Edexcel International A Level (IAL) Maths: Pure 3

Your notes

Numerical Methods

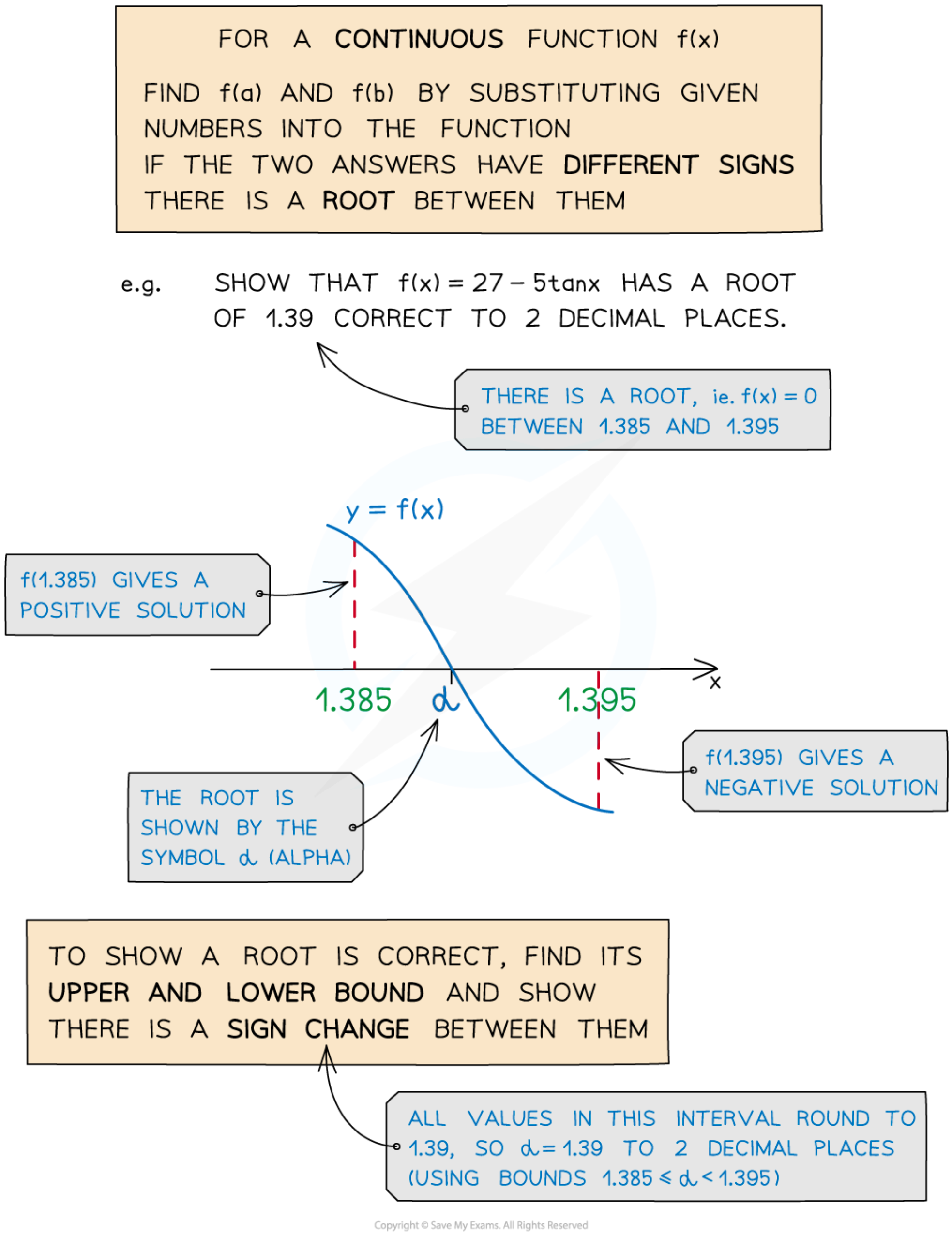
Contents

Your notes

Change of sign method

What does a change of sign mean?

- A sign change between f(a) to f(b) means there must be a root between a and b
- When an equation cannot be solved using the usual **analytical methods**, we can use change of sign to find **approximate solutions**
- We can say a root lies within a particular interval or is correct to a given accuracy
- Bounds can be used to show a root is correct
- Using sign change to find a root is only appropriate for continuous functions in a small interval



Your notes

Examiner Tips and Tricks

- Remember this will only work if the function is continuous and the interval is small enough.
- Sign change questions may be part of bigger numerical methods questions.

Worked Example

The graph of function $f(x) = 3xe^x - 5$ crosses the x-axis at the point P (p, 0).

a) Show that $0.7 < p < 0.8$.

b) Show that $p = 0.771$ to 3 decimal places.

$f(x) = 3xe^x - 5$

a) $f(0.7) = 3 \times 0.7e^{0.7} - 5 = -0.771119...$ NEGATIVE

$f(0.8) = 3 \times 0.8e^{0.8} - 5 = 0.34129...$ POSITIVE

SUBSTITUTE VALUES INTO f(x)

CHANGE OF SIGN MEANS $0.7 < p < 0.8$

b) $p = 0.771$ (3dp)

$0.7705 \leqslant p < 0.7715$

FIND UPPER AND LOWER BOUND OF p

SUBSTITUTE BOUNDS INTO f(x)

$f(0.7705) = -0.0052...$ NEGATIVE

$f(0.7715) = 0.0062...$ POSITIVE

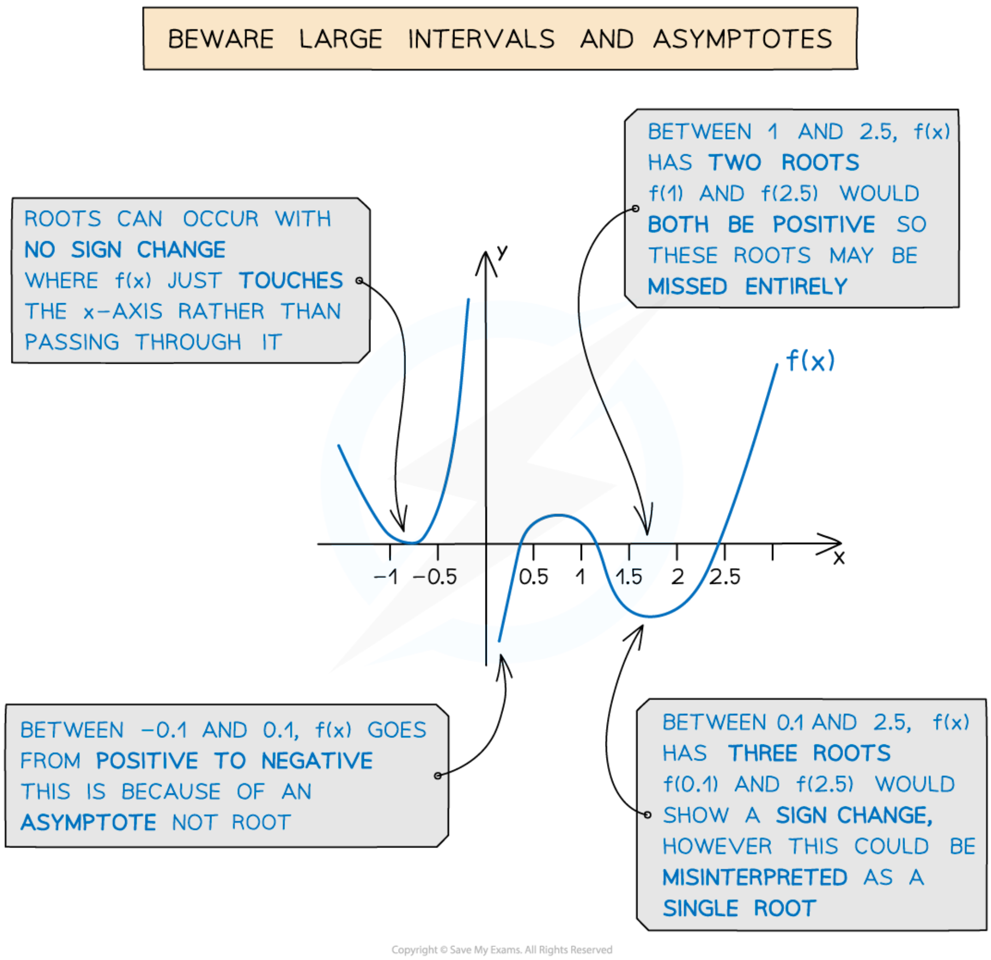
SOLUTION MUST BE $p = 0.771$ TO 3dp

Your notes

Failure of the change of sign method

When can the change of sign method fail?

- There are certain circumstances where a **change of sign** fails to appropriately reflect the number or existence of roots
- If the interval is **too large** there may be more than one root within it
  - An **even number of roots** mean roots are missed entirely as a sign change is not identified
  - An **odd number of roots (> 1)** may mean not all roots are identified
- In a discontinuous funciton, a sign change may occur which is not caused by a root but by an asymptote
- If a function **touches the x-axis** but does not pass through it there would be a root but no sign change



Your notes

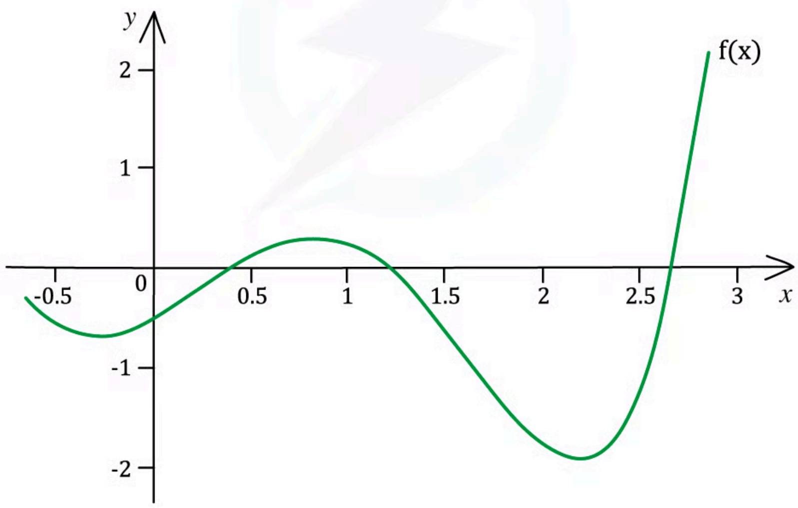
Worked Example

The graph shows the function of $f(x) = xe^x - 2x^3 - 0.5$ for $-0.5 \leq x \leq 3$.

a) State how many roots the equation $xe^x - 2x^3 - 0.5 = 0$ has in the interval $-0.5 \leq x \leq 3$.

b) Give a reason why using a sign change method to check for a root over the interval $-0.5 \leq x \leq 3$ will fail.

c) By using an appropriate interval find the first positive root to 1 decimal place for the function $f(x) = xe^x - 2x^3 - 0.5$.



Your notes

a) 3 ROOTS

GRAPH CROSSES x−AXIS 3 TIMES

b) INTERVAL IS TOO BIG, SIGN CHANGE WILL REFLECT 3 ROOTS NOT JUST 1

c) $0.4e^{0.4} - 2(0.4)^3 - 0.5 = -0.031...$

$0.5e^{0.5} - 2(0.5)^3 - 0.5 = 0.074...$

SIGN CHANGE METHOD WORKS BEST FOR SMALL INTERVALS

$0.4 < x < 0.5$

$0.45e^{0.45} - 2(0.45)^3 - 0.5 = 0.23$

$0.4 < x < 0.45$ $\quad x = 0.4$ (1 dp)

SIGN CHANGE BETWEEN 0.4 AND 0.45 MEANS ROOT = 0.4 (1 dp)

Your notes

$x = g(x)$ iteration

What is iteration?

- When an equation cannot be solved using the usual **analytical methods**, we can still find **approximate solutions** to a certain degree of accuracy
- **Iteration** is one way to do this, by repeatedly using each answer as the new starting value for a function, we can achieve an ever more accurate answer

What does $x_{n+1} = g(x_n)$ mean?

- Iterations are shown using the notation $x_{n+1} = g(x_n)$
- This is a **recurrence relation** where, starting with a number (x_n), we will get an answer x_{n+1} which we can then reuse in the original function
- Equations need to be rearranged into an **iterative formula** – ie. the form $x = g(x)$

REARRANGING

WHEN **REARRANGING AN EQUATION** INTO AN **ITERATIVE FORMULA** $g(x)$, PAY ATTENTION TO THE **"SHOW THAT"** PART OF THE QUESTION TO ENSURE YOU REARRANGE IT INTO THE CORRECT FORMAT

SHOW THAT $x^3 - x^2 - 9 = 0$

CAN BE REARRANGED INTO $x = \sqrt{\dfrac{9}{x-1}}$

$$x^3 - x^2 - 9 = 0$$

SEPARATING LIKE TERMS CAN BE A GOOD PLACE TO START AS 9 IS ON ITS OWN IN THE ITERATIVE FORMULA

$$\Rightarrow x^3 - x^2 = 9$$

FACTORISING OFTEN HELPS, ALSO THE QUESTION HAS A $\sqrt{\text{SQUARE ROOT}}$ SO GETTING x^2 ON ITS OWN IS HELPFUL

$$x^2(x-1) = 9$$

$$x^2 = \frac{9}{x-1}$$

DIVIDING BY $x-1$ THAN SQUARE ROOTING BOTH SIDES LEAVES THE APPROPRIATE ITERATIVE FORMULA

$$x = \sqrt{\frac{9}{x-1}}$$

$x^3 - x^2 - 9 = 0$ ALSO REARRANGES TO $\sqrt{x = x^3 - 9}$ SO BE CAREFUL TO GET THE CORRECT FUNCTION

ITERATION

GIVEN A STARTING VALUE x_0, YOU CAN **SUBSTITUTE** EACH ANSWER INTO THE **ITERATIVE FORMULA.**
THE NOTATION $x_1, x_2 \ldots$ IS USED TO **TRACK EACH ITERATION.**
ANSWERS MAY **CONVERGE** ON A PARTICULAR NUMBER OR **DIVERGE.**

USING THE ITERATING FORMULA

$x_{n+1} = \sqrt{\dfrac{9}{x_n - 1}}$ STARTING $x_0 = 2$

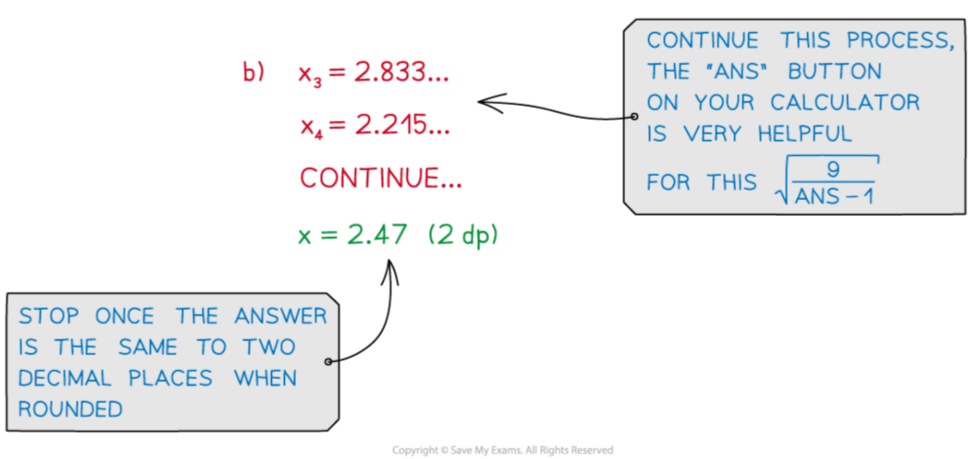
a) FIND x_1 AND x_2

b) SOLVE $x^3 - x^2 - 9 = 0$ TO 2dp

a) $x_1 = \sqrt{\dfrac{9}{2-1}} = 3$

$x_2 = \sqrt{\dfrac{9}{3-1}} = 2.121\ldots$

SUBSTITUTE x_0 INTO FORMULA AND REPEAT WITH NEW ANSWER



What are staircase diagrams and cobweb diagrams?

- Iterations can be shown on diagrams called **staircase** or **cobweb** diagrams
- These can be drawn by **plotting the graphs** of **$y = x$** against **$y = g(x)$** from your iterative formula

Your notes

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DRAWING STAIRCASE AND COBWEB DIAGRAMS

STEP 1: SKETCH GRAPHS OF $y = x$ AND $g(x)$ FROM YOUR ITERATIVE FORMULA (POINTS WHERE THE TWO GRAPHS MEET ARE ROOTS)

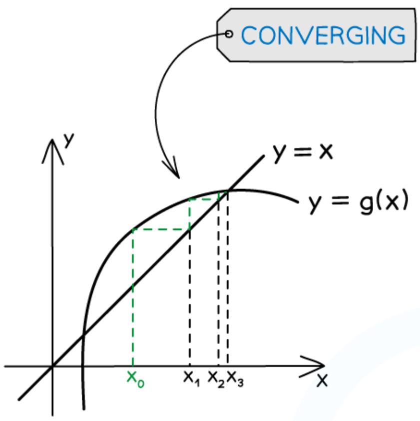
STEP 2: MARK x_0 ONTO THE x-AXIS AND DRAW A **VERTICAL LINE** TO MEET THE **CURVE** $y = g(x)$

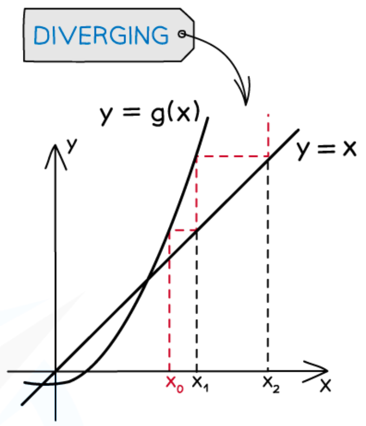
STEP 3: FROM THIS POINT DRAW A **HORIZONTAL LINE** TO MEET THE **LINE** $y = x$. THIS IS x_1.

STEP 4: CONTINUE BY DRAWING ANOTHER **VERTICAL LINE** TO THE **CURVE** $y = g(x)$ THEN A HORIZONTAL LINE TO THE LINE $y = x$. EACH TIME YOU REPEAT THIS STEP IS **ANOTHER ITERATION.**

STEP 5: IF THE LINES ARE GETTING **CLOSER** TO THE ROOT THE ITERATION IS **CONVERGING,** IF THEY ARE GETTING **FURTHER** AWAY IT IS **DIVERGING.**

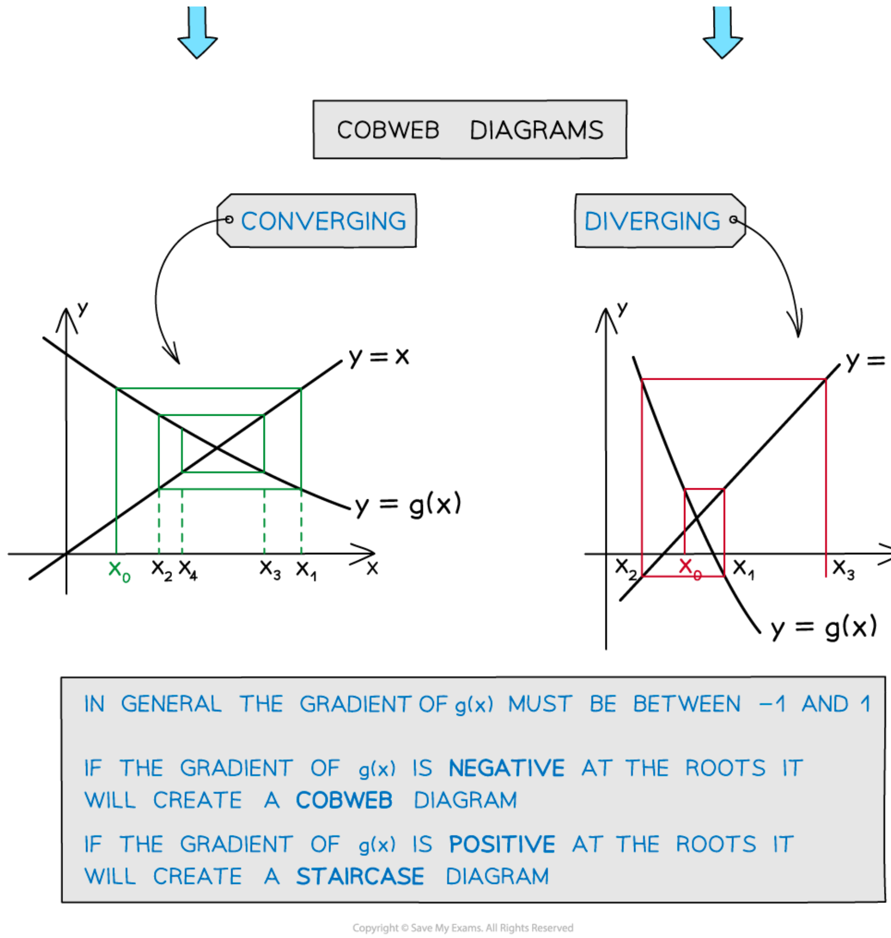
STAIRCASE DIAGRAMS





FOR **CONVERGING DIAGRAMS,** EACH ITERATION FROM x_0 WILL GET **CLOSER TO THE ROOT** (POINT WHERE THE TWO GRAPHS INTERSECT)
FOR COBWEB DIAGRAMS THE SOLUTIONS WILL ALTERNATE ABOVE AND BELOW THE ROOT BUT THEY WILL STILL BE GETTING CLOSER EACH TIME

FOR **DIVERGING DIAGRAMS,** EACH ITERATION FROM x_0 WILL GET **FURTHER FROM THE ROOT**



Your notes

Examiner Tips and Tricks

- You must show all your steps when rearranging an equation into an iterative formula
- Working backwards can often be helpful to figure out how an equation has been rearranged but you must write your answer as if you worked forwards
- Use ANS button on your calculator to calculate repeated iterations
- Keep track of your iterations using x_2, x_3... notation
- Iteration may be part of bigger numerical methods questions

Worked Example

a) Show that $3x^2 - x^3 + 3 = 0$ can be written in the form $x = 3 + \frac{3}{x^2}$.

b) Use the iteration formula $x_{n+1} = 3 + \frac{3}{x_n^2}$ and $x_0 = 2$ to find a solution to 2 decimal places.

c) Sketch a diagram to show the convergence of the sequence for x_1, x_2 and x_3.

a) $3x^2 - x^3 + 3 = 0$

IT CAN BE HARD TO KNOW WHERE TO START, THERE IS NO x^3 IN FINAL ANSWER. BEGIN BY MAKING THAT POSITIVE

$3 = x^3 - 3x^2$

$3 = x^2(x - 3)$

SEPARATE x TERMS TO FACTORISE

WE NEED TO TURN x^3 AND x^2 INTO x^2 AND x SO x^2 MUST BE THE FACTOR

$\frac{3}{x^2} = x - 3$

$3 + \frac{3}{x^2} = x$

REARRANGE UNTIL FINAL ITERATIVE FORMULA IS REACHED (SOMETIMES WORKING BACKWARDS WILL HELP)

b) $x_0 = 2$

$x_1 = 3 + \frac{3}{2^2} = 3.75$

SUBSTITUTE $x = 2$ INTO ITERATIVE FORMULA

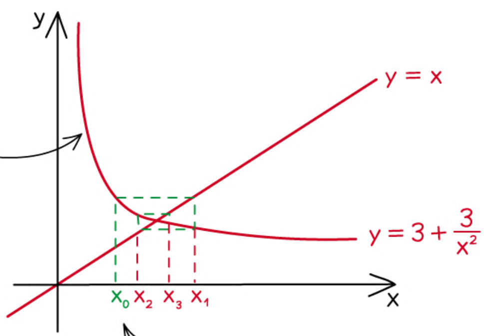
$x_2 = 3.213$

USE $3 + \frac{3}{\text{ANS}^2}$ TO CONTINUE THIS UNTIL x is THE SAME TO 2 dp

$x = 3.28$ (2 dp)

c)

SKETCH $y = x$ AND $y = 3 + \frac{3}{x^2}$



DRAW VERTICAL LINE FROM x_0 THEN HORIZONTAL AND REPEAT FOR x_1, x_2, x_3