Edexcel International AS Maths: Pure 2

Your notes

Laws of Logarithms

Contents

Your notes

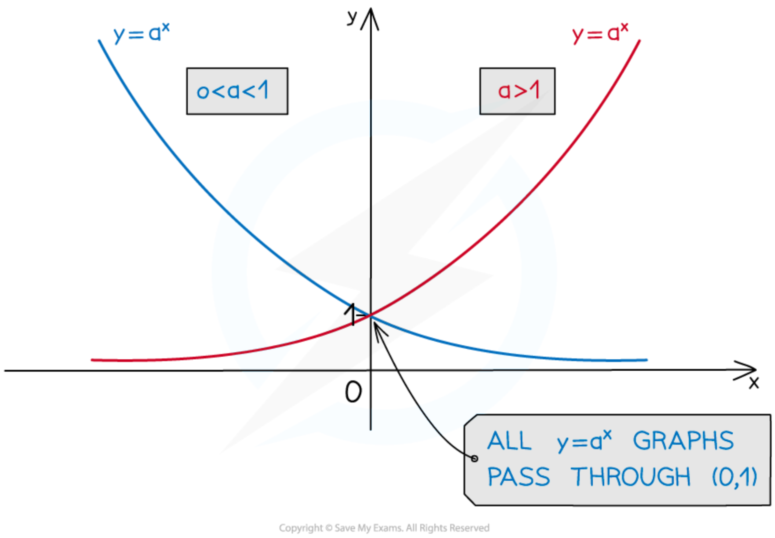
Exponential functions

What are exponential functions?

- Exponential functions of the form **$y = a^x$** with **$a > 0$** are considered at A level

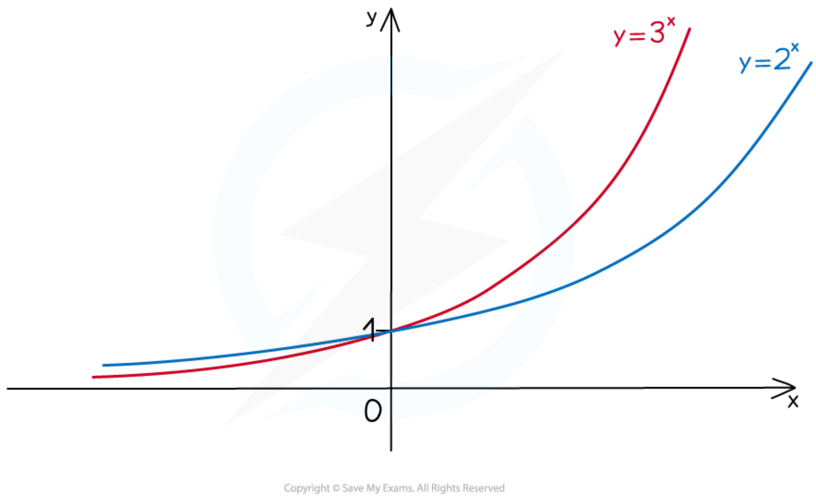
What are exponential graphs?

- All graphs of the form **$y = a^x$** will pass through (0, 1) because **$a^0 = 1$**
- The **x**-axis is an **asymptote**



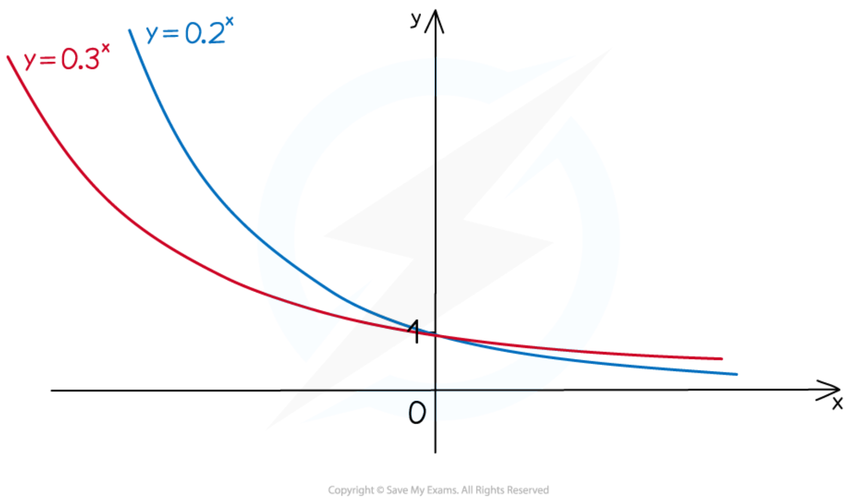
What is exponential growth?

- Where **$x < 0$** the higher value of **a** is the "lower" graph
  - e.g. $3^x < 2^x$ when $x < 0$
- Where **$x > 0$** the higher value of **a** is the "higher" graph
  - e.g. $3^x > 2^x$ when $x > 0$
- **$a > 1$** is **exponential growth** (see Exponential Growth & Decay)



What is exponential decay?

- Where **x < 0** the higher value of **a** is the "lower" graph
  - e.g. $0.3^x < 0.2^x$ when $x < 0$
- Where **x > 0** the higher value of **a** is the "higher" graph
  - e.g. $0.3^x > 0.2^x$ when $x > 0$
- **0 < a < 1** is **exponential decay**



- You may like to think about why **a = 1** is **not** considered... If a = 1, y = 1^x = 1 for all values of x

Worked Example

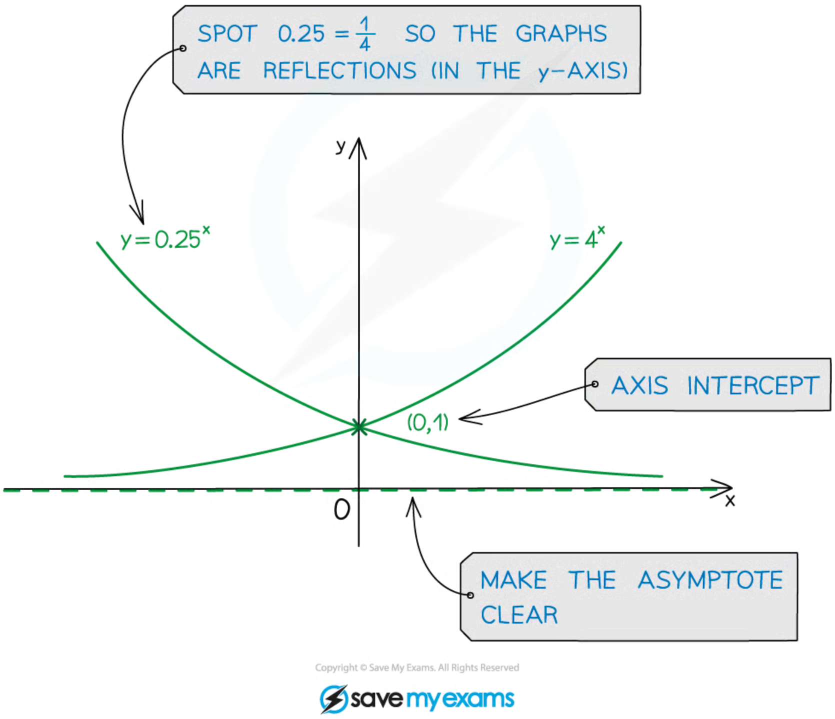
Your notes

Your notes

On the same diagram sketch the graphs of

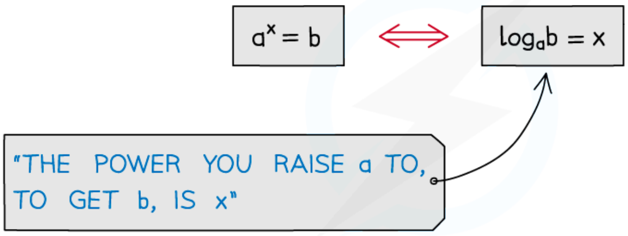
$y = 4^x$ and $y = 0.25^x$.

Mark any points where the graphs cross the coordinate axes and make any asymptotes clear.



Logarithmic functions

What are logarithmic functions?



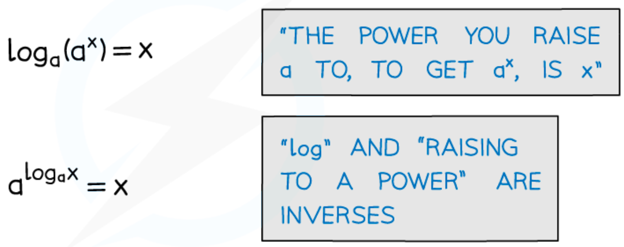
- **$a = b^x$** and **$\log_b a = x$** are equivalent statements
- **$a > 0$**
- **b** is called the base
- Every time you write a logarithm statement say to yourself what it means
  - **$\log_3 81 = 4$**

    "the power you raise 3 to, to get 81, is 4"
  - **$\log_p q = r$**

    "the power you raise **p** to, to get **q**, is **r**"

How are logarithms and powers inverses of each other?

- As a logarithm is the inverse of raising to a power



How do I use the inverse property of logs and powers?

e.g. WRITE $\dfrac{2^{\log_3 81} + 3^{\log_{10} 1000}}{5^{\log_5 5}}$ IN THE FORM $\frac{a}{b}$

WHERE a AND b ARE INTEGERS TO BE FOUND.

$2^{\log_3 81} = 2^4 = 16$ "THE POWER YOU RAISE 3 TO, TO GET 81, IS 4"

$3^{\log_{10} 1000} = 3^3 = 27$ "THE POWER YOU RAISE 10 TO, TO GET 1000, IS 3"

$5^{\log_5 5} = 5^1 = 5$ $\log_5 5 = \log_5 5^1 = 1$

$$\therefore \frac{2^{\log_3 81} + 3^{\log_{10} 1000}}{5^{\log_5 5}} = \frac{16 + 27}{5}$$

$$= \frac{43}{5}$$

- Recognising the rules of logarithms allows expressions to be simplified

Your notes

e.g. WITHOUT USING A CALCULATOR, EVALUATE $\log_2 64\sqrt{2}$

RECOGNISE 64 AND $\sqrt{2}$ ARE POWERS OF 2

LET $x = \log_2 64\sqrt{2}$

"THE POWER YOU RAISE 2 TO, TO GET $64\sqrt{2}$, IS x"

$2^x = 64\sqrt{2}$

REWRITE THE STATEMENT

$2^x = 2^6 \times 2^{\frac{1}{2}}$

$2^x = 2^{6+\frac{1}{2}}$

$2^x = 2^{\frac{13}{2}}$

USING LAWS OF INDICES

$x = \frac{13}{2}$

$\therefore \log_2 64\sqrt{2} = \frac{13}{2}$

- Recognition of common powers helps in simple cases
  - Powers of 2: $2^0 = 1, 2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, \ldots$
  - Powers of 3: $3^0 = 1, 3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81, \ldots$
  - The first few powers of 4, 5 and 10 should also be familiar
- For more awkward cases a calculator is needed

e.g. USE A CALCULATOR TO EVALUATE $\log_{10}32 + \log_2 12 - \log_3 19$ GIVING YOUR ANSWER TO 3 SIGNIFICANT FIGURES

log OR $\log_{\square}\square$ WITH BASE 10

$\log_{\square}\square$ WITH BASE 2

$\log_{\square}\square$ WITH BASE 3

$\log_{10}32 + \log_2 12 - \log_3 19 = 2.409968\ldots$

$= 2.41$ (3 sf)

- Calculators can have, possibly, different logarithm buttons

Your notes

- This button allows you to type in any number for the base

- Shortcut for base 10 although SHIFT button needed
- Before calculators, logarithmic values had to be looked up in printed tables

What notation is used with logarithms?

- 10 is a common base
  - $\log_{10} x$ is abbreviated to **log x** or **lg x**
- $(\log x)^2 \neq \log x^2$

Worked Example

Your notes

Solve the equation $2^{2x+3} - 41\left(2^x\right) + 5 = 0$,

giving your answers correct to 3 significant figures.

$2^{2x+3} - 41(2^x) + 5 = 0$

HIDDEN QUADRATIC

$2^3 \times 2^{2x} - 41(2^x) + 5 = 0$

$8(2^x)^2 - 41(2^x) + 5 = 0$

REWRITE

LET $y = 2^x$: $8y^2 - 41y + 5 = 0$

SUBSTITUTION NOT ESSENTIAL

$(8y-1)(y-5) = 0$

USE CALCULATOR TO SOLVE

$y = \frac{1}{8}$ $\quad y = 5$

$2^x = \frac{1}{8}$ $\quad 2^x = 5$

SPOT THIS ?!

$x = \log_2 \frac{1}{8}$ $\quad x = \log_2 5$

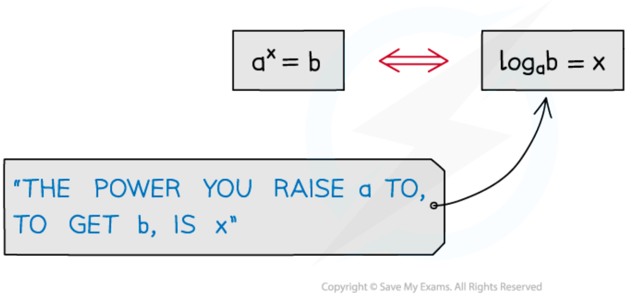
USE LOGARITHMS

$x = -3$ $\quad x = 2.32192...$

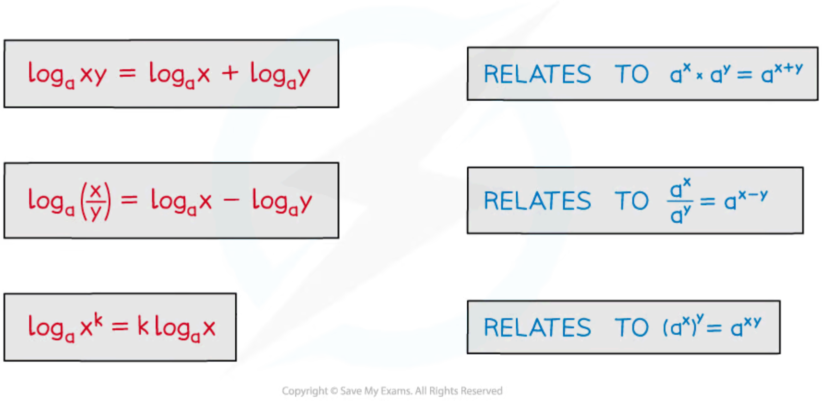
$x = 2.32$ (3sf)

Laws of logarithms

What are the laws of logarithms?

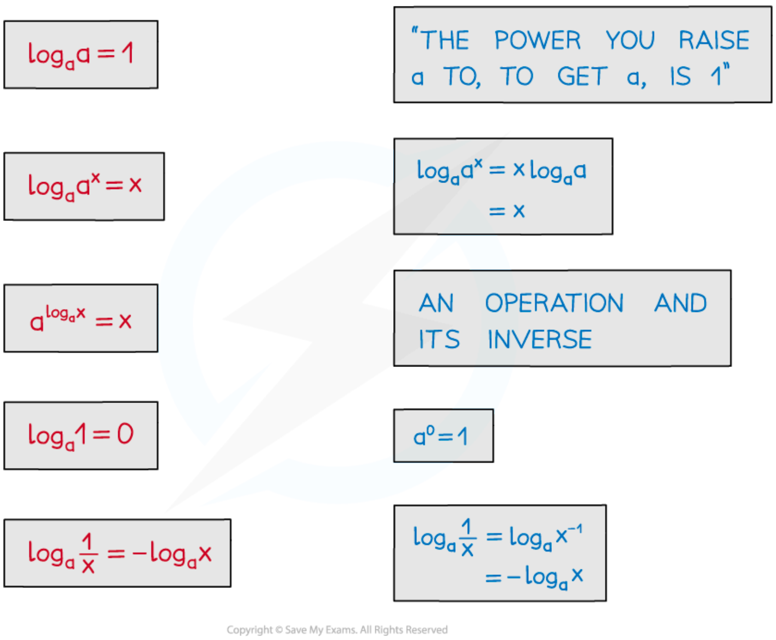


- There are many laws or rules of indices, for example
  - $a^m \times a^n = a^{m+n}$
  - $(a^m)^n = a^{mn}$
- There are equivalent laws of logarithms (for a > 0)
  - $\log_a xy = \log_a x + \log_a y$
  - $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
  - $\log_a x^k = k \log_a x$



- There are also some particular results these lead to
  - $\log_a a = 1$

- $\log_a a^x = x$
- $a^{\log_a x} = x$
- $\log_a 1 = 0$
- $\log_a \left(\frac{1}{x}\right) = -\log_a x$



- Two of these were seen in the notes Logarithmic Functions
- Beware ...
  - ... **log (x + y) ≠ log x + log y**

How do I use the laws of logarithms?

- Laws of logarithms can be used to ...
  - ... simplify expressions
  - ... solve logarithmic equations
  - ... solve exponential equations

Your notes

Your notes

e.g. WRITE $3\log_2(2x+3) + \log_2 5 - 2\log_2(x+1)$

AS A SINGLE LOGARITHM

$$3\log_2(2x+3) + \log_2 5 - 2\log_2(x+1) = \log_2(2x+3)^3 + \log_2 5 - \log_2(x+1)^2$$

$\boxed{k\log_a x = \log_a x^k}$

$$= \log_2 5(2x+3)^3 - \log_2(x+1)^2$$

$\boxed{\log_a x + \log_a y = \log_a(xy)}$

$$= \log_2 \frac{5(2x+3)^3}{(x+1)^2}$$

$\boxed{\log_a x - \log_a y = \log_a \frac{x}{y}}$

Examiner Tips and Tricks

- Remember to check whether your solutions are valid. log (x+k) is only defined if x > -k. You will lose marks if you forget to reject invalid solutions.

Worked Example

Solve the equation $\log_2(x+12) = 3\log_2 x - 2$

$\log_2(x+12) = 3\log_2 x - 2$

$\log_2(x+12) = \log_2 x^3 - 2$ ← $k\log_a x = \log_a x^k$

$\log_2 x^3 - \log_2(x+12) = 2$ ← LOGARITHMS ON SAME SIDE

$\log_2\left(\frac{x^3}{x+12}\right) = 2$ ← SINGLE LOGARITHM

$\frac{x^3}{x+12} = 2^2$ ← REWRITE

$x^3 = 4(x+12)$

$x^3 - 4x - 48 = 0$

REARRANGE AND SOLVE USING CALCULATOR

$x = 4$

NOTE: OTHER TWO SOLUTIONS CALCULATOR GIVES ARE IMAGINARY

Change of base

What is the change of base formula?

- We can rewrite a **logarithm** as a **multiple** of a logarithm of **any different positive base** using the formula

$$\log_a x = \frac{\log_b x}{\log_b a}$$

How do I use the change of base formula?

- This formula had more use when calculators were less advanced
  - Some old calculators only had a button for logarithm of base 10
  - To calculate $\log_5 7$ on these calculators you would have to enter

Your notes

- $\dfrac{\log_{10}7}{\log_{10}5}$

- The formula is only needed in a small number of cases
  - This is given in the **formulae booklet** in case it is needed
- The formula can be useful when evaluation a logarithm where the two numbers are **powers of a common number**
  - $\log_4 8 = \dfrac{\log_2 8}{\log_2 4} = \dfrac{3}{2}$
- The formula can be useful when you are solving equations and two logarithms have **different bases**
  - For example, if you have $\log_3 k$ and $\log_9 n$ within the same equation
    - You can rewrite $\log_9 n$ as $\dfrac{\log_3 n}{\log_3 9}$ which simplifies to $\dfrac{1}{2}\log_3 n$
    - Or you can rewrite $\log_3 k$ as $\dfrac{\log_9 k}{\log_9 3}$ which simplifies to $2\log_9 k$
- The formula also allows you to derive and use a formula for switching the numbers:

$$\log_a x = \frac{1}{\log_x a}$$

- Using the fact that $\log_x x = 1$

Examiner Tips and Tricks

- It is very rare that you will need to use the change of base formula
- Only use it when the bases of the logarithms are different

Your notes

Exponential equations

What are exponential equations?

- An equation where the unknown is a power
- In simple cases the solutions can be "spotted"

e.g. $5^{2x} = 125$ ← SPOT THAT 125 IS A POWER OF 5

$2x = 3$

$x = \frac{3}{2}$

- See Exponential Functions

How do I solve exponential equations in the form $a^{f(x)} = b$?

- If the value of *b* can be written as a power of *a* ($b = a^k$)
  - Write $a^{f(x)} = a^k$
  - Solve $f(x) = k$
- If the value of *b* can not be written as a power of *a*
  - Apply logs of base *a* to both sides to get:
    - $f(x) = \log_a b$
  - Solve for *x*

How do I solve exponential equations in the form $a^{f(x)} = b^{g(x)}$?

- If the value of *b* can be written as a power of *a* ($b = a^k$)
  - Use the index law to rewrite $b^{g(x)}$
    - $b^{g(x)} = (a^k)^{g(x)} = a^{kg(x)}$

- Write $a^{f(x)} = a^{kg(x)}$
- Solve $f(x) = kg(x)$

- If the value of *b* can not be written as a power of *a*
  - Apply logs of the same base (any base will work) to both sides to get:
    - $\log(a^{f(x)}) = \log(b^{g(x)})$
  - Use the laws of logarithms to bring the power to the front:
    - $f(x)\log a = g(x)\log b$
  - log *a* and log *b* are just numbers so rearrange and solve for *x*
- If either side is multiplied by a constant ($pa^{f(x)}$)
  - Do **not** write as $(pa)^{f(x)}$ – this is **incorrect**
  - Still take logs of both sides but you will need to use another law of logs
    - $\log(pa^{f(x)}) = \log p + \log pa^{f(x)}$
  - log *p* is just a constant so can be solved in the same way as above

How do I solve exponential equations with three terms?

- If the equation has three terms such as $a^{f(x)} + a^{g(x)} + c = 0$
  - Use the index laws in reverse to split up the powers
    - $a^{Ax+B} = a^{Ax} \times a^{B}$
  - Try and use a substitution to transform the equation into a quadratic
    - See Further Solving Quadratics (Hidden Quadratics)
    - Look out for terms where one power is double another power
  - Solve the quadratic
  - For each solution find the corresponding value(s) of *x*

Your notes

Worked Example

Your notes

Solve $7 \times 2^x = 5^{4x-3}$

Write your answer in the form $\frac{a \log_{10} 5 + \log_{10} p}{b \log_{10} 5 - \log_{10} q}$, where a and b are integers and p and q are prime numbers.

TAKE LOGS OF BOTH SIDES

$\log_{10}(7 \times 2^x) = \log_{10}(5^{4x-3})$

APPLY LAWS $\log(A \times B) = \log A + \log B$ AND $\log(A^k) = k\log A$

$\log_{10} 7 + x\log_{10} 2 = (4x - 3)\log_{10} 5$ EXPAND

$\log_{10} 7 + x\log_{10} 2 = 4x\log_{10} 5 - 3\log_{10} 5$

ISOLATE THE x TERMS ON ONE SIDE

$\log_{10} 7 + 3\log_{10} 5 = 4x\log_{10} 5 - x\log_{10} 2$

REARRANGE FOR x

$3\log_{10} 5 + \log_{10} 7 = (4\log_{10} 5 - \log_{10} 2)x$

$$x = \frac{3\log_{10} 5 + \log_{10} 7}{4\log_{10} 5 - \log_{10} 2}$$

Examiner Tips and Tricks

- Pay attention to how the question asks you to write your answer
  - it could ask for exact form, a specific form or rounded to a specified degree of accuracy