Your notes

# Edexcel International A Level (IAL) Maths: Pure 3

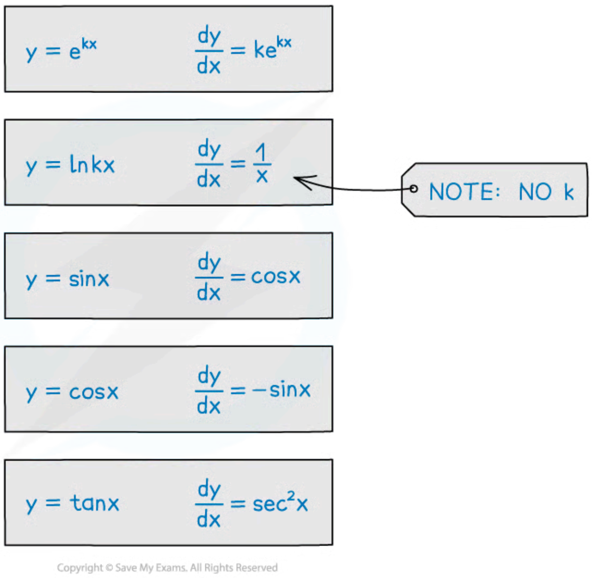
## Further Integration

### Contents

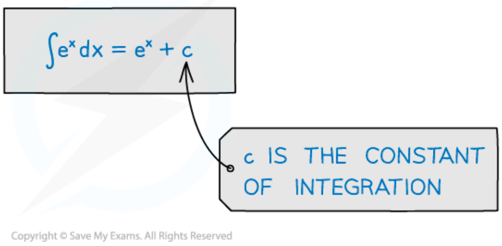
# Integrating other functions

## Is integration the reverse of differentiation?

- Yes, but remember "+c", the constant of integration ...
  - ... unless finding a definite integral
- Recognising common results helps to make integration easier (See Differentiating Other Functions)




## How do I integrate $e^x$ and $e^{kx}$?




- The gradient of $\mathbf{e^{kx}}$ is $\mathbf{ke^{kx}}$
  - ie $\mathbf{y = e^{kx}}$, $\mathbf{dy/dx = ke^{kx}}$
- The reverse applies when integrating

- This is an example of **reverse chain rule**

$$\int ke^{kx}\,dx = e^{kx} + c$$



## How do I integrating 1/x?

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

MODULUS OF x
x>0 IN ln x BUT NOT IN $\frac{1}{x}$



- Remember $\frac{1}{x} = x^{-1}$
  - The method for integrating powers does **not** apply if the power is -1

## How do I integrating sin x and cos x?

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$



- Note the minus in the integral of **sin x**

## How do I integrate tan x?

- The integral of **tan x** is **ln|sec x| + c**

### Examiner Tips and Tricks

- Make sure you have a copy of the formula booklet during revision but don't try to remember everything in the formula booklet.

Your notes

- However, do be familiar with the **layout** of the formula booklet – you'll be able to locate quickly whatever you are after, and you do not want to be searching every line of every page!
- For formulae you think you have remembered, use the booklet to double-check.

Your notes

### Worked Example

Find the following integrals

(a) $\int 3e^{3x}\,dx$

(b) $\int_{-6}^{-3} \frac{1}{x}\,dx$

(c) $\int \sin x\,dx$

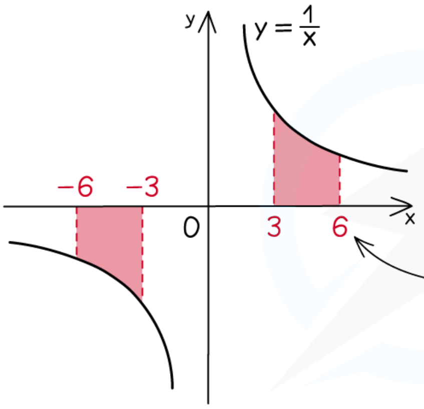
a) $\int 3e^{3x}dx = e^{3x} + c$

IN THE FORM $ke^{kx}$

Your notes

b) $\int_{-6}^{-3} \frac{1}{x}\,dx = \left[\ln|x|\right]_{-6}^{-3}$

$= \left[\ln x\right]_{3}^{6}$




THE SYMMETRY OF THE GRAPH SHOWS THESE ARE EQUAL

$= \ln 6 - \ln 3$

$= \ln \frac{6}{3}$

$= \ln 2$



c) $\int \sin x\,dx = -\cos x + c$

BEWARE OF THE MINUS!

Your notes

# Reverse chain rule

## What is the chain rule?

- The Chain Rule is a way of differentiating two (or more) functions

FOR $y = f(u)$, WHERE $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$



- In many simple cases the above formula/substitution is not needed

1. $y = (3x)^5$

$$\frac{dy}{dx} = 5(3x)^4 \times 3$$

$$\frac{dy}{dx} = 15(3x)^4$$

2. $y = e^{5x}$

$$\frac{dy}{dx} = e^{5x} \times 5$$

$$\frac{dy}{dx} = 5e^{5x}$$

ALSO FROM $e^{kx} \longrightarrow ke^{kx}$

3. $y = \sin^4 2x$

$$\frac{dy}{dx} = 4\sin^3 2x \times \cos 2x \times 2$$

$$\frac{dy}{dx} = 8\cos 2x \sin^3 2x$$

CHAIN RULE APPLIED TWICE



- The same can apply for the reverse – integration

## How do I integrate using the reverse chain rule?

Your notes

1. $\int 3\cos 3x \, dx = \sin 3x + c$

$\frac{d}{dx}[\sin 3x] = 3\cos 3x$

BY CHAIN RULE

2. $\int 8e^{8x} dx = e^{8x} + c$

$\frac{d}{dx}[e^{8x}] = 8e^{8x}$

3. $\int 10(2x+1)^4 dx = (2x+1)^5 + c$

$$\frac{d}{dx}\left[(2x+1)^5\right] = 5(2x+1)^4 \times 2$$
$$= 10(2x+1)^4$$



- In more awkward cases it can help to write the numbers in before integrating
- STEP 1: Spot the 'main' function
- STEP 2: 'Adjust' and 'compensate' any numbers/constants required in the integral
- STEP 3: Integrate and simplify

1. $\int \cos 3x \, dx$

STEP 1: SPOT THE "MAIN" FUNCTION

cos – WHICH WOULD COME FROM sin

CHAIN RULE SAYS TO MULTIPLY BY THE DERIVATIVE OF 3x, WHICH IS 3

THERE IS NO 3 IN THE INTEGRAL SO I NEED TO ADJUST AND COMPENSATE FOR IT

STEP 2: "ADJUST" AND "COMPENSATE" ANY NUMBERS REQUIRED IN THE INTEGRAL

$$= \frac{1}{3}\int 3\cos 3x \, dx$$

3cos3x IS WHAT I'D EXPECT TO SEE FROM DIFFERENTIATING sin3x USING CHAIN RULE

STEP 3: INTEGRATE AND SIMPLIFY

$$= \frac{1}{3}\sin 3x + c$$



Your notes

### Examiner Tips and Tricks

- If in doubt you can always use a substitution.
- Differentiation is easier than integration so if stuck try the opposite, eg. **sin** and **cos** are linked (remember that minus!) so if integrating a **sin** function, start by differentiating the corresponding **cos** function.
- Lastly, check your final answer by differentiating it.

### Worked Example

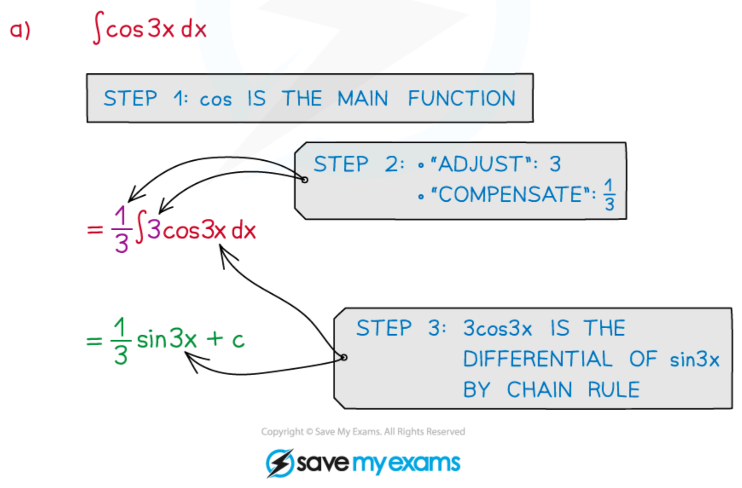
Your notes

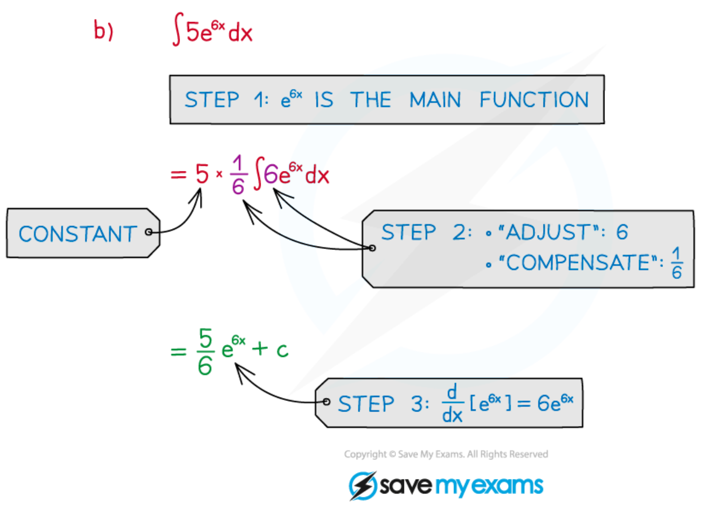
Find the following integrals

(a) $\int \cos 3x \, dx$

(b) $\int 5e^{6x} \, dx$

(c) $\int (2x+5)^3 \, dx$



Your notes

c) $\int (2x+5)^3 dx$

STEP 1: $(...)^3$ IS THE MAIN FUNCTION

$= \frac{1}{4} \times \frac{1}{2} \int 4 \times 2 \times (2x+5)^3 dx$

STEP 2: • "ADJUST": 4 FROM POWERS
2 FROM CHAIN RULE
• "COMPENSATE": $\frac{1}{4}$ AND $\frac{1}{2}$

$= \frac{1}{8}(2x+5)^4 + c$

STEP 3: $\frac{d}{dx}[(2x+5)^4] = 4(2x+5)^3 \times 2 = 8(2x+5)^3$

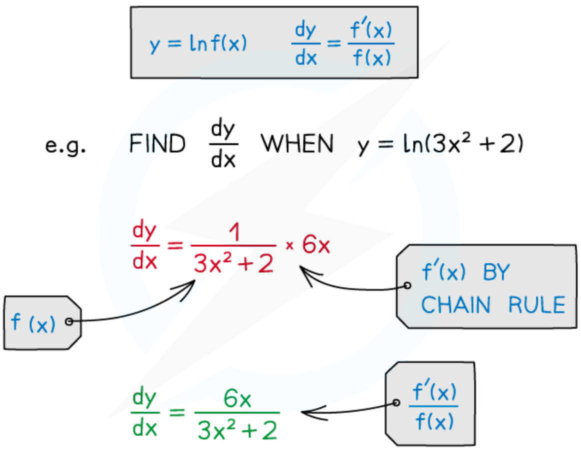


- AFTER SOME PRACTICE, YOU MAY FIND STEP 2 IS NOT NEEDED.
- BUT DO USE IT ON MORE AWKWARD QUESTIONS.
- ACCURACY IS THE MOST IMPORTANT THING.

# Integrating f'(x)/f(x)

## How do I integrate 1/x?

- The technique for integrating fractions depends on the type of fraction
- For **polynomial denominators** see Integration using Partial Fractions
- If $\frac{dy}{dx} = \frac{1}{x}$ then **y = ln |x| + c** – see Integrating Other Functions
- The type of fraction dealt with here is a specific case of Reverse Chain Rule



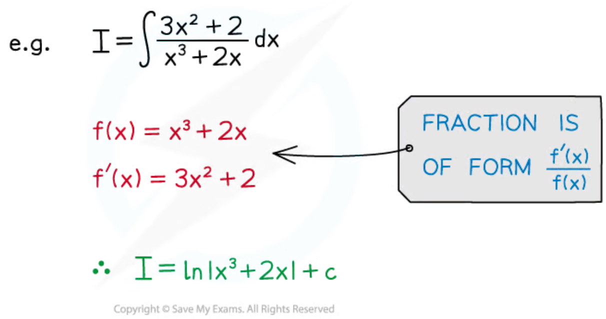



## How do I integrate f'(x)/f(x)?

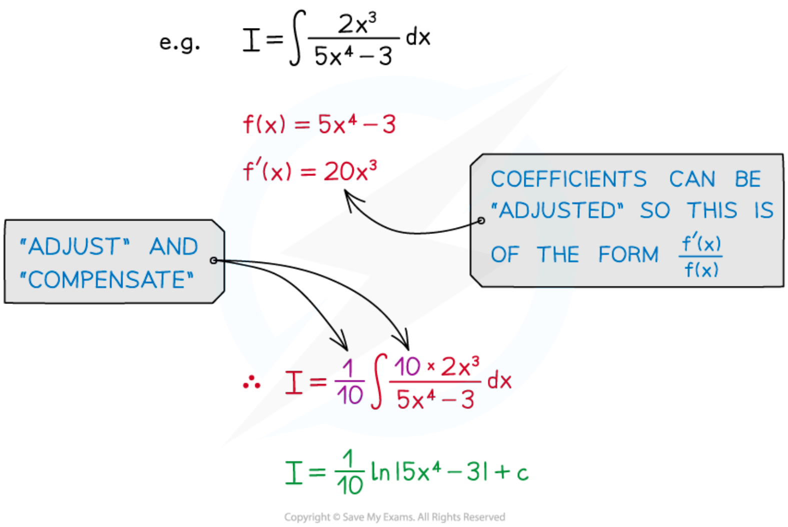



- “The top is ‘almost’ the derivative of the bottom”
  - 'almost' here meaning 'a multiple of' (see below)
- The integral will involve **ln |f(x)|** - ie **ln** of the bottom
  - Due to reverse chain rule

Your notes




## How do I adjust a fraction into the form f'(x)/f(x)?




- There may be coefficients to 'adjust' and 'compensate' for

### Examiner Tips and Tricks

- If you're unsure if the fraction is of the form **f'(x)/f(x)**, differentiate the denominator.
- Compare this to the numerator but you can ignore any **coefficients**.
- If the coefficients do not match then 'adjust' and 'compensate' for them.

### Worked Example

Your notes

**?** Show $\int_{\frac{\pi}{12}}^{\frac{\pi}{2}} \frac{\sin 4x}{1+2\cos 4x}\,dx = \frac{1}{8}\ln\frac{2}{3}$

$$I = \int_{\frac{\pi}{12}}^{\frac{\pi}{2}} \frac{\sin 4x}{1+2\cos 4x}\,dx$$

$$f(x) = 1 + 2\cos 4x$$

$$f'(x) = 2 \times -\sin 4x \times 4$$

$$= -8\sin 4x$$

IGNORING COEFFICIENTS, I IS OF THE FORM $\frac{f'(x)}{f(x)}$

"ADJUST" AND "COMPENSATE"

$$I = -\frac{1}{8}\int_{\frac{\pi}{12}}^{\frac{\pi}{2}} \frac{-8\sin 4x}{1+2\cos 4x}\,dx$$

$$I = \left[-\frac{1}{8}\ln|1+2\cos 4x|\right]_{\frac{\pi}{12}}^{\frac{\pi}{2}}$$

"ln OF BOTTOM"

$$I = \left(-\frac{1}{8}\ln|1+2\cos 2\pi|\right) - \left(-\frac{1}{8}\ln\left|1+2\cos\frac{\pi}{3}\right|\right)$$



$$I = \left(-\frac{1}{8}\ln 3\right) - \left(-\frac{1}{8}\ln 2\right)$$

$$I = \frac{1}{8}\left(\ln 2 - \ln 3\right)$$

SWAPPED TERMS SO FIRST IS POSITIVE

$$\therefore\ I = \frac{1}{8}\ln\frac{2}{3}$$

Your notes

# Integrating with trigonometric identities

## What are trigonometric identities?

ESSENTIAL TRIGONOMETRIC IDENTITIES

$$\sin^2 x + \cos^2 x \equiv 1$$

$$\tan x \equiv \frac{\sin x}{\cos x}$$



- Some are given in the formulae booklet
  - Be sure to note the difference between the ± and ∓ symbols!

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad \left(A \pm B \neq \left(k + \tfrac{1}{2}\right)\pi\right)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

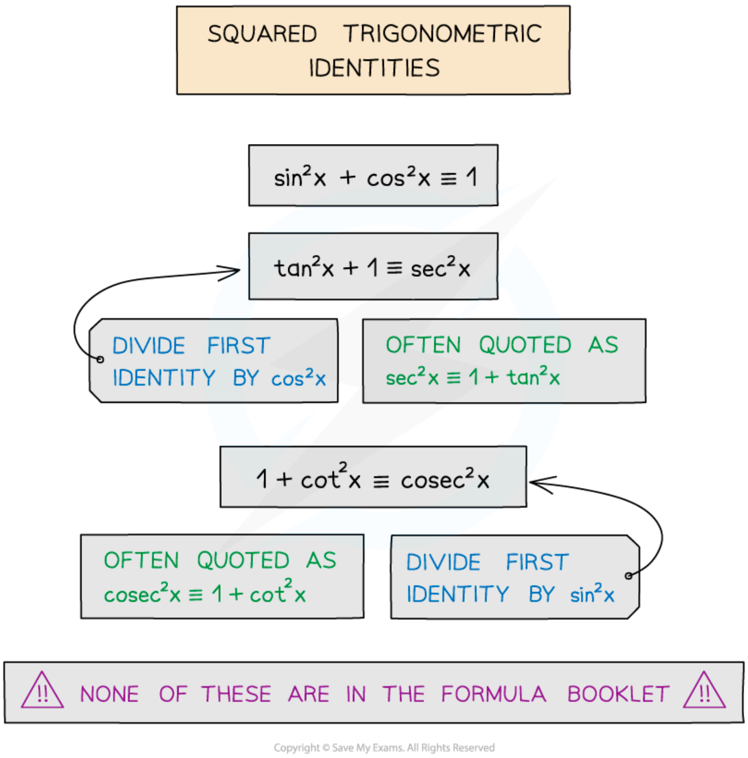
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$



## How do I know which trigonometric identities to use?

- There is no set method
- This is a matter of experience, familiarity and recognition
  - Practice as many questions as possible
  - Be familiar with trigonometric functions that can be integrated easily

- Be familiar with common identities – especially **squared** terms
- **$\sin^2 x$**, **$\cos^2 x$**, **$\tan^2 x$**, **$\mathrm{cosec}^2 x$**, **$\sec^2 x$**, **$\cot^2 x$** all appear in identities




Your notes

## How do I integrate $\tan^2 x$, $\cot^2 x$, $\sec^2 x$ and $\mathrm{cosec}^2 x$?

- The **integral** of **$\sec^2 x$** is **$\tan x$ (+c)**
  - This is because the **derivative** of $\tan x$ is $\sec^2 x$
- The **integral** of **$\mathrm{cosec}^2 x$** is **$-\cot x$ (+c)**
  - This is because the **derivative** of $\cot x$ is $-\mathrm{cosec}^2 x$
- The **integral** of **$\tan^2 x$** can be found by using the identity to rewrite $\tan^2 x$ before integrating:
  - $1 + \tan^2 x = \sec^2 x$
- The **integral** of **$\cot^2 x$** can be found by using the identity to rewrite $\cot^2 x$ before integrating:
  - $1 + \cot^2 x = \mathrm{cosec}^2 x$

## How do I integrate expressions involving sin x and cos x?

- For functions of the form **sin kx**, **cos kx** ... see Integrating Other Functions
- **sin kx × cos kx** can be integrated using the identity for **sin 2A**

- sin 2A = 2sinAcosA

e.g. $\int \sin x \cos x \, dx$

$\boxed{\sin 2A = 2\sin A\cos A}$

$= \int \frac{1}{2}\sin 2x \, dx$

$= -\frac{1}{2} \times \frac{1}{2}\int -2\sin 2x \, dx$

"ADJUST" AND "COMPENSATE"

$= -\frac{1}{4}\cos 2x + c$

AN EQUIVALENT ANSWER CAN BE FOUND USING THE FOLLOWING TECHNIQUE



- **$\sin^n$ kx cos kx** or **sin kx $\cos^n$ kx** can be integrated using **reverse chain rule** or **substitution**
- Notice no identity is used here but it looks as though there should be!

e.g. $\int \sin x \cos^3 x \, dx$

$= -\frac{1}{4}\int -4\sin x \cos^3 x \, dx$

REVERSE CHAIN RULE

"ADJUST" AND "COMPENSATE"

$-\sin x$ IS DERIVATIVE OF $\cos x$

$= -\frac{1}{4}\cos^4 x + c$

NOTICE NO IDENTITY USED IN THIS QUESTION



- $\sin^2$ kx and $\cos^2$ kx can be integrated by using the identity for cos 2A
  - For $\sin^2$A, cos 2A = 1 - 2$\sin^2$A
  - For $\cos^2$A, cos 2A = 2$\cos^2$A – 1

Your notes

e.g. $\int \sin^2 x \, dx$

$$\boxed{\begin{array}{l} \cos 2A = 1 - 2\sin^2 A \\ \text{SO} \quad \sin^2 A = \frac{1}{2}(1 - \cos 2A) \end{array}}$$

$$= \int \frac{1}{2}(1 - \cos 2x)\,dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \int \left[\frac{1}{2} - \frac{1}{2} \times \frac{1}{2}(2\cos 2x)\right] dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$



## How do I integrate tan x?

$$\boxed{\int \tan x \, dx = \ln|\sec x| + c}$$

$$\boxed{\int \tan kx \, dx \text{ IS IN THE FORMULA BOOKLET}}$$



- This is a standard result from the formulae booklet

## How do I integrate other expressions involving trig functions?

- The formulae booklet lists many standard trigonometric derivatives and integrals
  - Check both the “Differentiation” and “Integration” sections
  - For integration using the "Differentiation" formulae, remember that the integral of **f'(x)** is **f(x)** !

Your notes

Your notes

DIFFERENTIATION

| $f(x)$ | $f'(x)$ |
|---|---|
| $\tan kx$ | $k\sec^2 kx$ |
| $\sec kx$ | $k\sec kx\tan kx$ |
| $\cot kx$ | $-k\,\mathrm{cosec}^2 kx$ |
| $\mathrm{cosec}\,kx$ | $-k\,\mathrm{cosec}\,kx\cot kx$ |

INTEGRATION (+ CONSTANT)

| $f(x)$ | $\int f(x)dx$ |
|---|---|
| $\sec^2 kx$ | $\frac{1}{k}\tan kx$ |
| $\tan kx$ | $\frac{1}{k}\ln\lvert\sec kx\rvert$ |
| $\cot kx$ | $\frac{1}{k}\ln\lvert\sin kx\rvert$ |
| $\mathrm{cosec}\,kx$ | $-\frac{1}{k}\ln\lvert\mathrm{cosec}\,kx+\cot kx\rvert$, $\frac{1}{k}\ln\left\lvert\tan\left(\frac{1}{2}kx\right)\right\rvert$ |
| $\sec kx$ | $\frac{1}{k}\ln\lvert\sec kx+\tan kx\rvert$, $\frac{1}{k}\ln\left\lvert\tan\left(\frac{1}{2}kx+\frac{1}{4}\pi\right)\right\rvert$ |



- Experience, familiarity and recognition are important – practice, practice, practice!
- Problem-solving techniques

Your notes

e.g. $$\int\left(\frac{2\sin^2 x}{1+\cos 2x}+1\right)dx$$

• MIXTURE OF x's AND 2x's
• $\sin^2 x$ IS GOOD–COMMON IDENTITY
• $1+\cos 2x$ IS NOT A GOOD DENOMINATOR

$$=\int\left(\frac{2\sin^2 x}{1+(2\cos^2 x-1)}+1\right)dx$$

$$\boxed{\cos 2A \equiv 2\cos^2 A - 1}$$

$$=\int\left(\frac{2\sin^2 x}{2\cos^2 x}+1\right)dx$$

$$=\int(\tan^2 x+1)dx$$

$$=\int\sec^2 x\,dx$$

$$\boxed{\sec^2 x \equiv 1+\tan^2 x}$$

$$=\tan x + c$$

A STANDARD RESULT — BUT UNDER DIFFERENTIATION IN THE FORMULAE BOOKLET



### Examiner Tips and Tricks

- Make sure you have a copy of the formulae booklet during revision.
- Questions are likely to be split into (at least) two parts:
  - The first part may be to show or prove an identity
  - The second part may be the integration
- If you cannot do the first part, use a given result to attempt the second part.

### Worked Example

Your notes

(a) Show that $(1 + \cos 2x)\operatorname{cosec} 2x \equiv \cot x$

(b) Hence, or otherwise, find the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} (1 + \cos 2x)\operatorname{cosec} 2x \, dx \text{ in the form } a\ln b,$$

where $a$ and $b$ are constants to be found.

a) $\text{LHS} \equiv (1 + \cos 2x)\operatorname{cosec} 2x$

- NOTHING GAINED EXPANDING
- cosec IS $\frac{1}{\sin}$ AND cot IS A FRACTION

$$\text{LHS} \equiv \frac{1 + \cos 2x}{\sin 2x}$$

- 2x's AT MOMENT, NEED x's
- IDENTITIES FOR cos2x AND sin2x
- BUT WHICH cos2x?
- cot NEEDS cos ON TOP SO TRY
- $\cos 2x = 2\cos^2 x - 1$

$$\text{LHS} \equiv \frac{1 + (2\cos^2 x - 1)}{2\sin x \cos x}$$

$$\text{LHS} \equiv \frac{\cancel{2}\cos^{\cancel{2}} x}{\cancel{2}\sin x \cancel{\cos x}}$$

$$\text{LHS} \equiv \cot x \qquad \boxed{\cot x \equiv \frac{\cos x}{\sin x}}$$

Your notes

b) $\int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} (1 + \cos 2x)\operatorname{cosec} 2x \, dx$

$\equiv \int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \cot x \, dx \quad (*)$

IN FORMULAE BOOKLET

$= \left[\ln|\sin x|\right]_{\frac{\pi}{6}}^{\frac{3\pi}{4}}$

$= \left(\ln\left|\sin\frac{3\pi}{4}\right|\right) - \left(\ln\left|\sin\frac{\pi}{6}\right|\right)$

$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$= \ln\frac{1}{\sqrt{2}} - \ln\frac{1}{2}$

$= \ln 2^{-\frac{1}{2}} - \ln 2^{-1}$

$= -\frac{1}{2}\ln 2 + \ln 2$

$= \frac{1}{2}\ln 2$

(*) YOUR CALCULATOR WILL DO THIS BUT NOT GIVE YOU AN EXACT VALUE. YOU CAN SEE YOUR CALCULATOR TO CHECK BOTH DECIMAL ANSWERS.