Edexcel International AS Maths: Pure 2

Your notes

Circles

Contents

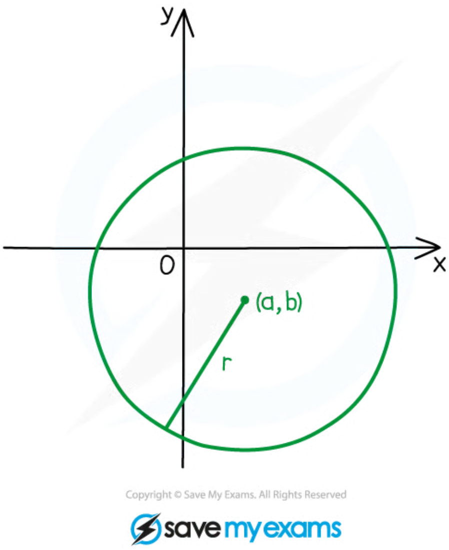
Your notes

Equation of a circle

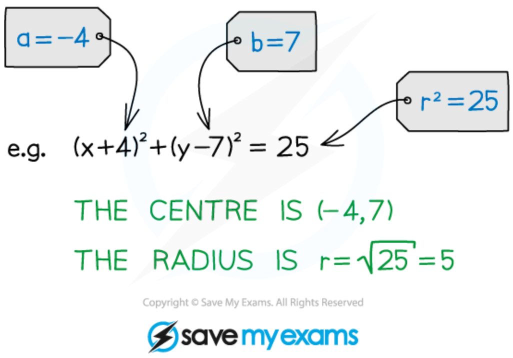
What is the equation of a circle?

- A circle with centre **(*a*, *b*)** and radius ***r*** has the equation

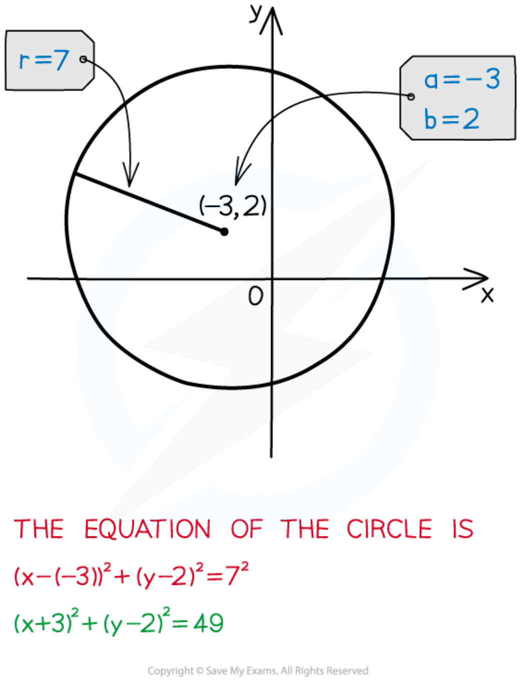
$$(x - a)^2 + (y - b)^2 = r^2$$



- You need to be able to find the centre and radius of a circle from its equation



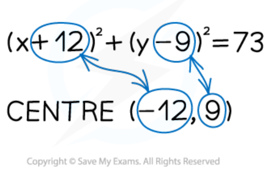
- You need to be able to find the equation of a circle given its centre and radius



Your notes

Examiner Tips and Tricks

- Remember that the numbers in the brackets have the opposite signs to the coordinates of the centre.



- Don't forget to take the **square root** of the right-hand side of the equation when finding the radius.

Worked Example

Your notes

A circle has centre $(-5, -7)$ and goes through the point $(1, 1)$.

Find the equation of the circle.

THE RADIUS IS THE DISTANCE FROM $(-5,-7)$ TO $(1,1)$

$r = \sqrt{(1-(-5))^2 + (1-(-7))^2}$

$= \sqrt{6^2+8^2} = \sqrt{100} = 10$

THE DISTANCE BETWEEN TWO POINTS (x_1, y_1) AND (x_2, y_2) IS $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

THE EQUATION OF THE CIRCLE IS

$(x-(-5))^2 + (y-(-7))^2 = 10^2$

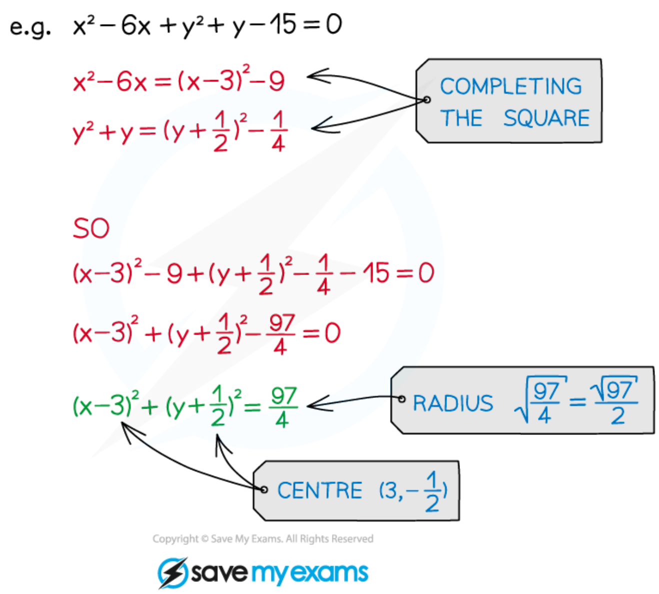
$(x+5)^2 + (y+7)^2 = 100$

$a = -5$
$b = -7$

Finding the centre & radius

How do I find the centre and radius from the equation of a circle?

- A circle equation in a different form can always be rearranged into $(x - a)^2 + (y - b)^2 = r^2$ form in order to find the centre and radius
- This will often involve **completing the square**



Examiner Tips and Tricks

Make sure you are able to rearrange circle equations given in the general form $x^2 + y^2 + 2fx + 2gy + c = 0$

Your notes

$x^2 + y^2 + 2fx + 2gy + c = 0$

$x^2 + 2fx + y^2 + 2gy + c = 0$

$(x+f)^2 - f^2 + (y+g)^2 - g^2 + c = 0$

$(x+f)^2 + (y+g)^2 = f^2 + g^2 - c$

RADIUS $\sqrt{f^2+g^2-c}$

CENTRE $(-f, -g)$

Worked Example

A circle has the equation

$$x^2 + 2x + y^2 - 6y + k = 0$$

and passes through the point $(5, 12)$.

Find the centre and radius of the circle.

CIRCLE PASSES THROUGH (5,12) SO

$(5)^2 + 2(5) + (12)^2 - 6(12) + k = 0$

$25 + 10 + 144 - 72 + k = 0$

$k = -107$

USE THE COORDINATES OF THE KNOWN POINT TO FIND THE VALUE OF k

EQUATION OF THE CIRCLE IS

$x^2 + 2x + y^2 - 6y - 107 = 0$

$(x+1)^2 - 1 + (y-3)^2 - 9 - 107 = 0$

$(x+1)^2 + (y-3)^2 = 117$

REARRANGE INTO $(x-a)^2 + (y-b)^2 = r^2$ FORM

CENTRE IS $(-1, 3)$

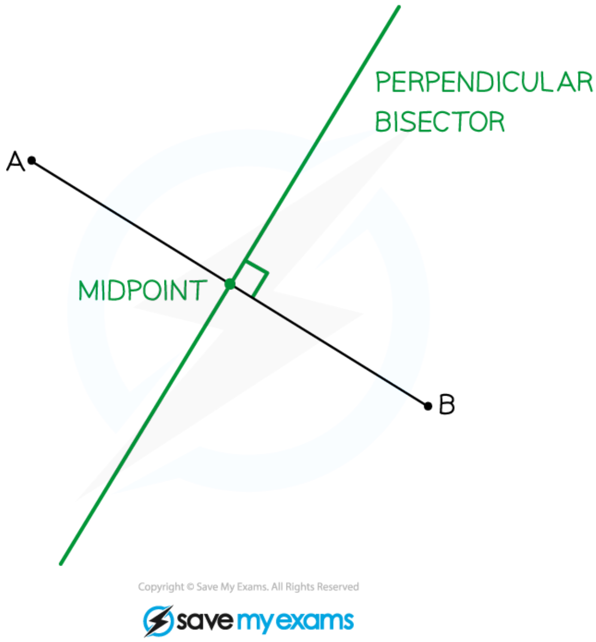
RADIUS IS $\sqrt{117}$

Your notes

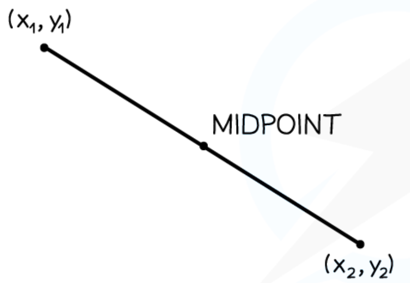
Bisection of chords

How can I find the equation of a perpendicular bisector?

- The **perpendicular bisector** of a line segment:
  - is perpendicular to the line segment
  - goes through the midpoint of the line segment



- The midpoint and gradient of the line segment between points (x_1, y_1) and (x_2, y_2) are given by the formulae



$$\text{MIDPOINT} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\text{GRADIENT} = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

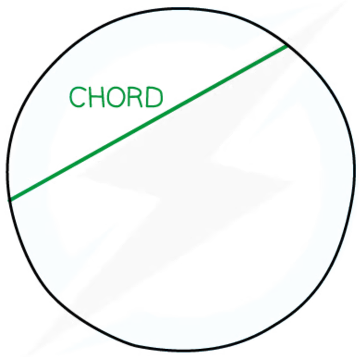
- The gradient of the perpendicular bisector is therefore

$$\text{GRADIENT OF PERPENDICULAR BISECTOR} = -\left(\frac{x_2 - x_1}{y_2 - y_1}\right)$$

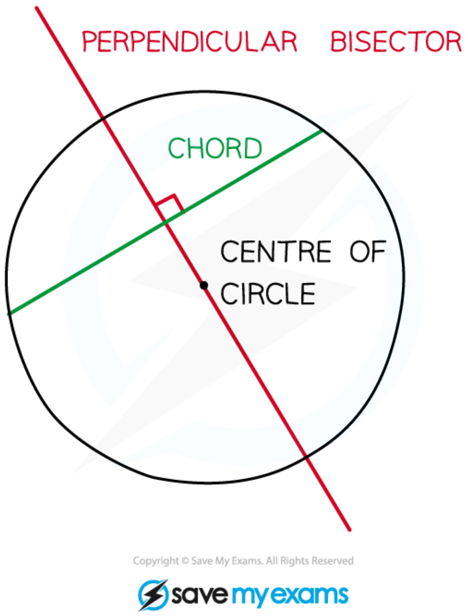
- The equation of the perpendicular bisector is the equation of the line with that gradient through the line segment's midpoint (see Equation of a Straight Line)

How can I use perpendicular bisectors to find the equation of a circle?

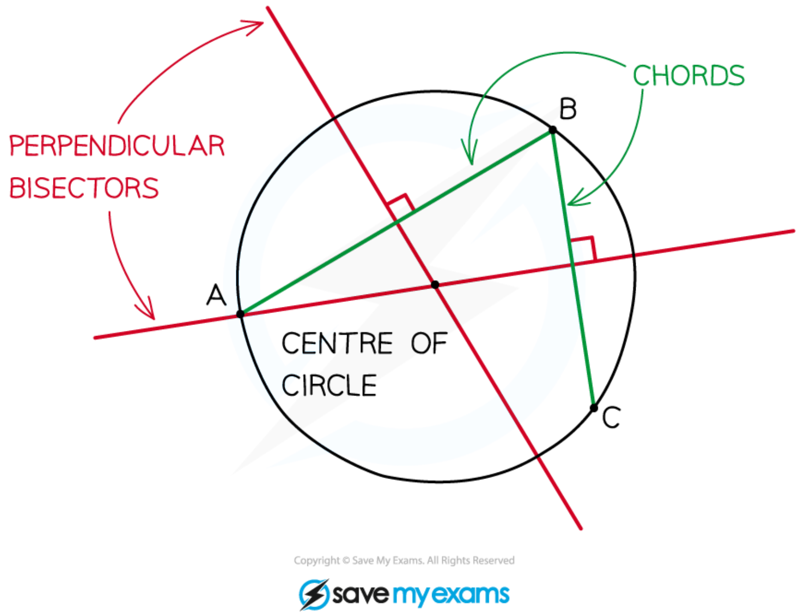
- A **chord** of a circle is a straight line segment between any two points on the circle



- The perpendicular bisector of a chord always goes through the centre of the circle



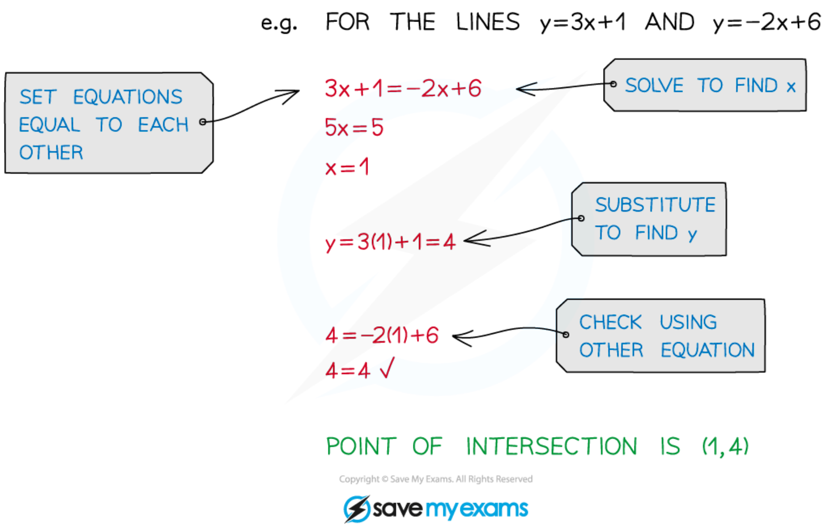
- If you know three points on a circle, draw any two chords between them – the perpendicular bisectors of the chords will meet at the centre of the circle



- Now that you know the centre of the circle and a point on the circle you can write the equation of the circle

Examiner Tips and Tricks

- To find the point of intersection of two straight lines, set the equations of the lines equal to each other and solve.



Worked Example

Your notes

Your notes

The points $A\ (0,\ 17)$, $B\ (8,\ 13)$ and $C\ (-10,\ 7)$ lie on the circumference of a circle. Use the properties of chords and perpendicular bisectors to find the equation of the circle.

PERPENDICULAR BISECTOR OF AB:

GRADIENT $= -\frac{8-0}{13-17} = 2$ ← $-\left(\frac{x_2-x_1}{y_2-y_1}\right)$

MIDPOINT OF AB $= \left(\frac{0+8}{2}, \frac{17+13}{2}\right) = (4, 15)$ ← $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$y = 2(x-4)+15 = 2x+7$ ← USE GRADIENT AND MIDPOINT TO FIND EQUATION OF PERPENDICULAR BISECTOR

PERPENDICULAR BISECTOR OF BC:

GRADIENT $= -\frac{-10-8}{7-13} = -3$

MIDPOINT OF BC $= \left(\frac{8+(-10)}{2}, \frac{13+7}{2}\right) = (-1, 10)$

$y = -3(x-(-1))+10 = -3x+7$

POINT OF INTERSECTION OF PERPENDICULAR BISECTORS:

$2x+7 = -3x+7$ ← SET EQUATIONS EQUAL

$5x = 0$ ← SOLVE FOR x

$x = 0$

$y = 2(0)+7 = 7$ ← SUBSTITUTE TO FIND y

∴ CENTRE OF CIRCLE IS (0,7)

EQUATION OF CIRCLE:

$(x-0)^2 + (y-7)^2 = r^2$ ← $(x-a)^2+(y-b)^2 = r^2$

$x^2 + (y-7)^2 = r^2$

BUT (0,17) IS ON THE CIRCLE SO

$(0)^2 + (17-7)^2 = r^2$ ← SUBSTITUTE THE COORDINATES

$100 = r^2$

OF ANY ONE POINT INTO THE EQUATION

THE EQUATION OF THE CIRCLE IS

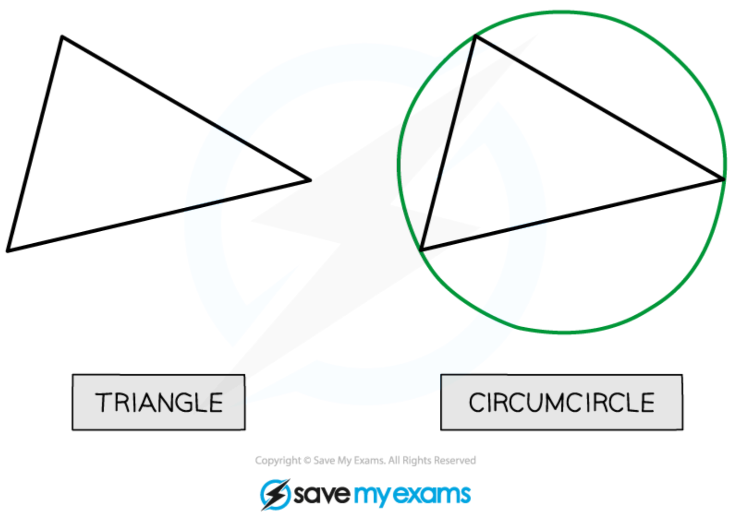
$x^2+(y-7)^2=100$

Your notes

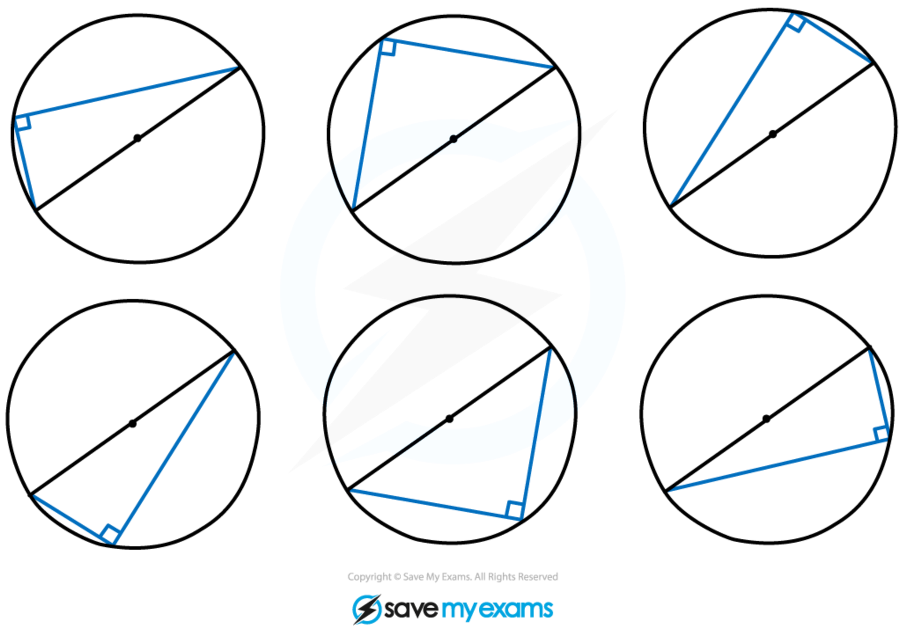
Angle in a semicircle

What are angles in a semicircle?

- It is always possible to draw a unique circle through the three vertices of a triangle – this is called the **circumcircle** of the triangle



- The **angle in a semicircle property** says that if a triangle is right-angled, then its hypotenuse is a diameter of its circumcircle

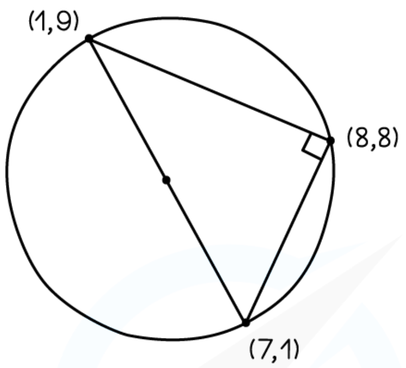


- It also says that **any** angle at the circumference in a semicircle is a right angle

How can I use angles in a semicircle to find the equation of a circle?

- Because the hypotenuse of a right-angled triangle is a diameter of the triangle's circumcircle you also know that:

    1. the radius of the circumcircle is half the length of the hypotenuse
    2. the centre of the circumcircle is the midpoint of the hypotenuse



CENTRE OF CIRCLE = MIDPOINT OF HYPOTENUSE

$= \left(\frac{1+7}{2}, \frac{9+1}{2}\right)$

$= (4,5)$

MIDPOINT OF LINE SEGMENT FROM (x_1, y_1) TO (x_2, y_2) IS

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

RADIUS $= \frac{1}{2} \times$ LENGTH OF HYPOTENUSE

$= \frac{1}{2}\left(\sqrt{(7-1)^2+(1-9)^2}\right)$

$= \frac{1}{2}\sqrt{100} = 5$

LENGTH OF LINE SEGMENT FROM (x_1, y_1) TO (x_2, y_2) IS

$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

- Once you know the radius and the centre you can write down the **equation of the circle**

EQUATION OF CIRCLE WITH CENTRE (4,5) AND RADIUS OF 5 IS

$(x-4)^2+(y-5)^2 = 5^2$

$(x-4)^2+(y-5)^2 = 25$

Your notes

Your notes

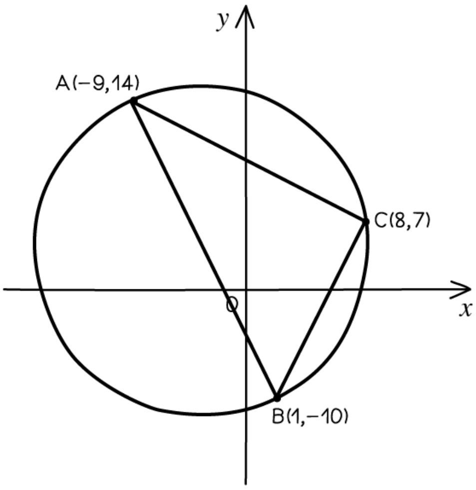
Examiner Tips and Tricks

- To show that a triangle is right-angled, show that the lengths of its sides satisfy Pythagoras' theorem.

Worked Example

Your notes

The three points $A\ (-9,\ 14)$, $B\ (1,\ -10)$ and $C\ (8,\ 7)$ lie on a circle, as shown in the diagram.



a) Show that AB is a diameter of the circle

b) Find an equation of the circle

a) LENGTHS OF THE SIDES ARE:

$AB^2 = (1-(-9))^2 + (-10-14)^2 = 676$

$AC^2 = (8-(-9))^2 + ((7-14))^2 = 338$

$BC^2 = (8-1)^2 + (7-(-10))^2 = 338$

BECAUSE WE'RE DEALING WITH PYTHAGORAS, IT'S EASIER TO KEEP EVERYTHING SQUARED HERE

$AC^2 + BC^2 = 338 + 338 = 676 = AB^2$

∴ BY PYTHAGORAS ABC IS A RIGHT-ANGLED TRIANGLE, AND AB IS THE HYPOTENUSE

∴ BY THE ANGLE IN A SEMICIRCLE PROPERTY, AB IS THE DIAMETER OF THE CIRCLE THROUGH A, B AND C

b) LENGTH OF $AB = \sqrt{676} = 26$

RADIUS OF CIRCLE $= \frac{1}{2} \times 26 = 13$

CENTRE OF CIRCLE = MIDPOINT OF AB

$= \left(\frac{-9+1}{2}, \frac{14+(-10)}{2}\right)$

$= (-4, 2)$

EQUATION OF THE CIRCLE IS

$(x-(-4))^2 + (y-2)^2 = 13^2$

$(x+4)^2+(y-2)^2=169$

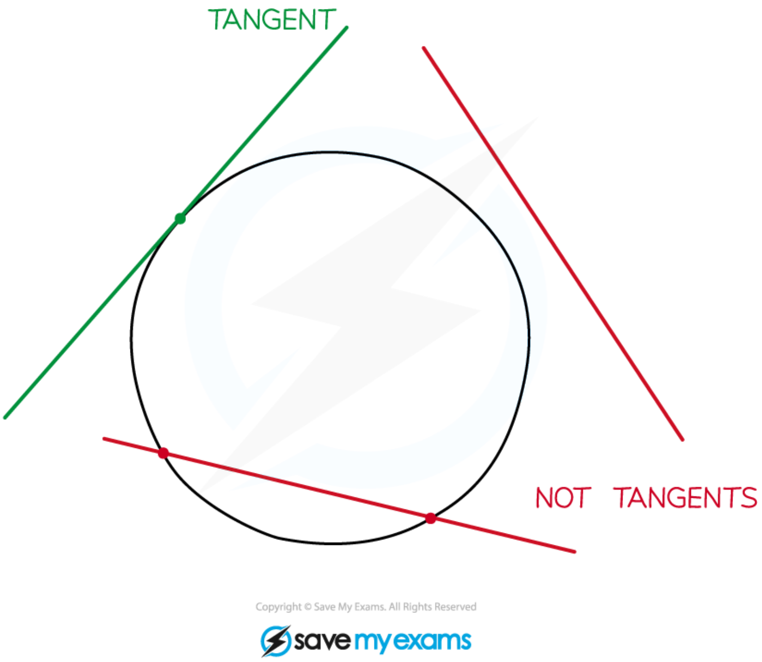
Your notes

Your notes

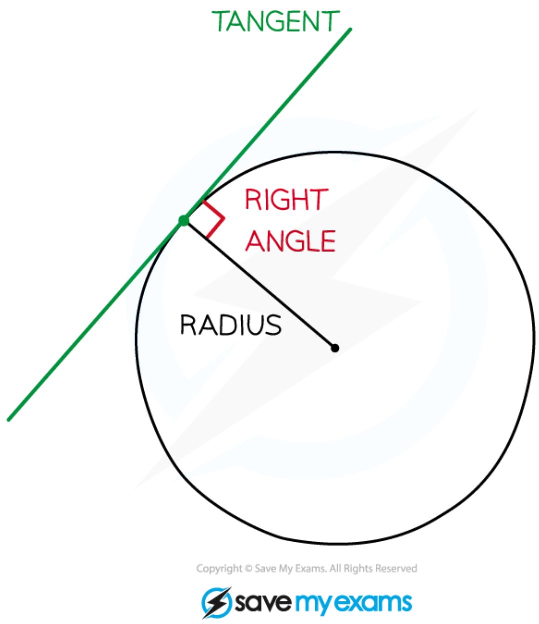
Radius & tangent

What is the relationship between a tangent and a radius?

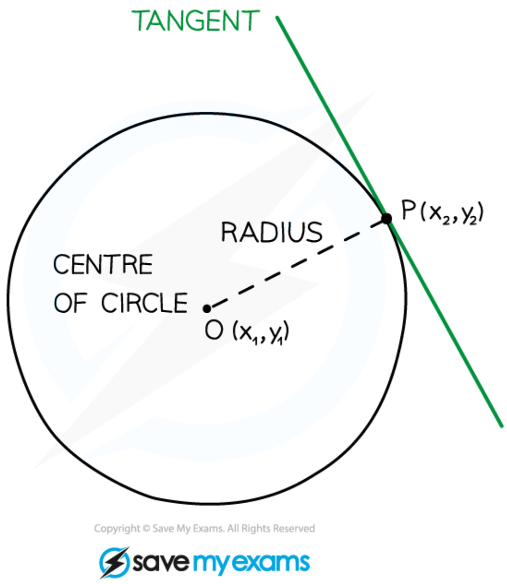
- A **tangent** is a line that **touches** a circle at a single point but doesn't cut across the circle



- A tangent to a circle is perpendicular to the radius of the circle at the point of intersection



How do I find the equation of a tangent to a circle?



- STEP 1: Find the gradient of the radius ***OP***

Your notes

$$\text{GRADIENT OF RADIUS} = \frac{y_2 - y_1}{x_2 - x_1}$$

- STEP 2: Find the gradient of the tangent

THE TANGENT IS PERPENDICULAR TO THE RADIUS, SO:

$$\text{GRADIENT OF TANGENT} = -\frac{x_2 - x_1}{y_2 - y_1}$$

- STEP 3: The equation of the tangent is the equation of the line with that gradient that goes through point ***P*** (see Equation of a Straight Line)

Examiner Tips and Tricks

- If you understand the formula in Step 2 above, you can find the gradient of the tangent without having to find the gradient of the radius first.

Worked Example

Your notes

The circle C has equation $(x-5)^2 + (y-2)^2 = 25$.

The point $P(2, 6)$ lies on C.

Find an equation for the tangent to C at point P.

STEP 1: C HAS CENTRE (5,2)

GRADIENT OF RADIUS AT $P = \frac{6-2}{2-5} = -\frac{4}{3}$ $\quad \frac{y_2-y_1}{x_2-x_1}$

STEP 2: GRADIENT OF TANGENT AT $P = \frac{3}{4}$ $\quad -\frac{x_2-x_1}{y_2-y_1}$

STEP 3: $y = \frac{3}{4}(x-2)+6$ $\quad$ LINE THROUGH (2,6) WITH GRADIENT $\frac{3}{4}$

$= \frac{3}{4}x - \frac{3}{2} + 6$

THE EQUATION OF THE TANGENT IS

$y = \frac{3}{4}x + \frac{9}{2}$