Edexcel International A Level (IAL) Maths: Pure 3

Your notes

Reciprocal & Inverse Trigonometric Functions

Contents

Definitions of reciprocal trigonometric functions

What are the reciprocal trigonometric functions?

- There are three reciprocal trig functions corresponding to the three regular trig functions

1. SECANT (sec)

$$\sec x = \frac{1}{\cos x}$$

(UNDEFINED FOR VALUES OF x FOR WHICH $\cos x = 0$)

2. COSECANT (cosec)

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

(UNDEFINED FOR VALUES OF x FOR WHICH $\sin x = 0$)

3. COTANGENT (cot)

$$\cot x = \frac{1}{\tan x}$$

(UNDEFINED FOR VALUES OF x FOR WHICH $\tan x = 0$; EQUAL TO ZERO FOR VALUES OF x FOR WHICH $\tan x$ IS UNDEFINED)

- Cotangent can also be written in terms of sine and cosine

$$\cot x = \frac{\cos x}{\sin x}$$

Examiner Tips and Tricks

- To solve equations with the reciprocal trig functions, convert them into the regular trig functions and solve in the usual way.
- Don't forget that both **tan** and **cot** can be written in terms of **sin** and **cos** (see the Worked Example below).
- You will sometimes see **csc** instead of **cosec** for cosecant.

Your notes

Worked Example

a) Prove that $\dfrac{1+\cot x}{1+\tan x} \equiv \cot x$.

b) Hence solve, in the interval $0 \leq x \leq 360°$,

the equation $\dfrac{1+\cot x}{1+\tan x} = 10$.

Give your answers to one decimal place.

a) $1+\cot x \equiv 1+\dfrac{\cos x}{\sin x} \equiv \dfrac{\sin x+\cos x}{\sin x}$

$1+\tan x \equiv 1+\dfrac{\sin x}{\cos x} \equiv \dfrac{\cos x+\sin x}{\cos x}$

REWRITE IN TERMS OF sin AND cos

SO $\dfrac{1+\cot x}{1+\tan x} \equiv \dfrac{\left(\dfrac{\sin x+\cos x}{\sin x}\right)}{\left(\dfrac{\cos x+\sin x}{\cos x}\right)} \equiv \dfrac{\cos x}{\sin x} \equiv \cot x$

$\dfrac{a/b}{a/c} = \dfrac{c}{b}$

b) WE NEED TO SOLVE $\cot x = 10$

$\dfrac{1}{\tan x} = 10$

$\tan x = \dfrac{1}{10}$

$x = 5.71°$

USE $\tan^{-1}$ ON YOUR CALCULATOR TO FIND THE PRINCIPAL VALUE OF x

OR $x = 180 + 5.71 = 185.71°$

THEN BE SURE TO FIND ANY OTHER VALUES IN THE REQUIRED INTERVAL

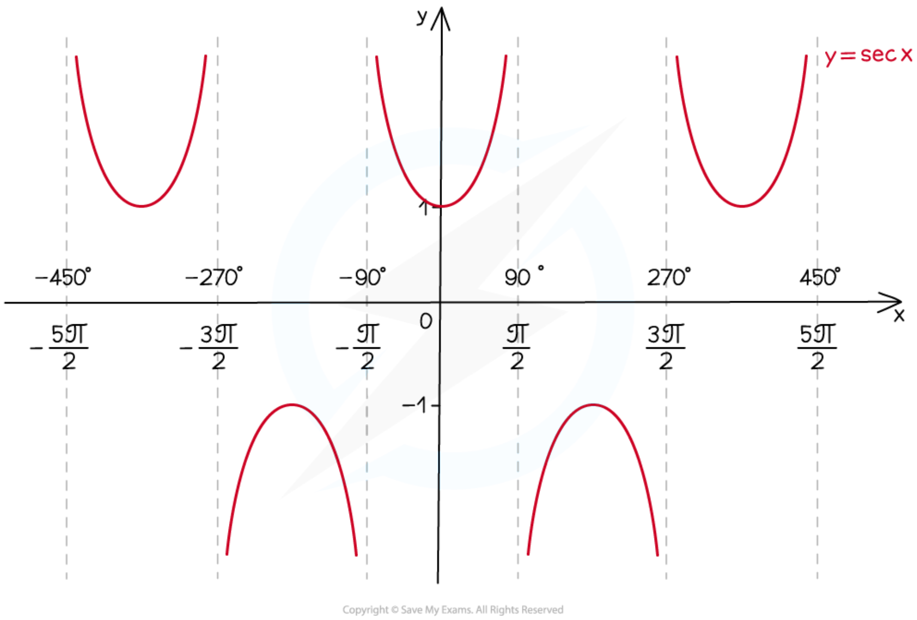
SO $x = 5.7°$ OR $185.7°$

(1 d.p.)

Graphs of reciprocal trigonometric functions

How do I sketch the graph of y = sec x?

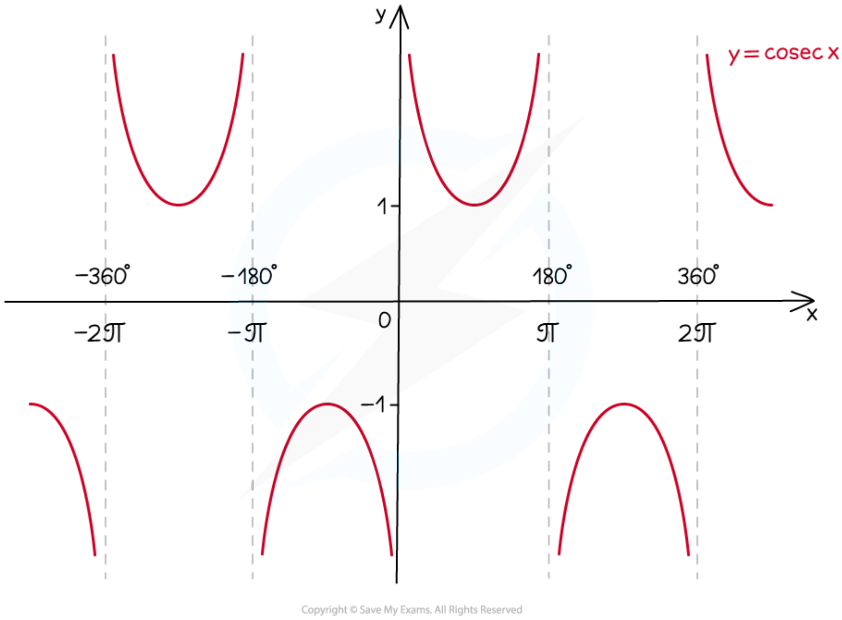
- The graph of ***y* = sec*x*** looks like this:



- ***y*-axis** is a **line of symmetry**
- has **period** (ie repeats every) **360°** or **2π** radians
- vertical **asymptotes** wherever **cos *x* = 0**
- **domain** is all *x* **except odd multiples of 90°** (90°, -90°, 270°, -270°, etc.)
- the **domain in radians** is all *x* **except odd multiples of π/2** (π/2, - π/2, 3π/2, -3π/2, etc.)
- **range** is ***y* ≤ -1 or y ≥ 1**

How do I sketch the graph of y = cosec x?

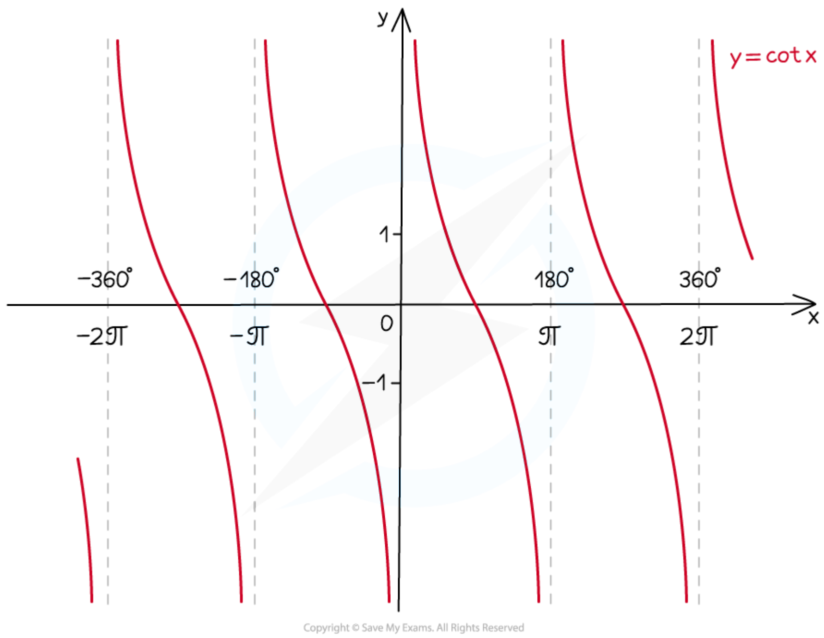
- The graph of ***y* = cosec *x*** looks like this:



- has **period** (ie repeats every) **360°** or **2π** radians
- vertical **asymptotes** wherever **sin *x*= 0**
- **domain** is all *x* **except multiples of 180°** (0°, 180°, -180°, 360°, -360°, etc.)
- the **domain in radians** is all *x* **except multiples of π** (0, π, - π, 2π, -2π, etc.)
- **range** is ***y*** **≤ -1 or y ≥ 1**

How do I sketch the graph of y = cot x?

- The graph of ***y*** **= cot *x*** looks like this:



- has **period** (ie repeats every) **180°** or **π** radians
- vertical **asymptotes** wherever **tan *x* = 0**
- **domain** is all *x* **except multiples of 180°** (0°, 180°, -180°, 360°, -360°, etc.)
- the **domain in radians** is all *x* **except multiples of π** (0, π, - π, 2π, -2π, etc.)
- **range** is $y \in \mathbb{R}$ (ie **cot** can take *any* real number value)

Examiner Tips and Tricks

- Make sure you know the shapes of the graphs for **cos**, **sin** and **tan**.
- The shapes of the reciprocal trig function graphs follow from those graphs plus the definitions **sec = 1/cos**, **cosec = 1/sin** and **cot = 1/tan**

Worked Example

Your notes

Your notes

a) Sketch, in the interval $-2\pi \leq x \leq 2\pi$,

the graph of $y = 2 - 3\sec x$.

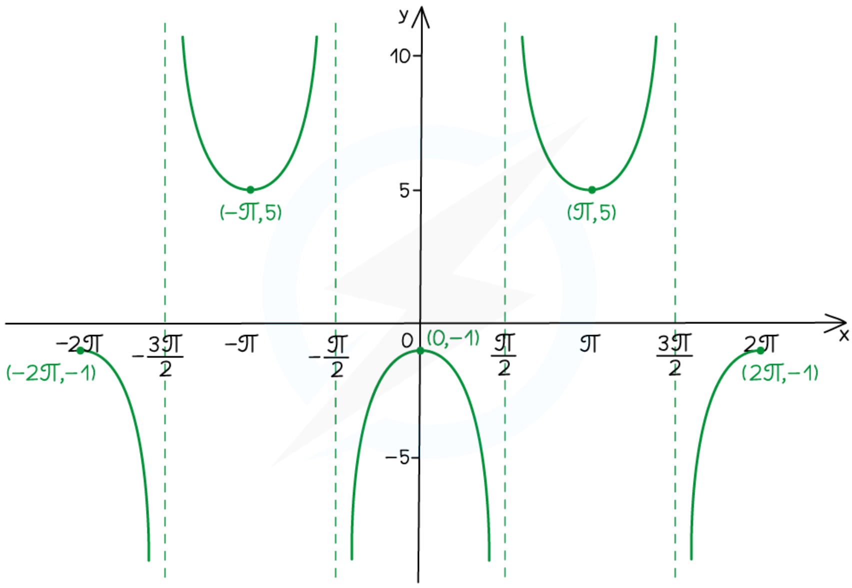
b) Hence deduce the range of values of k for which

the equation $2 - 3\sec x - k = 0$ has no solutions.

a)

THIS IS A TRANSFORMATION OF $y=\sec x$

- VERTICAL STRETCH, SCALE FACTOR 3
- THEN REFLECT IN x–AXIS
- FINALLY TRANSLATE VERTICALLY BY VECTOR $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$
- DON'T FORGET THE ASYMPTOTES



b) $2-3\sec x-k=0$

$\Rightarrow 2-3\sec x=k$

FROM GRAPH, THE RANGE OF $y=2-3\sec x$ IS $y\leqslant-1$ OR $y\geqslant5$

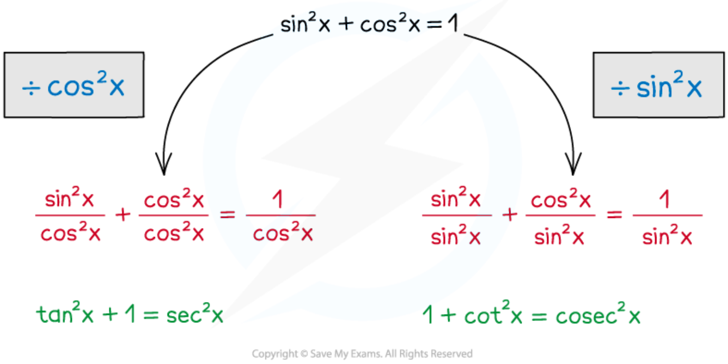
SO THE EQUATION HAS NO SOLUTIONS WHEN $-1<k<5$

Your notes

Identities with reciprocal trigonometric functions

What are the reciprocal trigonometric identities?

- There are two identities with **sec**, **cosec** and **cot** that you need to know and be able to use:
  - $\tan^2 x + 1 \equiv \sec^2 x$
  - $1 + \cot^2 x \equiv \mathrm{cosec}^2 x$
- These are not really 'new' identities – they can both be derived from $\sin^2 x + \cos^2 x \equiv 1$
  - To derive the identity for $\sec^2 x$ divide $\sin^2 x + \cos^2 x \equiv 1$ by $\cos^2 x$
  - To derive the identity for $\mathrm{cosec}^2 x$ divide $\sin^2 x + \cos^2 x \equiv 1$ by $\sin^2 x$



Examiner Tips and Tricks

- These identities are not given in the exam formulae booklet – make sure you know how to find them.
- They will be needed to prove some trigonometric identities and solve some trigonometric equations.

Worked Example

Your notes

Solve the equation $9\sec^2\theta - 11 = 3\tan\theta$,

in the interval $0 \leq \theta \leq 360°$.

Give your answers to 3 significant figures.

$(9\sec^2\theta - 9) - 2 = 3\tan\theta$

$\tan^2\theta + 1 = \sec^2\theta$
SO $\sec^2\theta - 1 = \tan^2\theta$

$9\tan^2\theta - 2 = 3\tan\theta$

$9\tan^2\theta - 3\tan\theta - 2 = 0$

THIS IS A QUADRATIC IN $\tan\theta$.
COMPARE $9y^2 - 3y - 2 = 0$

$(3\tan\theta - 2)(3\tan\theta + 1) = 0$

$\tan\theta = \frac{2}{3}$ OR $\tan\theta = -\frac{1}{3}$

USE $\tan^{-1}$ ON YOUR CALCULATOR TO FIND PRINCIPAL VALUES OF θ

IF $\tan\theta = \frac{2}{3}$

$\theta = 33.69$ OR $\theta = 180 + 33.69 = 213.69$

IF $\tan\theta = -\frac{1}{3}$

$\theta = -18.43$ OR $\theta = 360 - 18.43 = 341.57$

OR $= 180 - 18.43 = 161.57$

BUT THEN BE SURE TO FIND ALL OTHER VALUES OF θ IN THE REQUIRED RANGE

SO $= 33.7°, 162°, 214°, 342°$
(3 SIG. FIGS.)

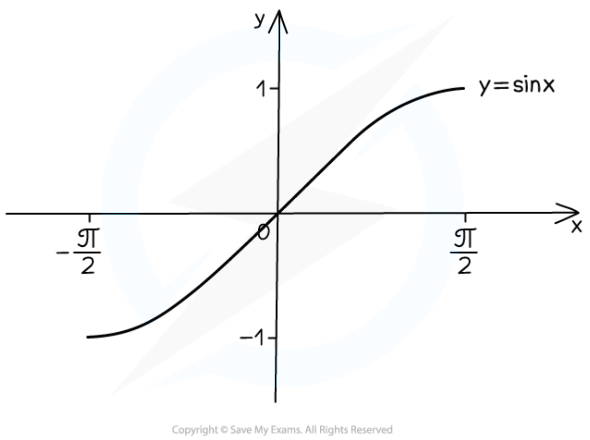
Inverse trigonometric functions

What are the inverse trigonometric functions?

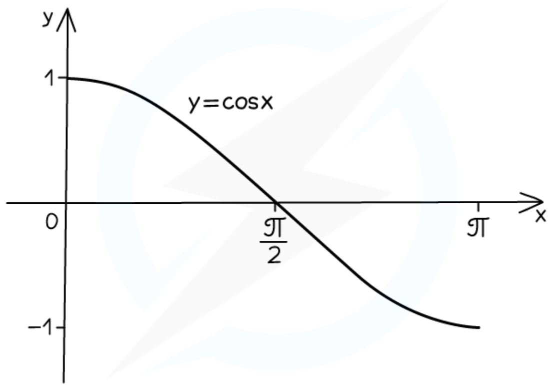
- These functions are the **inverse functions** of **sin**, **cos** and **tan**
  - **sin (arcsin *x*) = *x***
  - **cos (arccos *x*) = *x***
  - **tan (arctan *x*) = *x***
- The domains of **sin**, **cos**, and **tan** must first be restricted to make them one-to-one functions (only one-to-one functions have inverses)

What are the restricted domains?

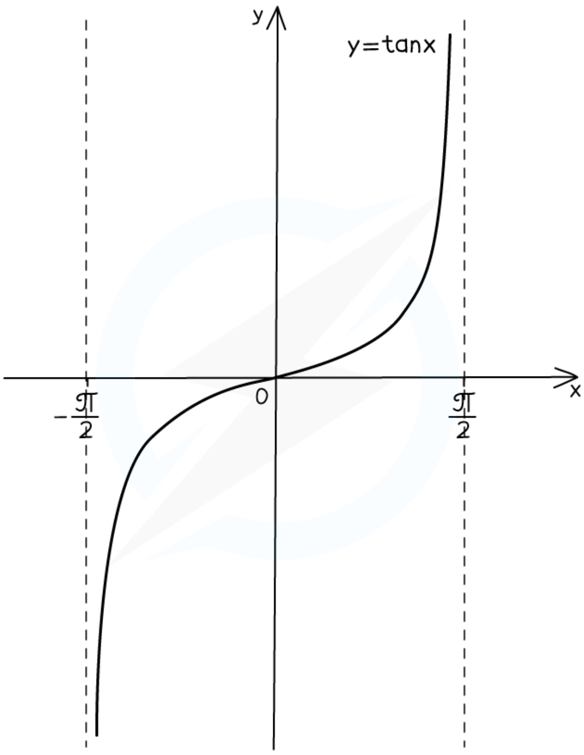
- domain of **sin *x*** is restricted to **$-\pi/2 \le x \le \pi/2$** ($-90° \le x \le 90°$)



- domain of **cos *x*** is restricted to **$0 \le x \le \pi$** ($0° \le x \le 180°$)



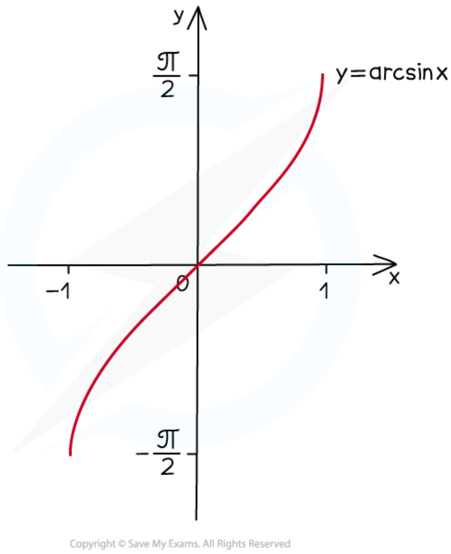
- domain of **tan *x*** is restricted to **−π/2 < *x* < π/2** (**−90° < *x* < 90°**)



How do I sketch the graph of y = arcsin x?

- The graph of ***y* = arcsin *x*** looks like this:

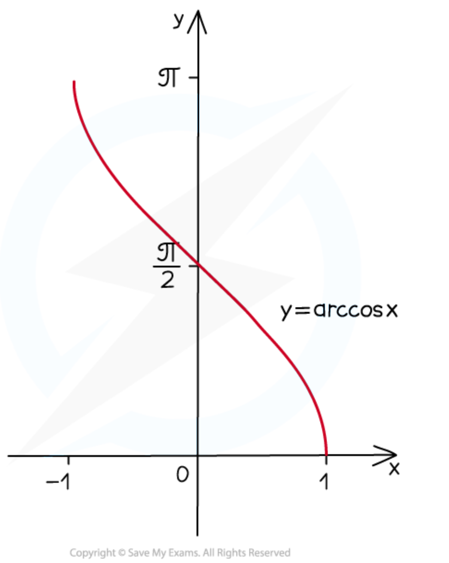
Your notes



- *the* **domain** is **-1 ≤ *x* ≤ 1**
- the **range** is **-π/2 ≤ arcsin *x* ≤ π/2** (**-90° ≤ arcsin *x* ≤ 90°**)

How do I sketch the graph of y = arccos x?

- The graph of ***y* = arccos *x*** looks like this:



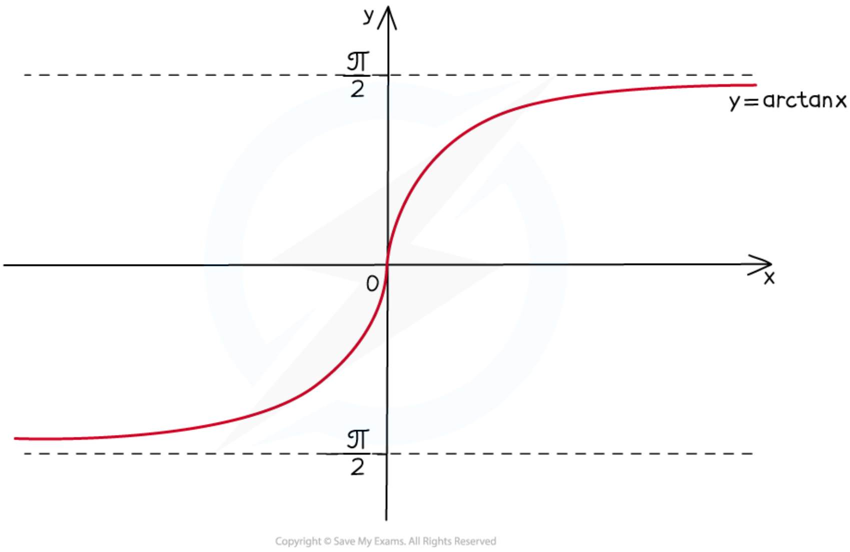
- *the* **domain** is **-1 ≤ *x* ≤ 1**
- the **range** is **0 ≤ arccos *x* ≤ π** (**0° ≤ arccos *x* ≤ 180°**)

How do I sketch the graph of y = arctan x?

- The graph of ***y* = arctan *x*** looks like this:

Your notes

Your notes



- the **domain** is $x \in \mathbb{R}$ (ie arctan x is defined for *all* real number values of x)
- the **range** is **$-\pi/2 <$ arctan $x < \pi/2$** ($-90° <$ arctan $x < 90°$)
- horizontal **asymptotes** at **$y = -\pi/2$** and **$y = \pi/2$**

Examiner Tips and Tricks

- Make sure you know the shapes of the graphs for **sin**, **cos** and **tan.**
- As inverses, the graphs of **arcsin**, **arccos** and **arctan** are reflections of **sin**, **cos** and **tan** in the line **$y = x$.**
- The values returned by the **$\sin^{-1}$**, **$\cos^{-1}$** and **$\tan^{-1}$** keys on your calculator are the values from the ranges of **arcsin**, **arccos** and **arctan.**

Worked Example

Your notes

Given that x satisfies the equation $\arccos x = k$,

where $\frac{\pi}{2} < k < \pi$,

a) state the range of possible values of x

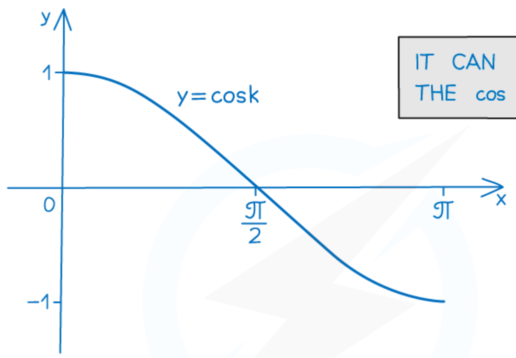
b) express, in terms of x,

i $\sin k$ ii $\tan k$

$\cos(\arccos x) = x$

a) IF $\arccos x = k$ THEN $x = \cos k$

FOR $\frac{\pi}{2} < k < \pi$, $0 > \cos k > -1$



IT CAN HELP TO SKETCH THE cos GRAPH HERE

SO $-1 < x < 0$

b) i) $\sin^2\theta + \cos^2\theta = 1$ $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$

SO $\sin k = \pm\sqrt{1 - \cos^2 k} = \pm\sqrt{1 - x^2}$

BUT $\frac{\pi}{2} < k < \pi$ SO $\sin k > 0$

$\sin k = \sqrt{1 - x^2}$

ii) $\tan k = \frac{\sin k}{\cos k}$

$\tan k = \frac{\sqrt{1 - x^2}}{x}$