Edexcel International AS Maths: Pure 2

Your notes

Sequences & Series

Contents

Your notes

Language of sequences & series

What is a sequence?

- A **sequence** is an ordered set of numbers with a rule for finding all the numbers in the sequence

e.g. 1, 4, 7, 10, 13, ... IS A SEQUENCE

THE RULE COULD BE GIVEN AS:
"START AT 1. EVERY NUMBER AFTER THAT IS THREE MORE THAN THE NUMBER BEFORE IT"

- The numbers in a sequence are called **terms**
- The terms of a sequence are often referred to by letters with a subscript

SO IN THE SEQUENCE ABOVE

$u_1 = 1$ $u_2 = 4$ $u_3 = 7$ $u_4 = 10$ $u_5 = 13$

THE GENERAL TERM OF THE SEQUENCE WOULD BE

$u_n = 3n - 2$

What is a series?

- You get a **series** by summing up the terms in a sequence

e.g. FOR THE SEQUENCE 1, 4, 7, 10, 13, ...

THE ASSOCIATED **SERIES** IS $1 + 4 + 7 + 10 + 13 + \ldots$

Your notes

- We use the notation S_n to refer to the sum of the first n terms in the series

  i.e. $S_n = u_1 + u_2 + u_3 + \ldots + u_n$

SO FOR THE SERIES ABOVE

$$S_1 = 1$$
$$S_2 = 1+4 = 5$$
$$S_3 = 1+4+7 = 12$$
$$S_4 = 1+4+7+10 = 22$$
$$S_5 = 1+4+7+10+13 = 35$$

What are increasing, decreasing and periodic sequences?

- A sequence is **increasing** if $u_{n+1} > u_n$ for all positive integers n
  - i.e. if every term is greater than the term before it
- A sequence is **decreasing** if $u_{n+1} < u_n$ for all positive integers n
  - i.e. if every term is less than the term before it

e.g.

3, 7, 11, 15, 19, ... IS AN INCREASING SEQUENCE

64, 32, 16, 8, 4, ... IS A DECREASING SEQUENCE

1, −2, 3, −4, 5, ... IS NEITHER AN INCREASING NOR A DECREASING SEQUENCE

- A sequence is **periodic** if the terms repeat in a cycle
- The **order** (or **period**) of a periodic sequence is the number of terms in each repeating cycle

e.g.

1, 2, 3, 1, 2, 3, 1, 2, 3, ... IS A PERIODIC SEQUENCE OF ORDER 3

2, −2, 2, −2, 2, −2, ... IS A PERIODIC SEQUENCE OF ORDER 2

0, 0, 0, 0, 0, 0, 0, ... IS A PERIODIC SEQUENCE OF ORDER 1

Your notes

Examiner Tips and Tricks

Look out for sequences defined by trigonometric functions – this can be a way of 'hiding' a periodic function.

$u_n = \cos(180n°)$

$u_1 = \cos(180°) = -1$

$u_2 = \cos(360°) = 1$

$u_3 = \cos(540°) = -1$

$u_4 = \cos(720°) = 1$

THIS IS THE SEQUENCE $-1, 1, -1, 1, -1, 1, \ldots$
(i.e. A PERIODIC SEQUENCE OF ORDER 2)

Worked Example

Your notes

Determine the order of the following periodic functions

a) $u_n = 3 + (-1)^n$

b) $u_n = \sin(90n°)$

WORK OUT THE TERMS AND LOOK FOR A PATTERN

a) $u_1 = 3+(-1)^1 = 3-1 = 2$

$u_2 = 3+(-1)^2 = 3+1 = 4$

$u_3 = 3+(-1)^3 = 3-1 = 2$

$u_4 = 3+(-1)^4 = 3+1 = 4$

THE SEQUENCE IS 2,4,2,4,2,4, ...

PERIODIC WITH ORDER 2

b) $u_1 = \sin 90° = 1$ $\quad u_2 = \sin 180° = 0$

$u_3 = \sin 270° = -1$ $\quad u_4 = \sin 360° = 0$

$u_5 = \sin 450° = 1$ $\quad u_6 = \sin 540° = 0$

$u_7 = \sin 630° = -1$ $\quad u_8 = \sin 720° = 0$

SEQUENCE IS 1, 0, −1, 0, 1, 0, −1, 0, ...

PERIODIC WITH ORDER 4

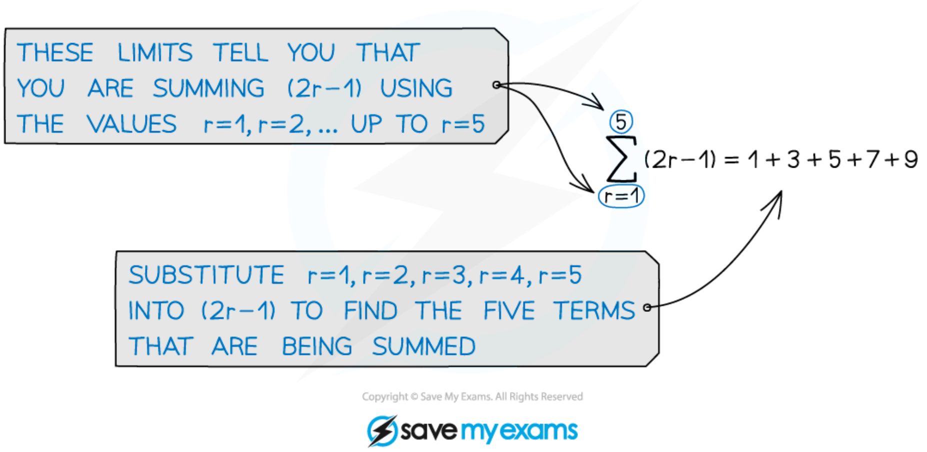
BE CAREFUL HERE. EVERY SECOND TERM IS A ZERO, BUT IT TAKES FOUR TERMS FOR THE PATTERN TO REPEAT

Your notes

Sigma notation

What is sigma notation?

- The symbol Σ is the capital Greek letter sigma – that's why it's called 'sigma notation'!
- 'Σ' stands for 'sum' – the expression to the right of the Σ tells you what is being summed, and the limits above and below tell you which terms you are summing



- Be careful – the limits don't have to start with 1!
  - For example: $\sum_{r=0}^{4}(2r+1)$ or $\sum_{r=7}^{11}(2r-13)$

How do I use sigma notation?

- Sigma notation can be used to represent both **arithmetic series** and **geometric series**
  - **Arithmetic** will have the form $A + Br$
  - **Geometric** will have the form $A \times B^r$
  - Writing out the first few terms will help you

Your notes

$$\sum_{r=1}^{5}(5r+1) = 6 + 11 + 16 + 21 + 26$$

THIS IS THE SUM OF THE FIRST FIVE TERMS OF THE ARITHMETIC SERIES WITH FIRST TERM 6 AND COMMON DIFFERENCE 5

$$\sum_{r=1}^{5}(4\times 3^{r}) = 12 + 36 + 108 + 324 + 972$$

THIS IS THE SUM OF THE FIRST FIVE TERMS OF THE GEOMETRIC SERIES WITH FIRST TERM 12 AND COMMON RATIO 3

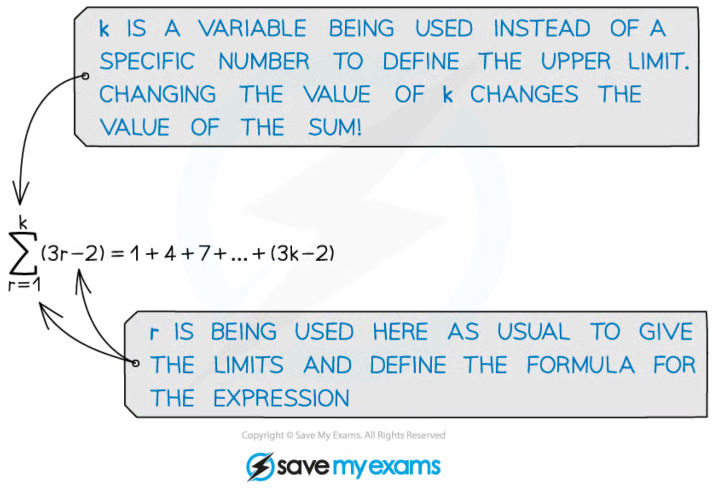
- To work out such a sum use the arithmetic and geometric series formulae
- As long as the expressions being summed are the same you can add and subtract in sigma notation
  - For example:

$$\sum_{r=1}^{6}(4r+7) + \sum_{r=7}^{11}(4r+7) = \sum_{r=1}^{11}(4r+7)$$

$$\sum_{r=1}^{100}(7\times 2^{r}) - \sum_{r=1}^{50}(7\times 2^{r}) = \sum_{r=51}^{100}(7\times 2^{r})$$

Examiner Tips and Tricks

Be careful when more than one letter appears in a sigma notation question.



Your notes

Worked Example

Your notes

Evaluate:

a) $\sum_{r=50}^{99} (2r+1)$

b) $\sum_{r=k-4}^{12} (3 \times 4^r)$ when $k = 7$

"EVALUATE" HERE MEANS "FIND THE VALUE OF"

a) THIS IS THE SUM $101 + 103 + 105 + \ldots + 199$

ARITHMETIC SERIES WITH $a = 101$, $d = 2$, AND LAST TERM $l = 199$

$S_n = \frac{n}{2}(a+l)$

$S_{50} = \frac{50}{2}(101 + 199) = 7500$

BE CAREFUL HERE! THERE ARE 50 NUMBERS (NOT 49) FROM $r=50$ TO $r=99$. SO $n=50$

b) FOR $k=7$ THIS IS

$$\sum_{r=3}^{12}(3 \times 4^r) = 3\times4^3 + 3\times4^4 + 3\times4^5 + \ldots + 3\times4^{12}$$
$$= 3\times4^3 + 4(3\times4^3) + 4^2(3\times4^3) + \ldots + 4^9(3\times4^3)$$
$$= 192 + 4(192) + 4^2(192) + \ldots + 4^9(192)$$

THAT IS THE FIRST 10 TERMS OF THE GEOMETRIC SERIES WITH $a = 192$ AND $r = 4$

$$S_{10} = \frac{192(4^{10}-1)}{4-1} = 67\,108\,800$$

$S_n = \frac{a(r^n-1)}{r-1}$

BE CAREFUL! THIS r IS THE COMMON RATIO OF THE GEOMETRIC SERIES. IT IS DIFFERENT FROM THE r IN THE SIGMA NOTATION.

Recurrence relations

What is a recurrence relation?

- A **recurrence relation** describes each term in a sequence as a function of the previous term – ie $u_{n+1} = f(u_n)$
- Along with the first term of the sequence, this allows you to generate the sequence term by term

e.g. FIND THE FIRST FOUR TERMS OF THE SEQUENCE DEFINED BY $u_{n+1} = u_n + 3, \ u_1 = 2$

$u_1 = 2 \quad u_2 = u_1 + 3 = 2 + 3 = 5$

$u_3 = u_2 + 3 = 5 + 3 = 8$

$u_4 = u_3 + 3 = 8 + 3 = 11$

THE FIRST FOUR TERMS ARE 2, 5, 8, 11

- Both **arithmetic sequences** and **geometric sequences** can be defined using recurrence relations
  - **Arithmetic** can be defined by $u_{n+1} = u_n + d, \ u_1 = a$
  - **Geometric** can be defined by $u_{n+1} = u_n \times r, \ u_1 = a$

$$u_{n+1} = u_n - 5, \quad u_1 = 9$$

THIS IS THE SEQUENCE $9, 4, -1, -6, -11, \ldots$
(i.e. ARITHMETIC SEQUENCE WITH FIRST TERM 9 AND COMMON DIFFERENCE -5)

$$u_{n+1} = 2u_n, \quad u_1 = 3$$

THIS IS THE SEQUENCE $3, 6, 12, 24, 48, \ldots$
(i.e. GEOMETRIC SEQUENCE WITH FIRST TERM 3 AND COMMON RATIO 2)

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- However, you can also define sequences that are neither arithmetic nor geometric

$$u_{n+1} = 2u_n + 3, \quad u_1 = 1$$

THIS IS THE SEQUENCE $1, 5, 13, 29, 61, \ldots$
(DON'T CONFUSE THIS WITH ARITHMETIC SEQUENCE $5, 7, 9, 11, \ldots$ DEFINED BY $u_n = 2n+3$)

$$u_{n+1} = (u_n)^2, \quad u_1 = 2$$

THIS IS THE SEQUENCE $2, 4, 16, 256, 65\,536, \ldots$

Examiner Tips and Tricks

- For arithmetic or geometric sequences defined by recurrence relations, you can sum the terms using the **arithmetic series** and **geometric series** formulae.
- To sum up the terms of other sequences, you may have to think about the series and find a clever trick.

Your notes

Worked Example

A sequence $u_1,\ u_2,\ u_3,\ \ldots$ is defined by

$$u_1 = 2$$

$$u_{n+1} = u_n^2 - 3$$

Find $\sum_{r=1}^{101} u_r$

$u_1 = 2 \qquad u_2 = 2^2 - 3 = 1 \qquad u_3 = 1^2 - 3 = -2$

$u_4 = (-2)^2 - 3 = 1 \qquad u_5 = 1^2 - 3 = -2$

WORK OUT THE TERMS AND LOOK FOR A PATTERN

SO THE SEQUENCE IS

$2, 1, -2, 1, -2, 1, -2, \ldots$

IN THE FIRST 101 TERMS THERE WILL BE

ONE 2 $\quad 1 \times 2 = 2$

FIFTY 1's $\quad 50 \times 1 = 50$

FIFTY −2's $\quad 50 \times (-2) = -100$

$2 + 50 + (-100) = -48$

SO $\sum_{r=1}^{101} u_r = -48$