Edexcel International A Level (IAL) Maths: Pure 3

Your notes

Further Differentiation

Contents

Differentiating other functions

How do I differentiate common functions?

- These are the common results
  - $\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln a$ for $a > 0$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\mathrm{cosec}^2 x$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosec}\ x) = -\mathrm{cosec}\ x \cot x$

How do I differentiate e^{kx}, a^x, ln(kx) and a^{kx}?

Your notes

1. IF $y = e^{kx}$, WHERE k IS A REAL CONSTANT, THEN $\frac{dy}{dx} = ke^{kx}$

2. IF $y = \ln x$ THEN $\frac{dy}{dx} = \frac{1}{x}$

3. IF $y = a^x$, WHERE a IS A REAL CONSTANT AND $a > 0$, THEN $\frac{dy}{dx} = (\ln a)a^x$

4. IF $y = \ln kx$, WHERE k IS A REAL CONSTANT, THEN $\frac{dy}{dx} = \frac{1}{x}$

i.e. IF $y = a^x$ THEN

WE CAN WRITE $a = e^{\ln a}$

REMEMBER THIS TRICK! e AND $\ln$ ARE INVERSES, SO THEY CANCEL EACH OTHER

THEN $a^x = (e^{\ln a})^x = e^{(\ln a)x}$

LAWS OF INDICES $(a^m)^n = a^{mn}$

SO $y = e^{(\ln a)x}$ AND

$\frac{dy}{dx} = (\ln a)e^{(\ln a)x} = (\ln a)a^x$

IF $y = e^{kx}$, $\frac{dy}{dx} = ke^{kx}$

$\ln a$ IS JUST A CONSTANT!

IF $y = \ln kx$ THEN

$y = \ln k + \ln x$

LAWS OF LOGARITHMS $\ln(ab) = \ln a + \ln b$

SO $\frac{dy}{dx} = \frac{1}{x}$

$\ln k$ IS A CONSTANT, SO ITS DERIVATIVE IS ZERO

Your notes

- And for a^{kx}:

5. IF $y = a^{kx}$, WHERE a AND k ARE REAL CONSTANT AND $a>0$,

THEN $\frac{dy}{dx} = k(\ln a)a^{kx}$

- This last formula can be derived from Formula 3 by using the **chain rule**

Examiner Tips and Tricks

- The formulae for some of these derivatives are not given in the formulae booklet – you need to know them.
- The formulae for $\mathbf{e}^{kx}$ and **ln** $\boldsymbol{x}$ are the ones you absolutely do need to know .
- The other formulae can be derived from those two as shown above, and remember – the derivative of **ln** $\boldsymbol{kx}$ is $\frac{1}{x}$, NOT $\frac{k}{x}$!

Worked Example

Your notes

A curve has the equation $y = e^{-3x} + 3^{\frac{x}{2}} - \ln 5x$.

Find the gradient of the curve at the point with x-coordinate 2.

$$\frac{dy}{dx} = -3e^{-3x} + \frac{1}{2}(\ln 3)3^{\frac{x}{2}} - \frac{1}{x}$$

FORMULA 1

FORMULA 5

FORMULA 4

WHEN $x = 2$,

$$\frac{dy}{dx} = -3e^{-6} + \frac{1}{2}(\ln 3)3^{1} - \frac{1}{2}$$

$$= \frac{1}{2}(3\ln 3 - 1) - 3e^{-6} \qquad (\approx 1.14)$$

Your notes

Chain rule

What is the chain rule?

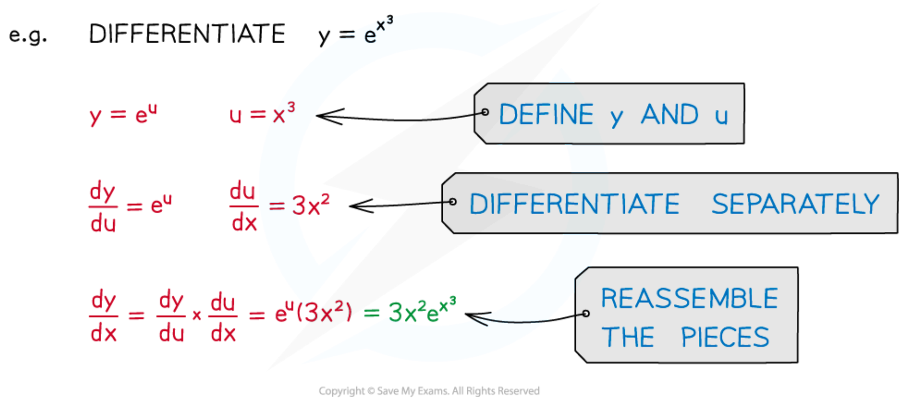
- The chain rule is a formula that allows you to differentiate **composite functions**
- If ***y*** is a function of ***u***, and ***u*** is a function of ***x***, then the **chain rule** tells us that:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

- Note that because ***u*** is a function of ***x***, ***y*** is also a function of ***x***
- In **function notation**, if $y = \mathrm{f}(\mathrm{g}(x))$ then the chain rule can be written as:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(g(x))\mathrm{g}'(x)$$

This allows us to differentiate more complicated expressions:



e.g. DIFFERENTIATE $y = \ln(7x^2)$

$f(x) = \ln x \qquad g(x) = 7x^2$ DEFINE $f(x)$ AND $g(x)$

$f'(x) = \frac{1}{x} \qquad g'(x) = 14x$ DIFFERENTIATE SEPARATELY

$f'(g(x)) = \frac{1}{7x^2}$

$\frac{dy}{dx} = f'(g(x))g'(x) = \left(\frac{1}{7x^2}\right)(14x) = \frac{2}{x}$ REASSEMBLE THE PIECES

What are the special cases of the chain rule?

- It allows us to differentiate a function raised to a power:
  - $$\frac{\mathrm{d}}{\mathrm{d}x}((\mathrm{f}(x)^n) = n(\mathrm{f}(x))^{n-1}\mathrm{f}'(x)$$

e.g. DIFFERENTIATE $y = (x^2 - 5x + 7)^7$ $n = 7$

$f(x) = x^2 - 5x + 7$

$f'(x) = 2x - 5$

$\frac{dy}{dx} = n(f(x))^{n-1}f'(x) = 7(x^2 - 5x + 7)^6(2x - 5)$

$\frac{dy}{dx} = 7(2x - 5)(x^2 - 5x + 7)^6$

- It allows us to differentiate functions that are not given in the form **y= f(x)**:
  - $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$$

e.g. FIND THE VALUE OF $\frac{dy}{dx}$ AT THE POINT (3,1) ON THE CURVE WITH EQUATION $y^3 + 2y^2 = x$

$$\frac{dx}{dy} = 3y^2 + 4y$$

DIFFERENTIATE x WITH RESPECT TO y

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{3y^2 + 4y}$$

INVERT TO FIND $\frac{dy}{dx}$

AT (3,1) $y = 1$ AND

$$\frac{dy}{dx} = \frac{1}{3(1)^2 + 4(1)} = \frac{1}{7}$$

- There is a useful case which is used to integrate certain functions:
  - $\frac{d}{dx}(\ln(f(x)) = \frac{f'(x)}{f(x)}$

Examiner Tips and Tricks

- The chain rule formulae are not in the exam formulae booklet – you have to know them.
- When using the chain rule be sure to keep your functions straight (ie which function is ***y*** and which is ***u***, or which is ***f*** and which is ***g***).

Worked Example

Your notes

For the curve with equation $y = 5^{x^4}$, find the gradient of the curve at the point $(1, 5)$.

Give your answer as an exact value.

$y = 5^u \qquad u = x^4$ ← DEFINE y AND u

$\frac{dy}{du} = (\ln 5)5^u \qquad \frac{du}{dx} = 4x^3$ ← DIFFERENTIATE SEPARATELY

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (\ln 5)5^u(4x^3)$

$\frac{dy}{dx} = (4\ln 5)x^3\left(5^{x^4}\right)$ ← REASSEMBLE

WHEN $x = 1$,

$\frac{dy}{dx} = (4\ln 5)(1)^3\left(5^{1^4}\right) = 20\ln 5$

THIS COULD BE DONE USING THE FUNCTION NOTATION VERSION OF THE CHAIN RULE, WITH $f(x) = 5^x$ AND $g(x) = x^4$

Product rule

What is the product rule?

- The product rule is a formula that allows you to differentiate a **product of two functions**
- If $y = u \times v$ where u and v are functions of x then the product rule is:

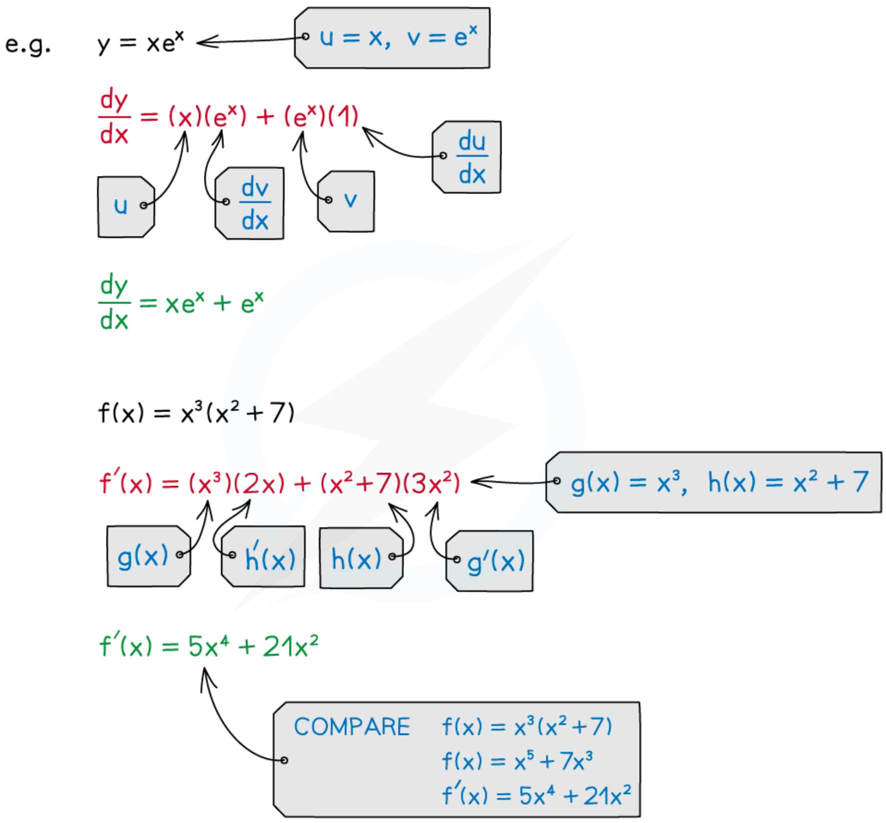
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

- In **function notation**, if $f(x) = g(x) \times h(x)$ then the product rule can be written as:

$$f'(x) = g(x)h'(x) + h(x)g'(x)$$

- The easiest way to remember the product rule is, for $y = u \times v$ where u and v are functions of x:

$$y' = uv' + vu'$$



Your notes

Examiner Tips and Tricks

- The product rule formulae are NOT in the formulae booklet – you need to know them.
- Don't confuse the **product** of two functions with a **composite function:**
  - The **product** of two functions is two functions **multiplied** together
  - A **composite** function is a **function of a function**

e.g. $f(x) = 3x^2$ $\quad g(x) = x - 3$

$f(x)g(x) = 3x^2(x-3) = 3x^3 - 9x^2$ PRODUCT

$fg(x) = f(g(x)) = 3(x-3)^2 = 3x^2 - 18x + 27$ COMPOSITE FUNCTION

- To differentiate composite functions you need to use the **chain rule**

Worked Example

Your notes

Differentiate with respect to x:

a) $(5\sin x - 7\cos x)e^{3x+2}$

b) $5x\ln 5x$

a) $u = 5\sin x - 7\cos x$ $\quad v = e^{3x+2}$

$\frac{du}{dx} = 5\cos x + 7\sin x$ $\quad \frac{dv}{dx} = 3e^{3x+2}$

DON'T FORGET MINUS SIGN WHEN YOU DIFFERENTIATE cos

USE CHAIN RULE

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = (5\sin x - 7\cos x)(3e^{3x+2}) + (e^{3x+2})(5\cos x + 7\sin x)$$

$$\frac{dy}{dx} = (22\sin x - 16\cos x)e^{3x+2}$$

b) $u = 5x$ $\quad v = \ln 5x$

$\frac{du}{dx} = 5$ $\quad \frac{dv}{dx} = \frac{1}{x}$

BE CAREFUL WITH THIS ONE! BECAUSE OF THE CHAIN RULE THE DERIVATIVE OF ln(ax) IS $\frac{1}{x}$ FOR **ANY** CONSTANT a

$$\frac{dy}{dx} = (5x)\left(\frac{1}{x}\right) + (\ln 5x)(5)$$

$$\frac{dy}{dx} = 5 + 5\ln 5x$$

Quotient rule

What is the quotient rule?

- The quotient rule is a formula that allows you to differentiate a **quotient of two functions** (ie one function divided by another)
- If $y = \frac{u}{v}$ where u and v are functions of x then the quotient rule is:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

- In **function notation**, if $f(x) = \frac{g(x)}{h(x)}$ then the quotient rule can be written as:

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$$

e.g. $y = \dfrac{x^2+1}{x-3}$ $\quad u = x^2+1, \quad v = x-3$

$$\frac{dy}{dx} = \frac{\overset{v}{(x-3)}\overset{\frac{du}{dx}}{(2x)} - \overset{u}{(x^2+1)}\overset{\frac{dv}{dx}}{(1)}}{\underset{v^2}{(x-3)^2}}$$

$$\frac{dy}{dx} = \frac{x^2-6x-1}{(x-3)^2}$$

$f(x) = \dfrac{\sin x}{\cos x}$ $\quad g(x) = \sin x, \quad h(x) = \cos x$

$$f'(x) = \frac{\overset{h(x)}{(\cos x)}\overset{g'(x)}{(\cos x)} - \overset{g(x)}{(\sin x)}\overset{h'(x)}{(-\sin x)}}{\underset{(h(x))^2}{(\cos x)^2}}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$f'(x) = \sec^2 x$ $\quad \dfrac{1}{\cos x} = \sec x$

$\dfrac{\sin x}{\cos x} = \tan x$ SO THIS IS THE DERIVATIVE OF $\tan x$

Examiner Tips and Tricks

- The quotient rule formula is in the formulae booklet – you don't have to memorise it.
- Be careful using the formula – because of the minus sign in the numerator the order of the functions is important.
- Look out for functions of the form $\mathrm{f}(x) = \mathrm{g}(x)(\mathrm{h}(x))^{-1}$
  - You could differentiate that using a combination of the **chain rule** and the **product rule** (and it can be good practice for you to try it!)
  - But it can also be seen as just a quotient rule question in disguise

Your notes

- $g(x)(h(x))^{-1} = \frac{g(x)}{h(x)}$

Your notes

Worked Example

Differentiate $f(x) = (\cos 2x)\left(x^2 + 2x + 1\right)^{-1}$

with respect to x.

$f(x) = \frac{\cos 2x}{x^2 + 2x + 1}$ ← REWRITE AS $f(x) = \frac{g(x)}{h(x)}$

$g(x) = \cos 2x$ $\quad$ $h(x) = x^2 + 2x + 1$

$g'(x) = -2\sin 2x$ $\quad$ $h'(x) = 2x + 2$

CHAIN RULE – AND REMEMBER MINUS SIGN WHEN DIFFERENTIATING cos

$f'(x) = \frac{(x^2 + 2x + 1)(-2\sin 2x) - (\cos 2x)(2x + 2)}{(x^2 + 2x + 1)^2}$

$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$

$f'(x) = \frac{-2\sin 2x(x^2 + 2x + 1) - 2\cos 2x(x + 1)}{(x^2 + 2x + 1)^2}$

$= \frac{-2\sin 2x(x + 1)^2 - 2\cos 2x(x + 1)}{(x + 1)^4}$ ← $x^2 + 2x + 1 = (x + 1)^2$

$f'(x) = \frac{-2(\cos 2x + (x + 1)\sin 2x)}{(x + 1)^3}$ ← SIMPLIFY IT

Differentiating reciprocal and inverse trig functions

How do I differentiate reciprocal trigonometric functions?

- The formulae for the derivatives of the reciprocal trigonometric functions are:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\operatorname{cosec}^2 x$$

- You can derive the derivatives for **sec**, **cosec**, and **cot** using the **chain rule** and the derivatives of the basic trigonometric functions

e.g. DIFFERENTIATE $y = \sec x$

$$y = \sec x = \frac{1}{\cos x} = (\cos x)^{-1}$$ USE DEFINITION OF sec

$$\frac{dy}{dx} = \underbrace{-(\cos x)^{-2}}_{n(f(x))^{n-1}}\underbrace{(-\sin x)}_{f'(x)}$$ USE CHAIN RULE FOR $(f(x))^n$

$$\frac{dy}{dx} = \frac{\sin x}{\cos^2 x} = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)$$

$$\frac{dy}{dx} = \sec x \tan x$$ $\tan x = \frac{\sin x}{\cos x}$

How do I differentiate inverse trigonometric functions?

- The formulae for the derivatives of the inverse trigonometric functions are:

Your notes

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

- You can derive the derivatives of **arcsin**, **arccos**, and **arctan** using the reciprocal form of the **chain rule** and the derivatives of the basic trigonometric functions

e.g. DIFFERENTIATE $y = \arccos x$

IF $y = \arccos x$
THEN $x = \cos y$ ← arccos AND cos ARE INVERSES

$\frac{dx}{dy} = -\sin y$ ← DIFFERENTIATE WITH RESPECT TO y

BUT $\sin^2 y + \cos^2 y = 1$ ← USE TRIG IDENTITY
SO $\sin y = \sqrt{1-\cos^2 y}$

AND $\cos y = x$
SO $\sin y = \sqrt{1-x^2}$

$\frac{dx}{dy} = -\sin y = -\sqrt{1-x^2}$

AND $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ ← USE RECIPROCAL FORM OF CHAIN RULE

SO $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

Examiner Tips and Tricks

- The formulae for the derivatives of **sec *x***, **cosec *x***, and **cot *x*** are in the formulae booklet – you don't need to memorise them

- However, you should know how to derive the derivatives of **sec**, **cosec**, and **cot** using the chain rule

- The formulae for the derivatives of **arcsin**, **arccos**, and **arctan** are not in the formulae booklet
    - You should know how to derive those derivatives using the reciprocal form of the chain rule

Your notes

Worked Example

Your notes

a) Given that $y = 5\text{cosec}^2 x$ find $\dfrac{dy}{dx}$

b) Show that the derivative of $\arctan x$ is $\dfrac{1}{1+x^2}$

a) $y = 5(\text{cosec}\, x)^2$ ← $5(f(x))^2$ USE CHAIN RULE

$\dfrac{dy}{dx} = 10\text{cosec}\, x\,(-\text{cosec}\, x \cot x)$ ← $5 \times 2(f(x))^{2-1} f'(x)$

$\dfrac{dy}{dx} = -10\text{cosec}^2 x \cot x$

b) LET $y = \arctan x$

THEN $x = \tan y$ ← tan AND arctan ARE INVERSES

SO $\dfrac{dx}{dy} = \sec^2 y$ ← DIFFERENTIATE WITH RESPECT TO y

$\dfrac{dx}{dy} = 1 + \tan^2 y$ ← USE TRIG IDENTITY

$\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{1+\tan^2 y}$ ← RECIPROCAL FORM OF CHAIN RULE

BUT $x = \tan y$ SO

$\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$