^{-12}$