Edexcel International A Level (IAL) Maths: Pure 3

Your notes

Functions

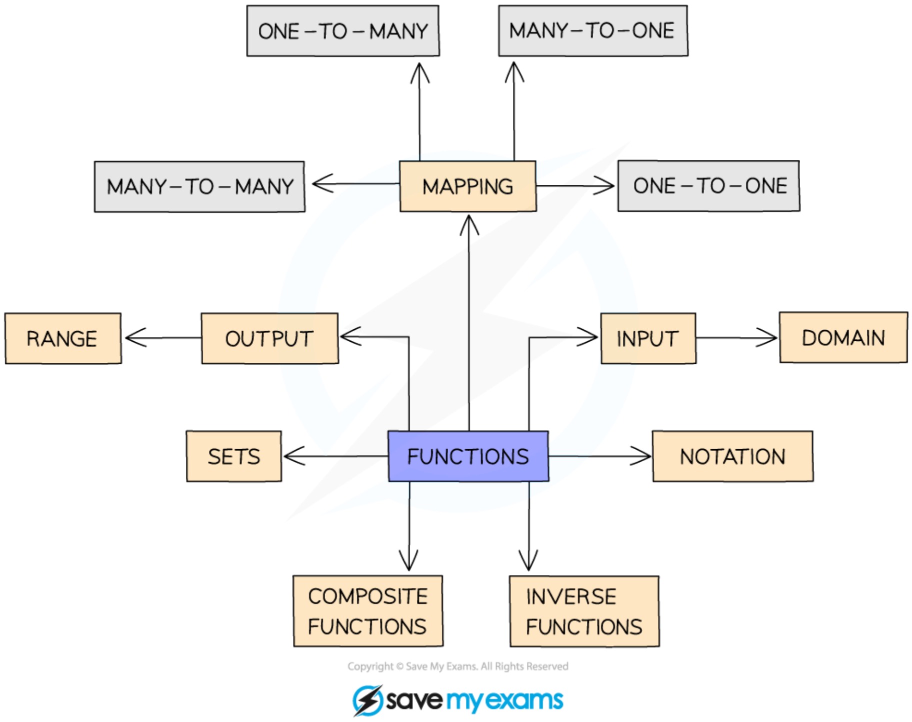
Contents

Your notes

Language of functions

What language is used with functions?

- The language of functions has many keywords associated with it that need to be understood



What are mappings?

- A **mapping** takes an '**input'** from one **set** of values to an '**output'** in another

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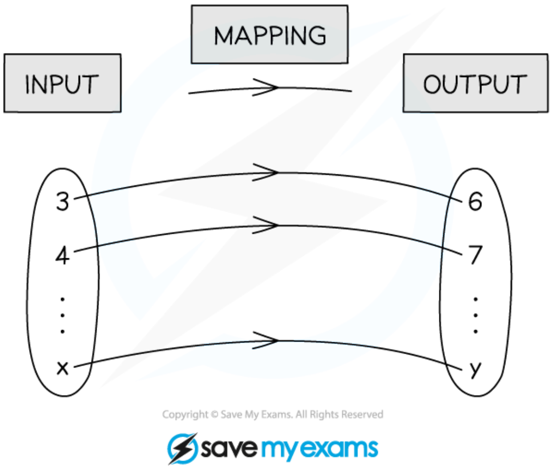
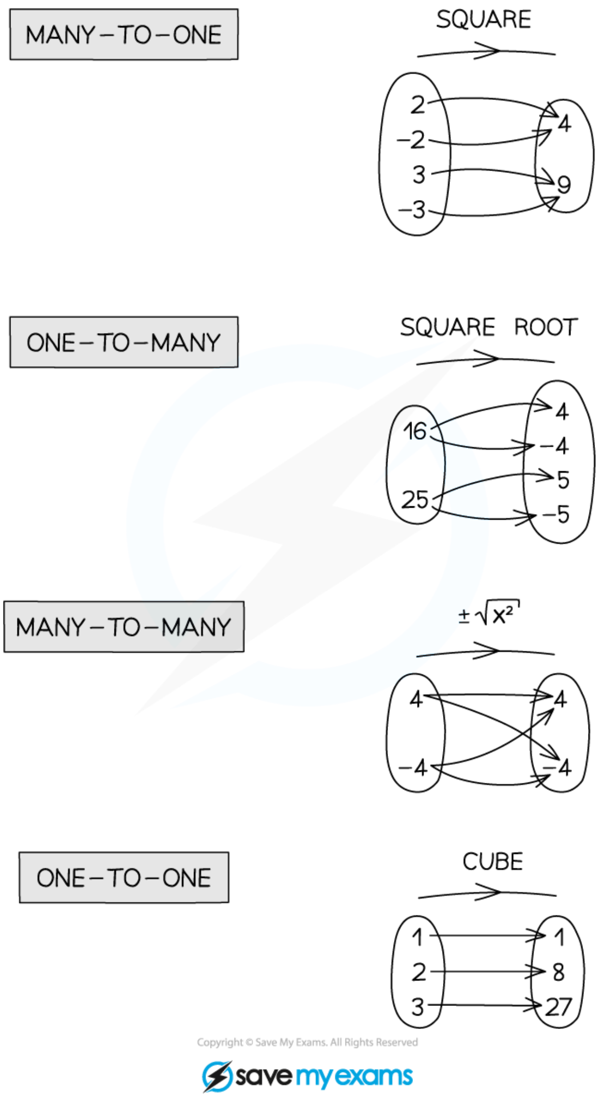
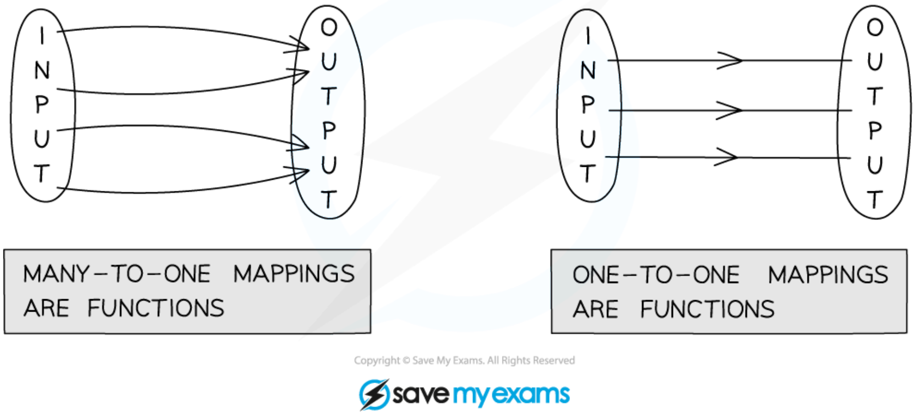


- Mappings can be
  - '**many**-to-**one**' (**many** 'input' values go to **one** 'output' value)
  - '**one**-to-**many**'
  - '**many**-to-**many**'
  - '**one**-to-**one**'



What is the difference between a mapping and a function?

- A **function** is a mapping where every 'input' value maps to a **single** 'output'
- **Many**-to-**one** and **one**-to-**one** mappings are functions
- **Mappings** which have **many possible outputs** are **not functions**

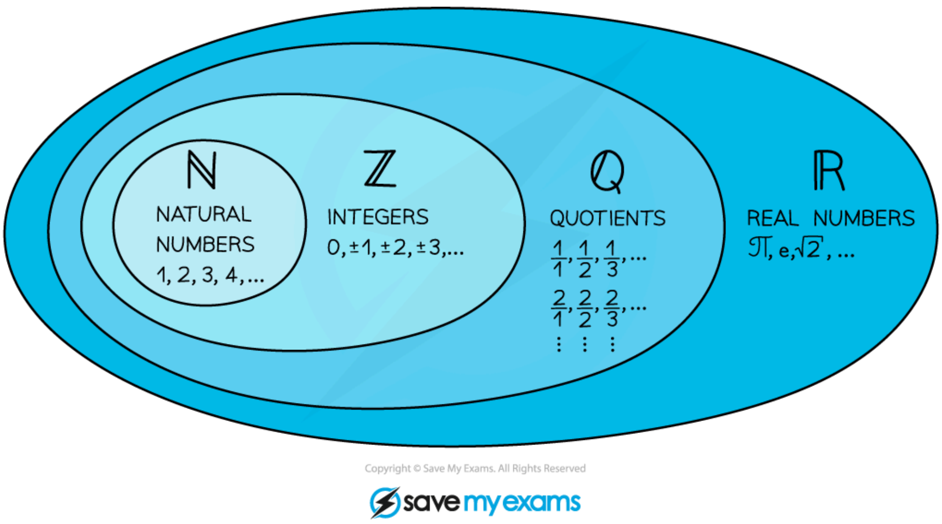


What is function notation?

- Functions are denoted by the notation **f(x)**, **g(x)**, etc
  - eg. $f(x) = x^2 - 3x + 2$
- Or the alternative notation
  - eg. $f: x \mapsto x^2 - 3x + 2$

What sets of numbers do I need to know?

- Functions often involve domains and ranges for specific sets of numbers
- All numbers can be organised into different sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$



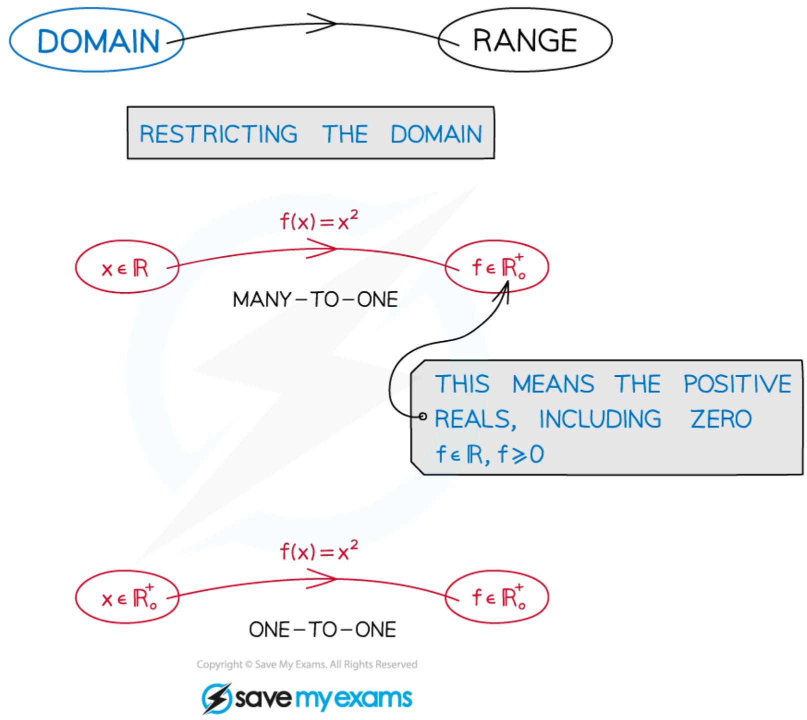
Your notes

- So $\mathbb{N}$ is a **subset** of $\mathbb{Z}$ etc
- $\mathbb{Z}^-$ would be the set of negative integers only

What is the domain of a function?

- The **domain** of a function is the set of values that are allowed to be the 'input'

- A function is only fully defined once its domain has been stated
- **Restrictions** on a domain can turn **many**-to-**one** functions into **one**-to-**one** functions

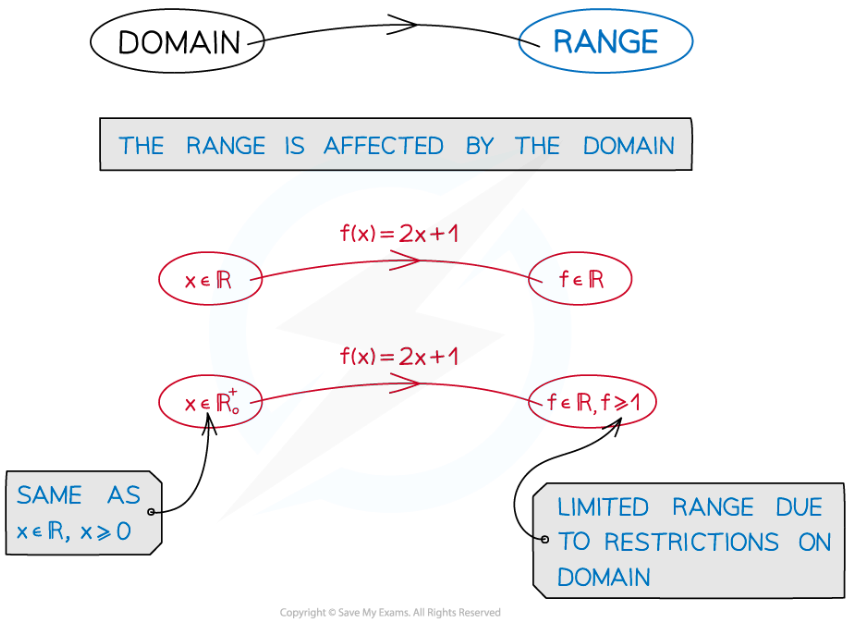


What is the range of a function?

- The **range** of a function is the set of values of all possible 'outputs'

Your notes

- The type of values in the range depend on the domain



Your notes

Worked Example

Your notes

A function is defined by $f : x \mapsto x^2 + 3x + 2, x \in \mathbb{R}, x \geq 0$

Find the range of f and sketch the graph of f against x.

POSITIVE QUADRATIC SO IT HAS A MINIMUM

$x^2 + 3x + 2 = (x + \frac{3}{2})^2 - \frac{9}{4} + 2$

$= (x + \frac{3}{2})^2 - \frac{1}{4}$

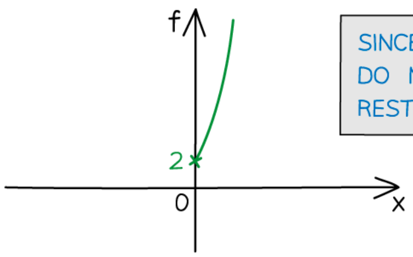
COMPLETE THE SQUARE TO FIND THE MINIMUM

$\therefore$ MINIMUM POINT IS $(-\frac{3}{2}, -\frac{1}{4})$

IT IS CRUCIAL YOU NOTICE THIS!

BUT $x \geqslant 0$ SO MINIMUM POINT OF f IS WHEN $x = 0$, $f = 2$

$\therefore$ RANGE OF f IS: $f \in \mathbb{R}$, $f \geqslant 2$



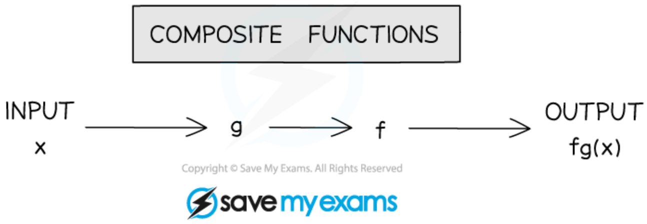
SINCE $x \geqslant 0$ YOU DO NOT NEED THE REST OF THE GRAPH!

Your notes

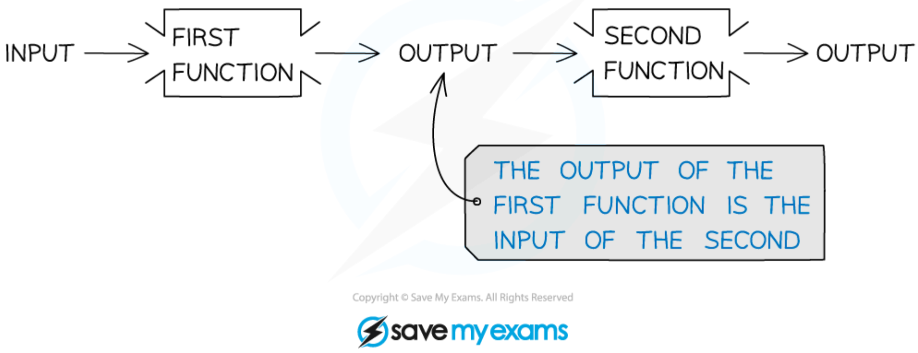
Composite functions

What is a composite function?

- A **composite function** is where one function is applied after another function



- The '**output**' of one function will be the '**input**' of the next one
- Sometimes called **function-of-a-function**
- A composite function can be denoted
  - $fg(x)$
  - $f(g(x))$
  - $f[g(x)]$
  - $(f \circ g)(x)$
  - All of these mean "f of $g(x)$"



How do I use composite functions?

$$\text{INPUT} \longrightarrow g \longrightarrow f \longrightarrow \text{OUTPUT}$$

$$x \longrightarrow g(x) \longrightarrow f(g(x)) \longrightarrow fg(x)$$

- **Recognise** the notation
  - **fg(x)** means **"f** of **g** of **x"**

The order matters

  - First apply *g* to *x* to get $g(x)$
  - Then apply *f* to the previous output to get $f(g(x))$
  - Always start with the function **closest to the variable**
  - $fg(x)$ is not usually equal to $gf(x)$

What are special cases of composite functions?

$f(x) = x^2 \qquad g(x) = 2x$

$fg(x) = f(2x) = (2x)^2 = 4x^2$

$gf(x) = g(x^2) = 2(x^2) = 2x^2$

$\boxed{fg \neq gf}$

$f(x) = 3x - 2 \qquad g(x) = 6x - 5$

$fg(x) = f(6x-5) = 3(6x-5) - 2 = 18x - 17$

$gf(x) = g(3x-2) = 6(3x-2) - 5 = 18x - 17$

$\boxed{fg = gf}$

- **fg(x)** and **gf(x)** are generally different but can sometimes be the same
- **ff(x)** is written as **$f^2(x)$**
- Inverse functions **$ff^{-1}(x) = f^{-1}f(x) = x$**

Your notes

Examiner Tips and Tricks

- **Domain** and **range** are important. In **fg(x)**, the 'output' (range) of **g must** be in the domain of **f(x)**, so **fg(x)** could exist, but **gf(x)** may not (or not for some values of x).

Worked Example

Two functions $f(x)$ and $g(x)$ both have domain $x \in \mathbb{R}$

(a) Given that $f(x+3) = x^3 + 9x^2 + 27x + 26$ find $f(x)$

(b) Given that $gf(x) = 3x^3 - 3$ find $g(x)$

(c) Find $fg(x)$

(You do not need to state domain and range in this question.)

(a) $f(x+3) = x^3 + 9x^2 + 27x + 26$

USE A SUBSTITUTION AS ITS NOT OBVIOUS!

LET $y = x+3$ SO $x = y-3$

$f(y) = (y-3)^3 + 9(y-3)^2 + 27(y-3) + 26$

$$\begin{aligned} &= y^3 - 9y^2 + 27y - 27 \\ &\quad + 9y^2 - 54y + 81 \\ &\quad + 27y - 81 \\ &\quad + 26 \end{aligned}$$

USE BINOMIAL TO EXPAND

LINE TERMS UP NEATLY SO IT'S EASIER TO SIMPLIFY

$= y^3 - 1$

$\therefore f(x) = x^3 - 1$

(b) $gf(x) = 3x^3 - 3 = 3(x^3 - 1)$

$= 3f(x)$

YOU COULD USE A SUBSTITUTION IF YOU DIDN'T SPOT THIS

$\therefore g(x) = 3x$

(c) $fg(x) = f(3x)$

g FIRST, THEN f OR "f OF g"

$= (3x)^3 - 1$

$\therefore fg(x) = 27x^3 - 1$

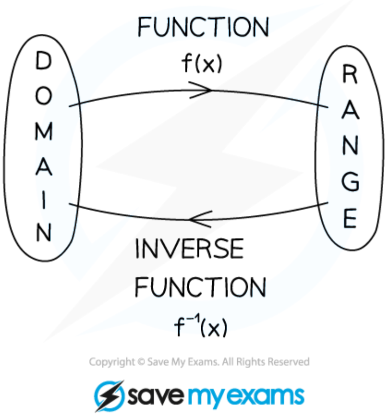
Your notes

Your notes

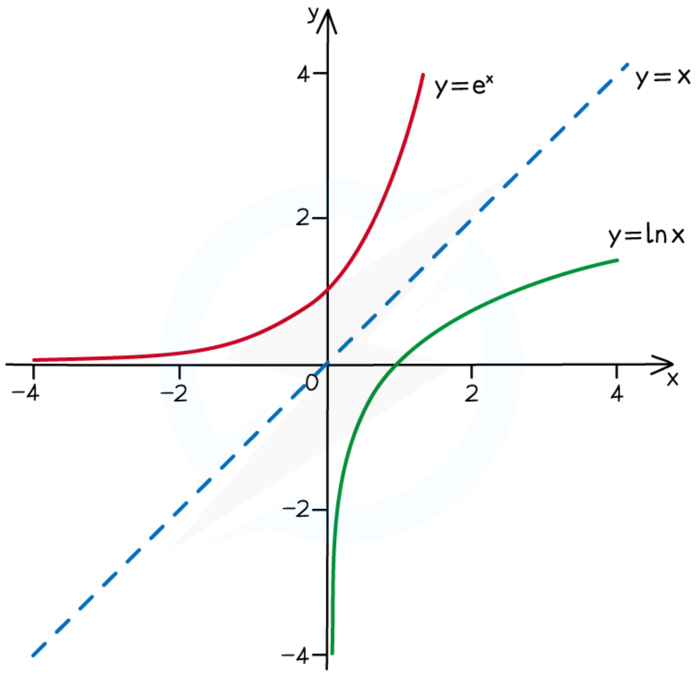
Inverse functions

What is an inverse function?

- An **inverse function** is the **opposite** to the original function
- An inverse function is denoted by $f^{-1}(x)$
- The **inverse** of a function only exists if the function is **one**-to-**one**
- Inverse functions are used to solve equations
    - The solution of $f(x) = 5$ is $x = f^{-1}(5)$

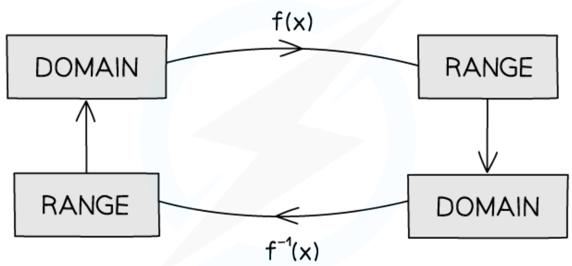


How do I sketch the graph of an inverse function?



- The graphs of a function and its inverse are reflections in the line **y = x**

What is the domain and range of an inverse function?



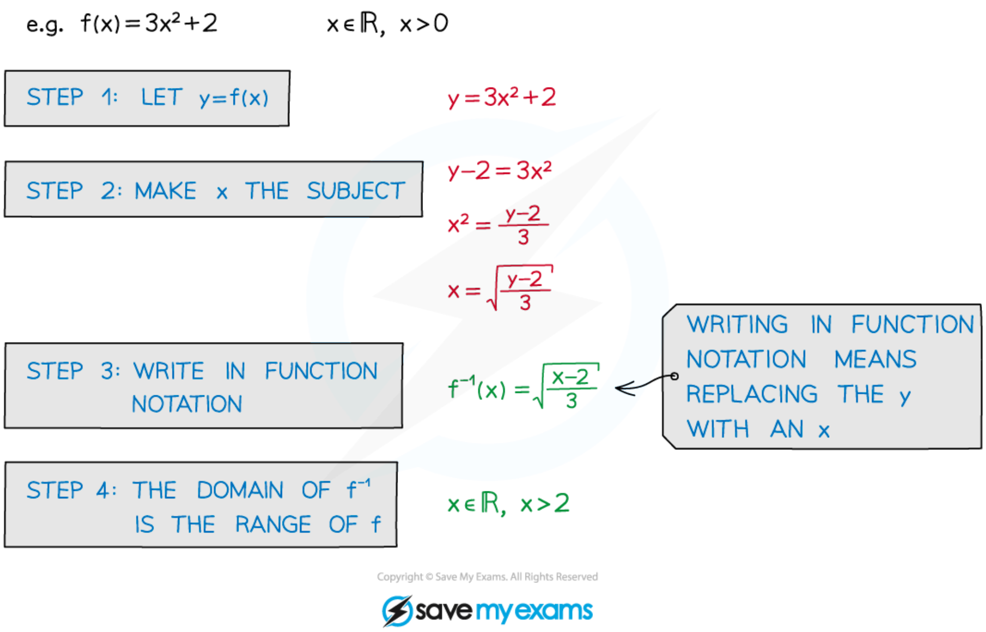
- The range of a function will be the domain of its inverse function
- The domain of a function will be the range of its inverse function

How do I find an inverse function using algebra?

- Set **y = f(x)** and make **x** the subject
- Then rewrite in function notation

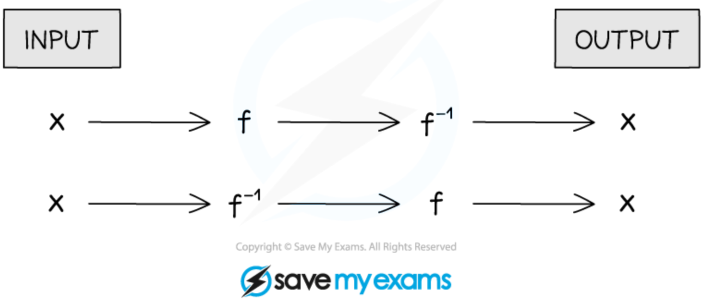
Your notes

- Domain is needed to fully define a function
- The range of **f** is the domain of $\mathbf{f^{-1}}$ (and vice versa)



How are inverse functions related to composite functions?

- A function (**f**) followed by its inverse ($\mathbf{f^{-1}}$) will return the input (**x**)
- $\mathbf{ff^{-1}(x) = f^{-1}f(x) = x}$ (for all values of x)



Worked Example

Your notes

A function is defined as $f(x) = 8e^{-0.2x}$, for $x \in \mathbb{R}, x \geq 0$

(a) Write down the range of $f(x)$

(b) Find $f^{-1}(x)$ and state its domain

(c) On the same diagram sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$

(a) $0 < f \leqslant 8$

NEGATIVE EXPONENTIAL DECAY
8 WILL BE A MAXIMUM
GRAPH NEVER REACHES 0

(b) $y = 8e^{-0.2x}$

STEP 1: $y = f(x)$

$\frac{y}{8} = e^{-0.2x}$

$\ln\left(\frac{y}{8}\right) = -0.2x$

$x = -5\ln\left(\frac{y}{8}\right)$

STEP 2: MAKE x THE SUBJECT

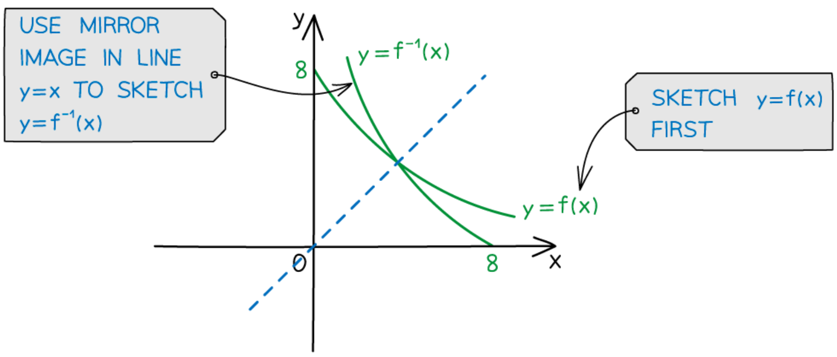
$\therefore\ f^{-1}(x) = -5\ln\left(\frac{x}{8}\right)$

STEP 3: FUNCTION NOTATION

$x \in \mathbb{R},\quad 0 < x \leqslant 8$

STEP 4: DOMAIN OF f^{-1} IS RANGE OF f

(c)



Your notes

Sketching graphs of modulus functions

What is the modulus function?

- The **modulus** function makes **any** 'input' **positive**
- **$|x| = x$ if $x \geq 0$ $|f(x)| = f(x)$ if $f(x) \geq 0$**
- **$|x| = -x$ if $x < 0$ $|f(x)| = -f(x)$ if $f(x) < 0$**
  - **For example:** $|5| = 5$ and $|-5| = 5$
- Sometimes called **absolute value**

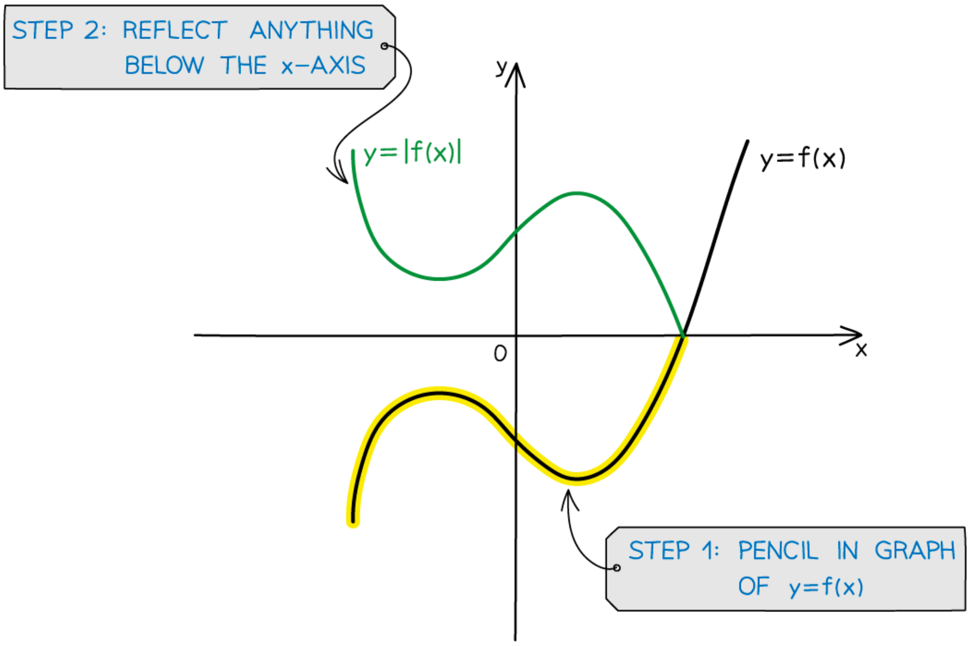
How do I sketch the straight line modulus graph y = a|x+p|+q?

- The graph will look like a "≣" if $a > 0$ or a "≣" if $a < 0$
- There will be a vertex at the point $(-p, q)$
- There could be 0, 1 or 2 roots
  - This depends on the location of the vertex and the orientation of the graph (≣ or ≣)
- Compare this to the **completed square form** of a **quadratic** $a(x+p)^2 + q$

How do I sketch the modulus of a function y = |f(x)|?

STEP 1 Pencil in the graph of **$y = f(x)$**

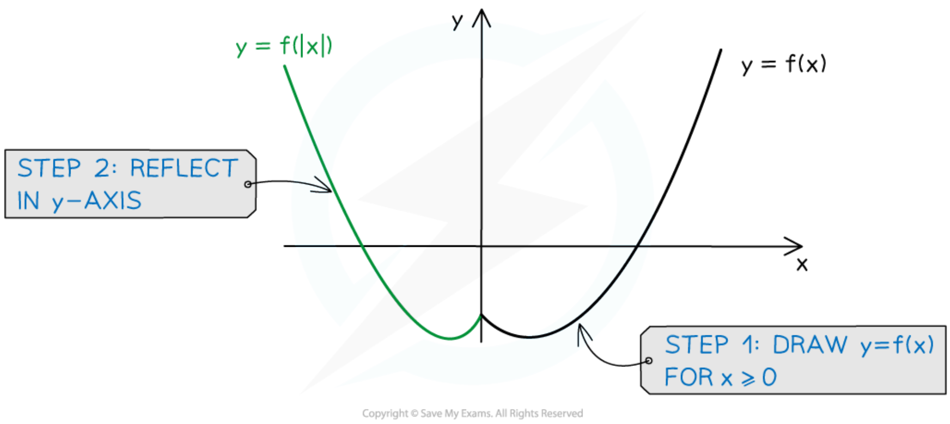
STEP 2 **Reflect** anything **below** the **x**-axis, in the **x**-axis, to get **$y = |f(x)|$**



How do I sketch functions of mod x, y = f(|x|)?

STEP 1 Sketch the graph of **y = f(x) only for *x* ≥ 0**

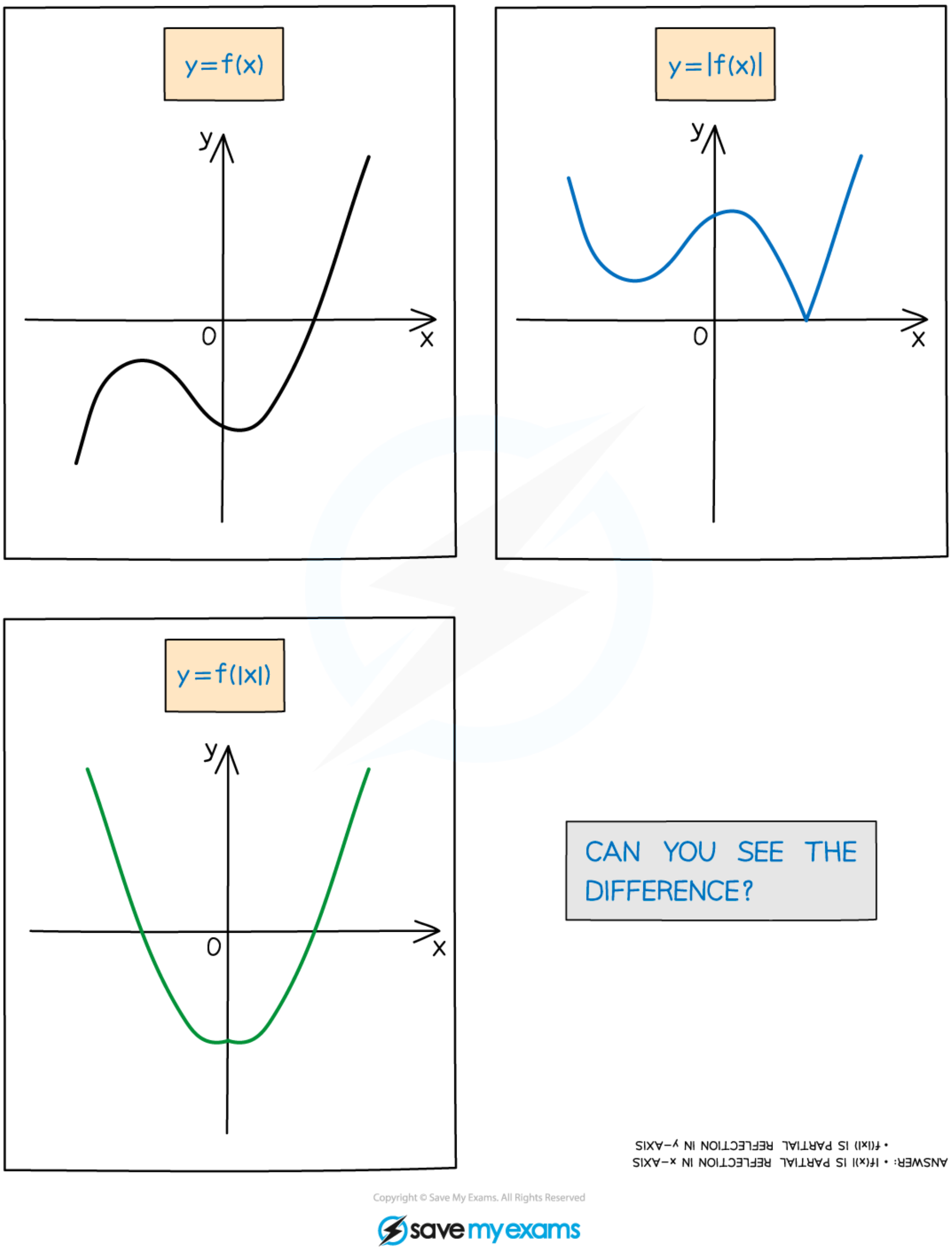
STEP 2 **Reflect** this in the ***y*–axis**



What is the difference between y = |f(x)| and y = f(|x|)?

- There is a difference between ***y* = |f(*x*)|** and ***y* = f(|*x*|)**
- The graph of ***y* = |f(*x*)| never** goes **below** the **x–axis**

- It does not have to have any lines of symmetry
- The graph of ***y* = f(|*x*|)** is **always symmetrical** about the ***y*-axis**
  - It can go below the *y*-axis



Your notes

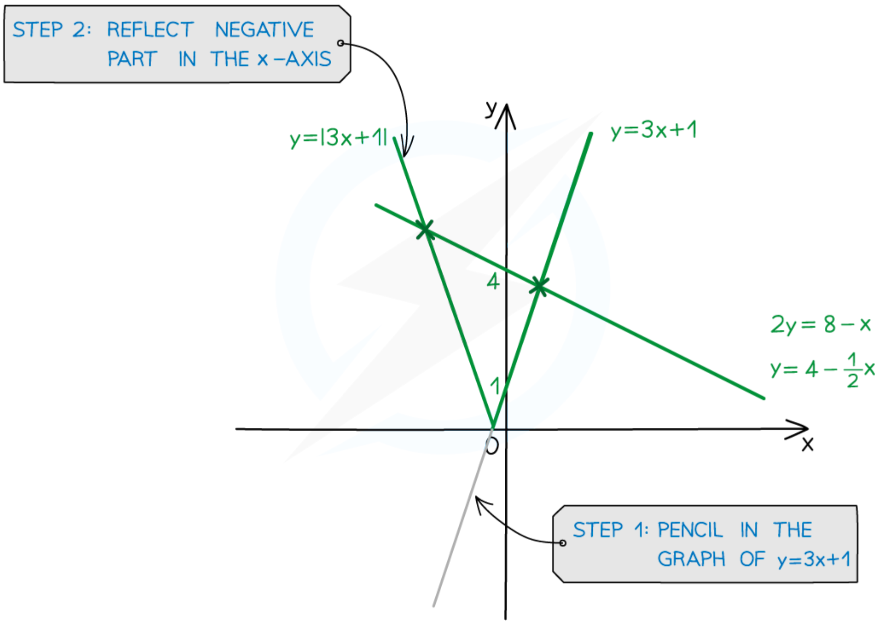
Worked Example

Your notes

(a) On the same diagram sketch the graphs of $y = |3x+1|$ and $2y = 8 - x$

(b) Hence write down how many solutions there are to the equation $4 - \frac{1}{2}x = |3x+1|$

(a)



(b) 2

THE GRAPHS CROSS TWICE

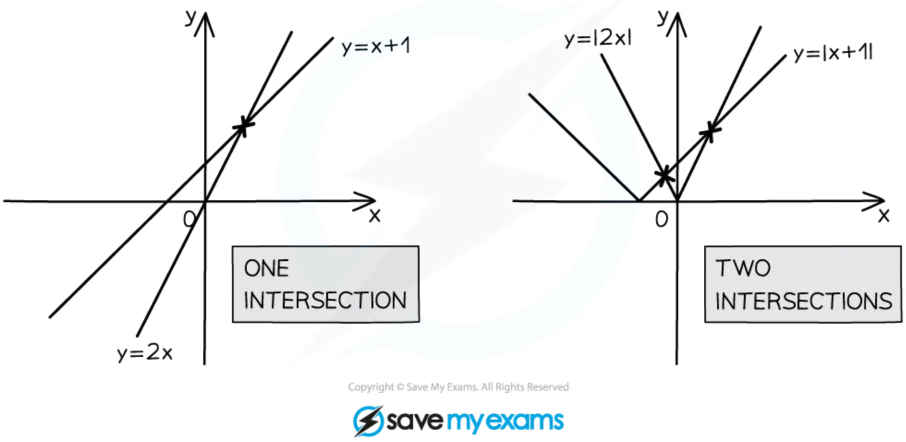
$4 - \frac{1}{2}x = -(3x+1)$

$4 - \frac{1}{2}x = 3x+1$

Solving equations with modulus functions

How do I sketch modulus equations?

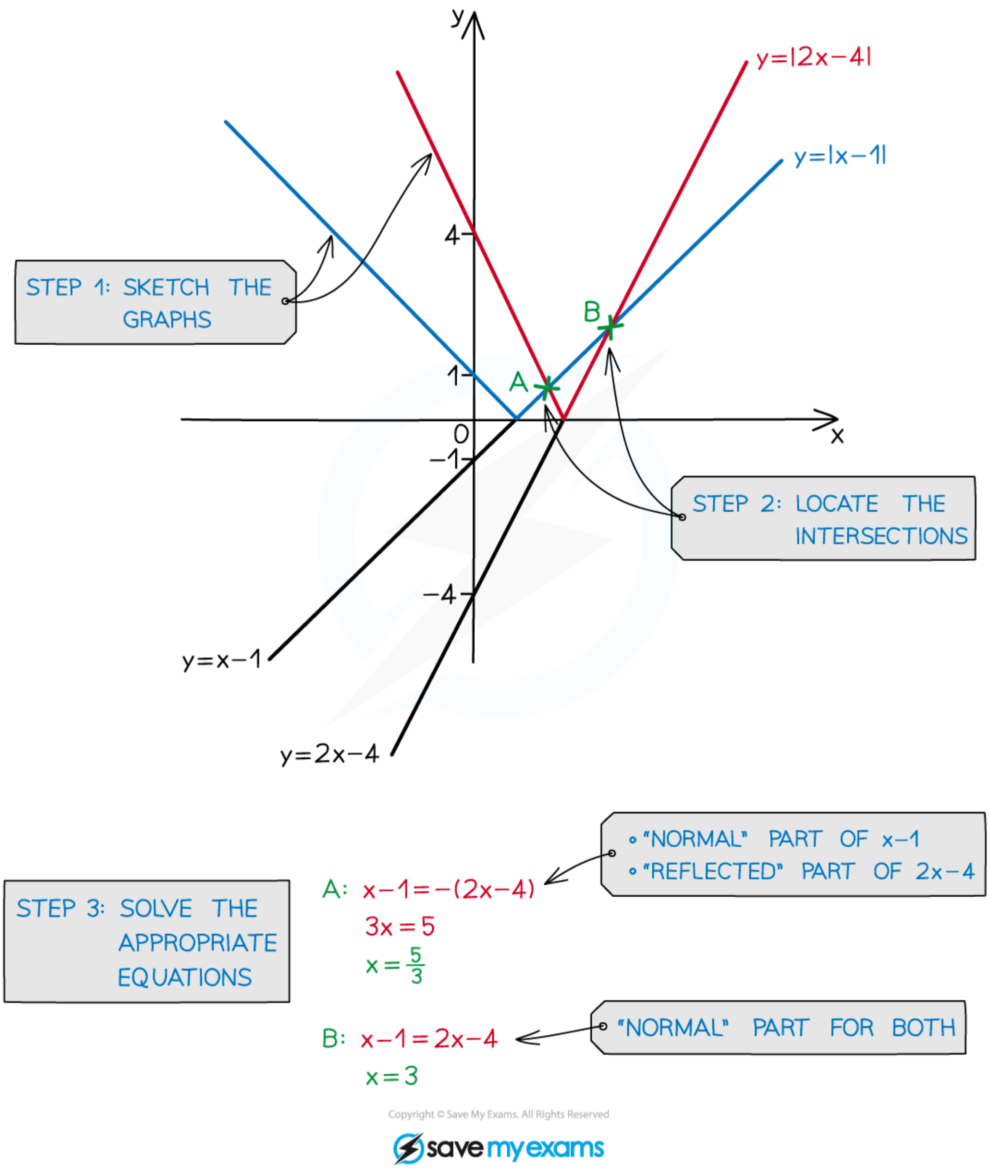
- Two non-parallel straight-line graphs would intersect **once**
- If a **modulus** is involved there **could** be **more** than one intersection
- **Deducing** where these **intersections** are is crucial to solving equations



How do I solve modulus equations?

- STEP 1 **Sketch** the graphs including any **modulus** (reflected) parts
  - (see Modulus Functions – Sketching Graphs)
- STEP 2 **Locate** the graph **intersections**
- STEP 3 **Solve** the appropriate equation(s) or inequality
  - For $|\mathrm{f}(x)| = |\mathrm{g}(x)|$ the two possible equations are $\mathrm{f}(x) = \mathrm{g}(x)$ and $\mathrm{f}(x) = -\mathrm{g}(x)$

e.g. SOLVE $|2x-4| = |x-1|$



Your notes

Examiner Tips and Tricks

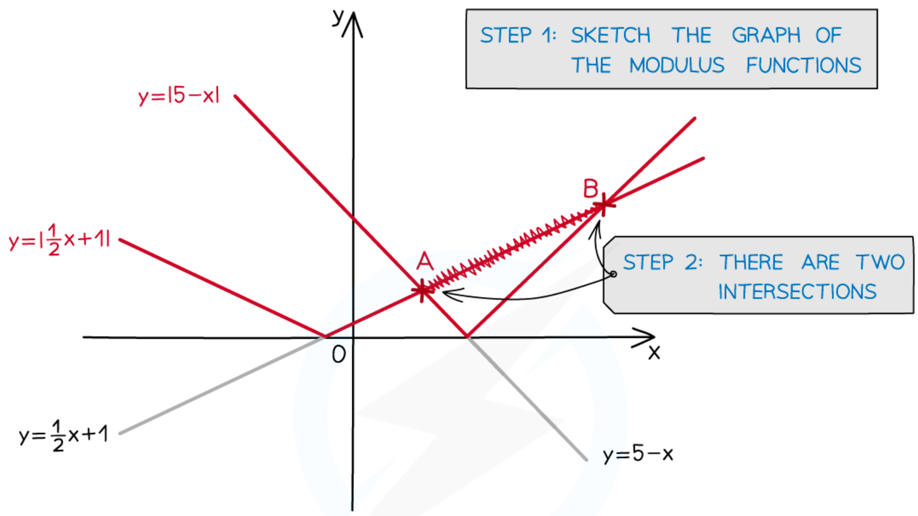
- Sketching the graphs is important as solving algebraically can lead to invalid solutions.
- For example, $x = 1$ is a solution to $x-4=2x-5$ but it is not a solution to $|x-4|=2x-5$ (substitute $x = 1$ into both sides and see why it does not work).

Worked Example

(a) Solve the equation $\left|\frac{1}{2}x + 1\right| = |5 - x|$

(b) Hence solve the inequality $\left|\frac{1}{2}x + 1\right| \geq |5 - x|$

(a)



A: $\frac{1}{2}x + 1 = 5 - x$

$\frac{3}{2}x = 4$

$x = \frac{8}{3}$

STEP 3: POINT A IS THE NORMAL PART OF BOTH EQUATIONS

B: $\frac{1}{2}x + 1 = -(5 - x)$

$6 = \frac{1}{2}x$

$x = 12$

STEP 3: POINT B IS THE NORMAL PART OF $\frac{1}{2}x+1$ BUT THE (REFLECTED) NEGATIVE PART OF $5-x$

$\therefore$ SOLUTIONS TO $\left|\frac{1}{2}x+1\right| = |5-x|$ ARE $x=\frac{8}{3}$ AND $x=12$

Your notes

Your notes

(b) $\frac{8}{3} \leq x \leq 12$

SHADE ON YOUR DIAGRAM THE PART OF THE GRAPH WHERE $|\frac{1}{2}x+1| \geq |5-x|$
THIS SHADING IS BETWEEN THE TWO INTERSECTIONS WHICH YOU FOUND IN PART a)