Edexcel International A Level (IAL) Maths: Pure 3

Your notes

Compound & Double Angle Formulae

Contents

Your notes

Compound angle formulae

What are the compound angle formulae?

- There are six **compound angle formulae** (also known as **addition formulae**), two each for **sin**, **cos** and **tan**:
- For **sin** the +/- sign on the left-hand side **matches** the one on the right-hand side

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

- For **cos** the +/- sign on the left-hand side is **opposite to** the one on the right-hand side

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

- For **tan** the +/- sign on the left-hand side **matches** the one in the **numerator** on the right-hand side, and is **opposite to** the one in the **denominator**

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B} \quad \tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

- You can **derive** the tan identity by:
  - Writing $\tan(A+B) \equiv \dfrac{\sin(A+B)}{\cos(A+B)}$
  - Dividing the numerator and denominator by $\cos A\cos B$

Examiner Tips and Tricks

- All these formulae are in the formulae booklet – you don't have to memorise them.

Worked Example

Your notes

a) Express $\tan(225° - 30°)$ in terms of $\tan 225°$ and $\tan 30°$.

b) Hence show that $\tan 195° = 2 - \sqrt{3}$.

a) $\tan(A-B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ ← FORMULA 6 ABOVE

$$\tan(225° - 30°) \equiv \frac{\tan 225° - \tan 30°}{1 + \tan 225° \tan 30°}$$

b) $\tan 225° = 1$ $\quad \tan 30° = \dfrac{\sqrt{3}}{3}$ ← YOU SHOULD KNOW THESE!

$$\tan(195°) = \tan(225° - 30°)$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + (1)\left(\frac{\sqrt{3}}{3}\right)} = \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}}$$

$$\left[\times \frac{3}{3}\right] \quad = \frac{3 - \sqrt{3}}{3 + \sqrt{3}}$$

$$= \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \times \frac{3 - \sqrt{3}}{3 - \sqrt{3}}$$ ← RATIONALISE THE DENOMINATOR

$$= \frac{12 - 6\sqrt{3}}{6}$$

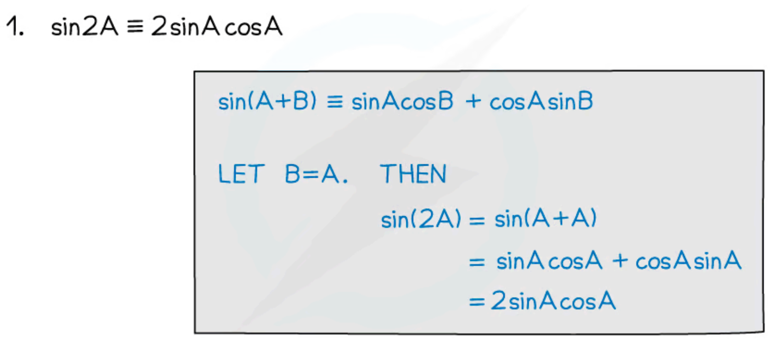
$$= 2 - \sqrt{3}$$

Your notes

Double angle formulae

What are the double angle formulae?

- The **double angle formulae** are derived from the 'A+B' versions of the **compound angle formulae**:
  - $\sin(2A) \equiv 2\sin A\cos A$
  - $\cos(2A) \equiv \cos^2 A - \sin^2 A$
    - $\cos(2A) \equiv 2\cos^2 A - 1$
    - $\cos(2A) \equiv 1 - 2\sin^2 A$
  - $\tan(2A) \equiv \dfrac{2\tan A}{1 - \tan^2 A}$



2. $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$

$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$

LET $B = A$. THEN

$$\begin{aligned} \cos(2A) &\equiv \cos(A+A) \\ &\equiv \cos A \cos A - \sin A \sin A \\ &\equiv \cos^2 A - \sin^2 A \end{aligned}$$

THE OTHER TWO FORMS COME FROM REARRANGING

$$\sin^2 A + \cos^2 A \equiv 1$$

AND SUBSTITUTING

3. $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

$\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

LET $B = A$. THEN

$$\begin{aligned} \tan(2A) &\equiv \tan(A+A) \\ &\equiv \frac{\tan A + \tan A}{1 - \tan A \tan A} \\ &\equiv \frac{2\tan A}{1 - \tan^2 A} \end{aligned}$$

Examiner Tips and Tricks

- The double angle formulae are **not** included in the formula booklet – you have to know them.
- If you forget them, you can always derive them from the compound angle formulae (which *are* in the formula booklet) as shown above.

Worked Example

Your notes

Your notes

a) Show that $7\sin 2\theta - 3\sin\theta = 0$ can be written in the form

$$a\sin\theta(b\cos\theta + c) = 0$$

stating the values of a, b and c.

b) Hence solve, for $0 \leq \theta < 360°$, the equation $7\sin 2\theta - 3\sin\theta = 0$. Give your answers to 1 decimal place.

a) $7\sin 2\theta = 7(2\sin\theta\cos\theta) = 14\sin\theta\cos\theta$

USE DOUBLE ANGLE FORMULA FOR sin

WE CAN REWRITE THE EQUATION AS

$14\sin\theta\cos\theta - 3\sin\theta = 0$

$\sin\theta(14\cos\theta - 3) = 0$

WITH $a = 1$ $b = 14$ $c = -3$

b) $\sin\theta(14\cos\theta - 3) = 0$

SO $\sin\theta = 0$ OR $\cos\theta = \frac{3}{14}$

IF $\sin\theta = 0$

$\theta = 0$ OR $\theta = 180$

YOU SHOULD KNOW THESE!

IF $\cos\theta = \frac{3}{14}$

$\theta = 77.63$ OR $\theta = 360 - 77.63 = 282.37$

USE CALCULATOR TO FIND PRINCIPAL VALUE

THEN FIND OTHER VALUES IN THE REQUIRED INTERVAL

THE SOLUTIONS ARE

$\theta = 0°, 77.6°, 180°, 282.4°$ (1 d.p.)

Your notes

Harmonic form

What is harmonic form?

- If ***a*** and ***b*** are positive constants, then
  - ***a* sin *x* + *b*cos *x*** can be rewritten as ***R* sin (*x* + α)**
  - ***a* sin *x* – *b*cos *x*** can be rewritten as ***R* sin (*x* – α)**
  - ***a* cos *x* + *b*sin *x*** can be rewritten as ***R* cos (*x* – α)**
  - ***a* cos *x* – *b*sin *x*** can be rewritten as ***R* cos (*x* + α)**
- The forms with ***R*** and **α** are called **harmonic forms**
  - It is the reverse of expanding the right-hand side out using compound angle formulae
  - It can be seen as factorising
- In all four cases ***R* > 0** and **0 < α < 90°**, with:

$$a = R\cos\alpha$$

$$b = R\sin\alpha$$

$$R = \sqrt{a^2 + b^2}$$

- To find **α**, first divide the **b** expression by the **a** expression
  - The ***R***'s cancel
  - Use the identity sin **α** / cos **α** = tan **α**
  - You get **b / a = tan α**
  - Then use $\mathbf{tan^{-1}}$ **(b/a)** to find **α**

How do I write trigonometric expressions in harmonic form?

- To rewrite the expression use the appropriate **compound angle formula** and equate coefficients e.g. $5\cos x - 12\sin x$

THIS IS CASE 4 ABOVE, WITH $a=5$ AND $b=12$

$\cos(x+\alpha) = \cos\alpha\cos x - \sin\alpha\sin x$

EXPAND USING COMPOUND ANGLE FORMULA FOR cos

SO $R\cos(x+\alpha) = R\cos\alpha\cos x - R\sin\alpha\sin x$

COMPARE TO ORIGINAL EXPRESSION:

$R\cos\alpha = 5 \qquad R\sin\alpha = 12$

$R\cos\alpha = a$
$R\sin\alpha = b$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{12}{5}$$

$$\alpha = \tan^{-1}\left(\frac{12}{5}\right) = 67.38^\circ$$

ALSO $(R\cos\alpha)^2 + (R\sin\alpha)^2 = 5^2 + 12^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 25 + 144$

$R^2(\cos^2\alpha + \sin^2\alpha) = 169$

$\cos^2\alpha + \sin^2\alpha = 1$

$R^2 = 169$

$R = 13$

$R = \sqrt{a^2 + b^2}$

$5\cos x - 12\sin x = 13\cos(x + 67.38^\circ)$

$a\cos x - b\sin x = R\cos(x+\alpha)$

Your notes

Examiner Tips and Tricks

- There is more than one way to rewrite a given expression, and the question in the exam will tell you which form to use.

Worked Example

Your notes

a) Express $\sin\theta - 4\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$.

b) Hence solve $\sin\theta - 4\cos\theta = 3$ for $0 \le \theta < 360°$.

a) $R\sin(\theta - \alpha) = R\cos\alpha \sin\theta - R\sin\alpha \cos\theta$

COMPOUND ANGLE FORMULA FOR sin

$R\cos\alpha = 1 \qquad R\sin\alpha = 4$

$R\cos\alpha = a$
$R\sin\alpha = b$

$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = 4$

$\alpha = \tan^{-1}(4) = 75.96°$

$R = \sqrt{1^2 + 4^2} = \sqrt{17}$

$R = \sqrt{a^2 + b^2}$

$\sin\theta - 4\cos\theta = \sqrt{17}\sin(\theta - 75.96°)$

b) $\sqrt{17}\sin(\theta - 75.96°) = 3$

$\sin(\theta - 75.96°) = \frac{3}{\sqrt{17}}$

$(\theta - 75.96°) = 46.69°$

OR $(\theta - 75.96°) = 180 - 46.69 = 133.31°$

USE $\sin^{-1}$ ON YOUR CALCULATOR TO FIND THE PRINCIPAL VALUE OF θ

THEN FIND OTHER SOLUTIONS IN THE REQUIRED RANGE

THE SOLUTIONS ARE

$\theta = 122.65°$ OR $209.27°$

Modelling with trigonometric functions

How do I model with trigonometric functions?

- Various real-life situations can be modelled using trigonometric functions
- You need to be able to interpret the equations used in the model
- If you need to identify maximum or minimum values of a formula, remember the bounds of the **sin** and **cos** functions:
  - $-1 \leq \sin x \leq 1$
  - $-1 \leq \cos x \leq 1$

e.g. WHAT ARE THE MINIMUM AND MAXIMUM VALUES OF $y = 15 - 23\sin 3x$

WHEN $\sin 3x = 1$, $y = 15 - 23(1) = -8$

WHEN $\sin 3x = -1$, $y = 15 - 23(-1) = 38$

SO $y_{max} = 38$ AND $y_{min} = -8$

- You may need to **simplify trigonometric expressions** to make the behaviour of an equation clearer

e.g. WHAT ARE THE MAXIMUM AND MINIMUM VALUES OF $y = 0.15\sin x + 0.2\cos x$?

$R\sin(x+\alpha) = R\cos\alpha \sin x + R\sin\alpha \cos x$

$R\cos\alpha = 0.15$ $\quad R\sin\alpha = 0.2$

$R = \sqrt{0.15^2 + 0.2^2} = 0.25$

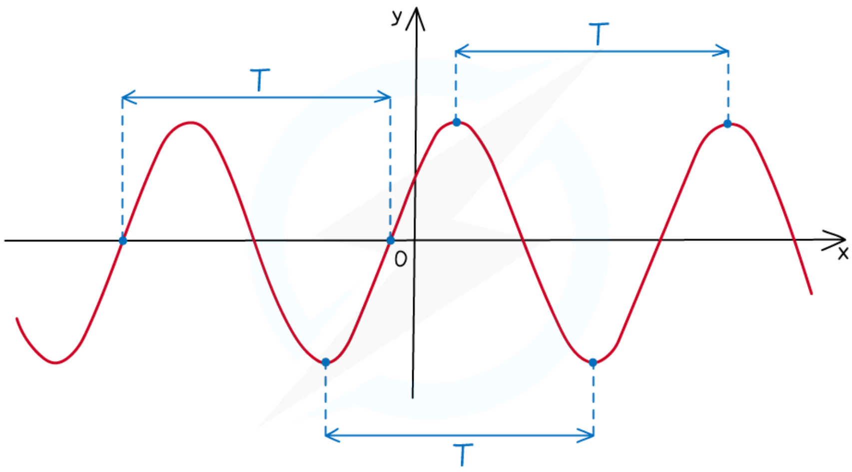
$\tan\alpha = \dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{0.2}{0.15} = \dfrac{4}{3}$

$\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ$ (1 d.p.)

SO $y = 0.25\sin(x + 53.1)$

$y_{max} = 0.25$ $\quad y_{min} = -0.25$

- You may also need to discuss the **period** of an equation
  - The period is often indicated by ***T***
  - For a periodic function in ***x*** like **sin** or **cos**, the **period** is how much ***x*** has to change by for the function to go through one complete cycle



- For functions of the form **cos (*qx*+ *r*)** or **sin (*qx* + *r*)** the period ***T*** is:

$$T = \frac{360}{q} \text{ (for angles in degrees)}$$

$$T = \frac{2\pi}{q} \text{ (for angles in radians)}$$

e.g. WHAT IS THE PERIOD OF $y = \cos(\frac{7}{8}x - 1.5)$ WHERE THE ANGLE IS MEASURED IN RADIANS?

THE PERIOD IS

$$T = \frac{2\pi}{\frac{7}{8}} = \frac{16\pi}{7}$$

Examiner Tips and Tricks

- The variable in these questions is often ***t*** for time.

Your notes

- Read the question carefully to make sure you know what you are being asked to solve.

Your notes

Worked Example

Your notes

An amusement park builds a large circular wheel as a tourist attraction. The height of a passenger on the wheel is modelled by the equation

$$H = 50 - 48\cos\left(\frac{\pi}{15}t + 0.3\right)$$

where H is the height of the passenger above the ground in metres, t is the number of minutes after boarding, and angles are given in radians.

a) What is the maximum height that the passenger reaches above the ground?

b) How many minutes after boarding the ride will the passenger reach that height for the first time?

c) How long does it take the wheel to make a complete revolution?

a) THE MAXIMUM VALUE OF H OCCURS WHEN THE COSINE IS EQUAL TO −1

$H_{max} = 50 - 48(-1) = 98\text{m}$

b) $\cos\left(\frac{\pi}{15}t + 0.3\right) = -1$

$\frac{\pi}{15}t + 0.3 = \cos^{-1}(-1) = \pi$

REMEMBER TO USE RADIANS!

$\frac{\pi}{15}t = \pi - 0.3$

$t = \frac{15}{\pi}(\pi - 0.3) = 13.6$ MINUTES (1 d.p.)

c) THE WHEEL COMPLETES ONE REVOLUTION EACH TIME

$\frac{\pi}{15}t$ GOES THROUGH 2π

$\frac{2\pi}{\left(\frac{\pi}{15}\right)} = 30$ MINUTES

PERIOD OF $\cos(qx+r)$ IS $T = \frac{2\pi}{q}$