Edexcel International AS Maths: Pure 2

Your notes

Polynomials

Contents

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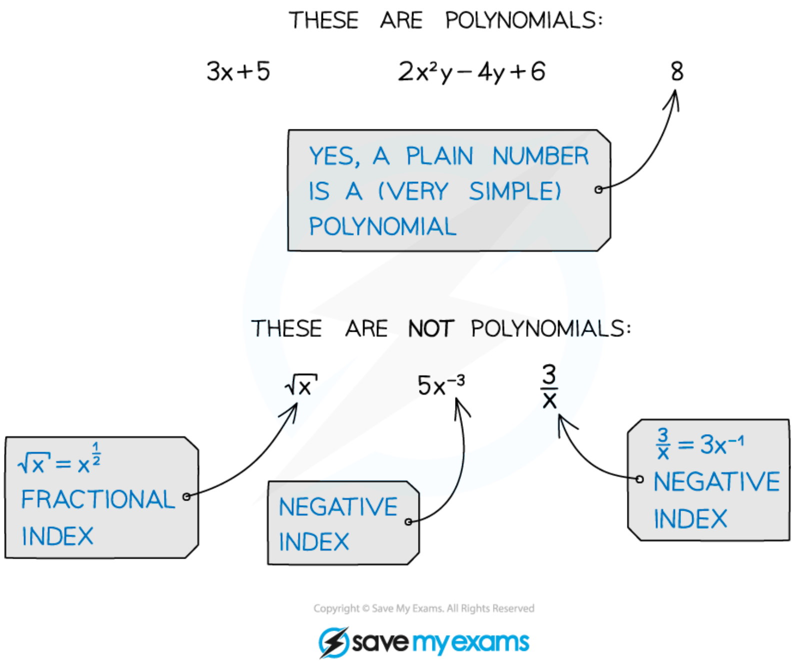
Polynomial division

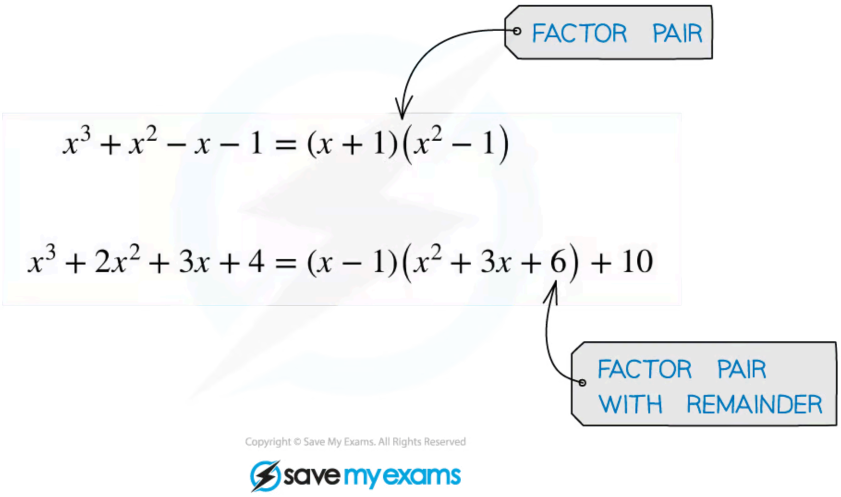
What is a polynomial?

- A **polynomial** is an algebraic expression consisting of a finite number of terms, with non-negative integer indices only



What is polynomial division?

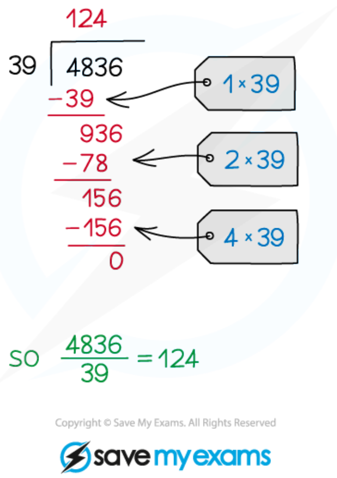
- Polynomial division is a method for splitting polynomials into factor pairs (with or without an accompanying remainder term)



- At A level you will most frequently use it to factorise polynomials, or when dealing with improper (ie 'top-heavy') algebraic fractions

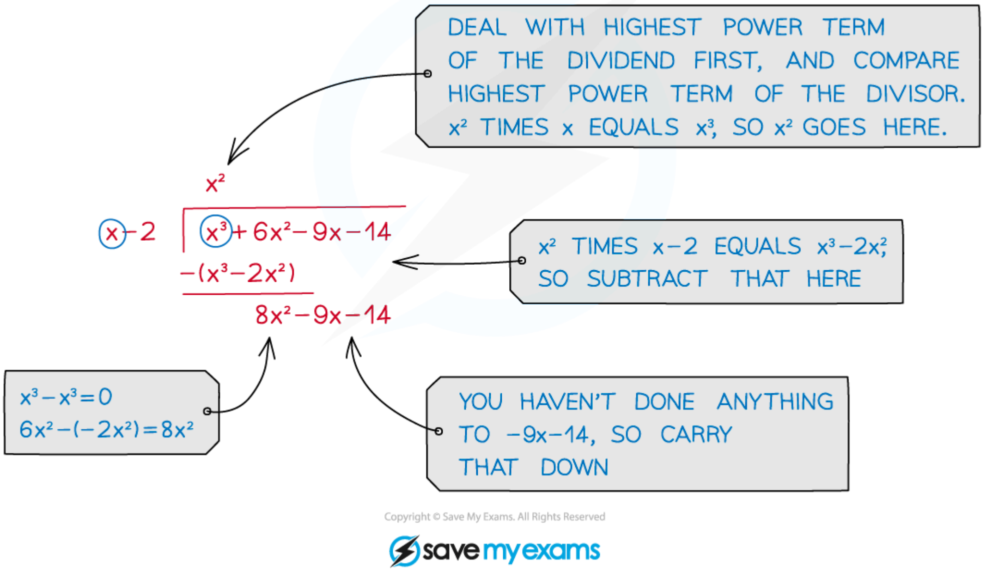
How do I divide polynomials?

- The method used for polynomial division is just like the long division method (sometimes called 'bus stop division') used to divide regular numbers:

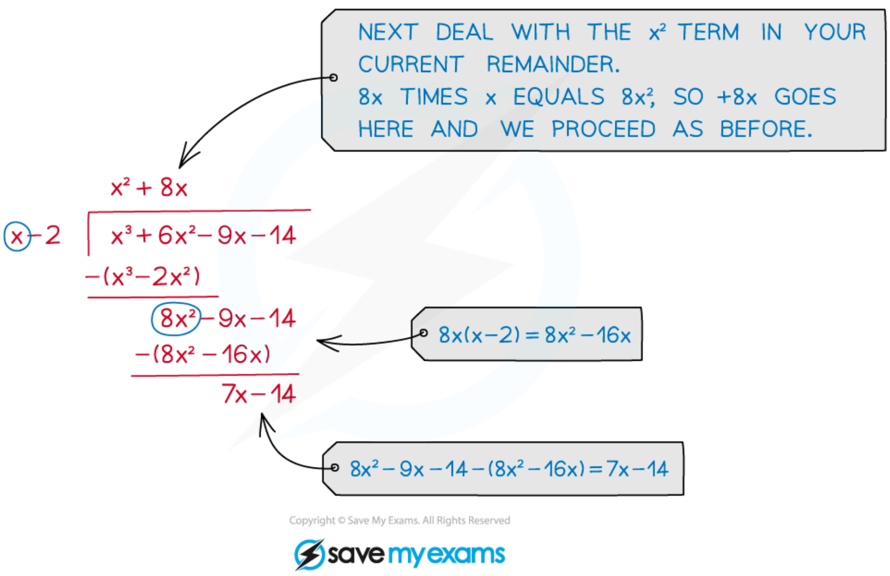


- At A level you will normally be dividing a polynomial dividend of degree 3 or 4 by a divisor in the form **($x \pm p$)**
- The answer to a polynomial division question is built up term by term, working downwards in powers of the variable (usually x)
- Start by dividing by the highest power term
- Write out this multiplied by the divisor and subtract

Divide the polynomial $f(x) = x^3 + 6x^2 - 9x - 14$ by $(x - 2)$.



- Continue to divide by each reducing power term and subtracting your answer each time



- Continue until you are left with zero

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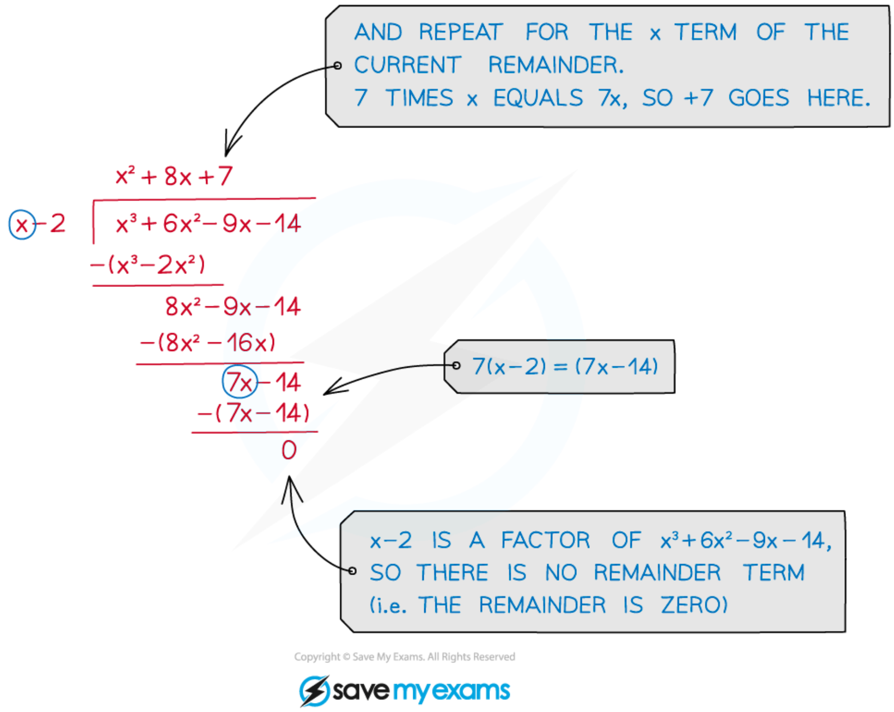


- If the divisor is not a factor of the polynomial then there will be a remainder term left at the end of the division

Worked Example

Your notes

For the polynomial $f(x) = x^3 + 6x^2 - 9x - 14$:

a) Divide $f(x)$ by $(x + 7)$.

b) Find the remainder when $f(x)$ is divided by $(x + 3)$.

a)

$$
\begin{array}{r|l}
 & x^2 - x - 2 \\
\hline
x+7 & x^3 + 6x^2 - 9x - 14 \\
 & -(x^3 + 7x^2) \\
 & \quad -x^2 - 9x - 14 \\
 & \quad -(-x^2 - 7x) \\
 & \qquad -2x - 14 \\
 & \qquad -(-2x - 14) \\
 & \qquad\qquad 0
\end{array}
$$

THE x^2 TERM OF THE REMAINDER HERE IS NEGATIVE, SO $-x$ GOES ON TOP

THE x TERM OF THE REMAINDER HERE IS NEGATIVE, SO -2 GOES ON TOP

$$\frac{x^3 + 6x^2 - 9x - 14}{x+7} = x^2 - x - 2$$

THIS MEANS THAT $x^3 + 6x^2 - 9x - 14 = (x+7)(x^2 - x - 2)$

b)

$$
\begin{array}{r|l}
 & x^2 + 3x - 18 \\
\hline
x+3 & x^3 + 6x^2 - 9x - 14 \\
 & -(x^3 + 3x^2) \\
 & \quad 3x^2 - 9x - 14 \\
 & \quad -(3x^2 + 9x) \\
 & \qquad -18x - 14 \\
 & \qquad -(-18x - 54) \\
 & \qquad\qquad 40
\end{array}
$$

THE x TERM OF THE REMAINDER HERE IS NEGATIVE, SO -18 GOES ON TOP

THE REMAINDER IS 40

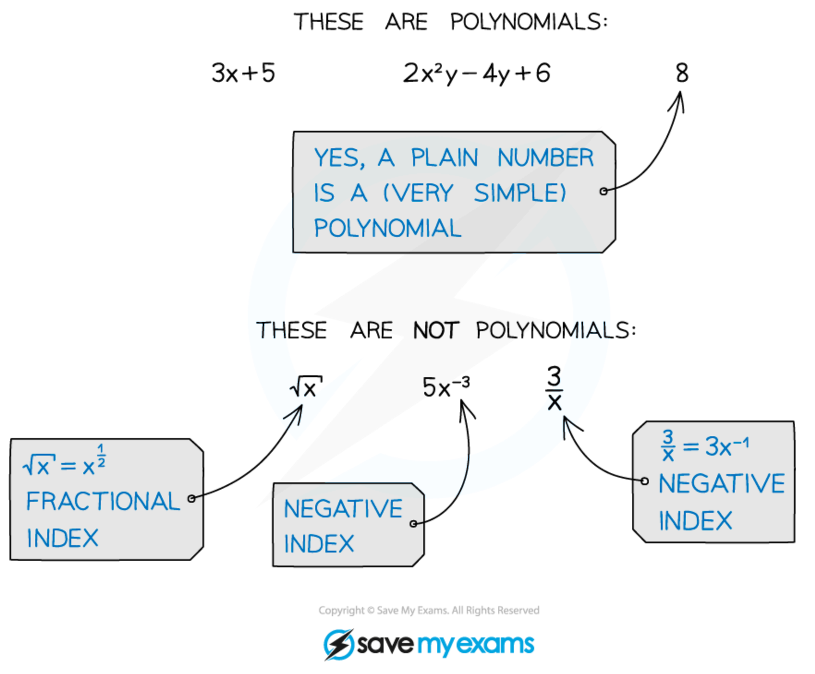
THIS MEANS THAT $x^3 + 6x^2 - 9x - 14 = (x+3)(x^2 + 3x - 18) + 40$

Your notes

Factor theorem

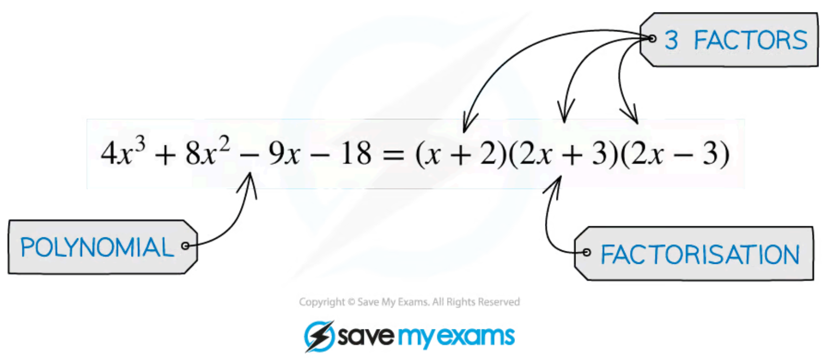
What is the factor theorem?

- The factor theorem is a very useful result about polynomials
- A **polynomial** is an algebraic expression consisting of a finite number of terms, with non-negative integer indices only



- At A level you will most frequently use the factor theorem as a way to simplify the process of factorising polynomials

$$4x^3 + 8x^2 - 9x - 18 = (x+2)(2x+3)(2x-3)$$



How do I apply the factor theorem?

Your notes

- For a polynomial **f(x)** the factor theorem states that:
  - If **f(*p*) = 0**, then **(*x* – *p*)** is a factor of **f(*x*)**

AND

  - If **(*x* – *p*)** is a factor of **f(x)**, then **f(*p*) = 0**

$$g(x) = 2x^3 - 7x^2 + 9$$
$$g(3) = 2(3)^3 - 7(3)^2 + 9$$
$$= 54 - 63 + 9$$
$$= 0$$

g(3) = 0

BY (i) THE FACTOR THEOREM TELLS US THAT (x−3) IS A FACTOR OF g(x)

$$h(x) = 4x^3 + 8x^2 - 9x - 18$$
$$= (x+2)\left(4x^2 - 9\right)$$

(x+2) = (x−(−2)) IS A FACTOR

BY (ii) THE FACTOR THEOREM TELLS US THAT h(−2) = 0 (TEST IT OUT!)

Examiner Tips and Tricks

- In an exam, the values of ***p*** you need to find that make **f(*p*) = 0** are going to be integers close to zero.
  - Try ***p*** = 1 and -1 first, then 2 and -2, then 3 and -3
  - It is very unlikely that you'll have to go beyond that.

Worked Example

For the polynomial $f(x) = x^3 + 6x^2 - 9x - 14$:

a) Use the factor theorem to show that $(x - 2)$ is a factor of $f(x)$.

b) Use the factor theorem to find another factor of $f(x)$.

a) $f(2) = (2)^3 + 6(2)^2 - 9(2) - 14$

$= 8 + 24 - 18 - 14 = 0$

FOR $(x-2)$ TO BE A FACTOR, $f(2)$ MUST BE EQUAL TO ZERO, NOT $f(-2)$

SO $(x-2)$ IS A FACTOR OF $f(x)$

b) $f(1) = (1)^3 + 6(1)^2 - 9(1) - 14$

TRY $x=1$ FIRST

$= 1 + 6 - 9 - 14 = -16$

$f(-1) = (-1)^3 + 6(-1)^2 - 9(-1) - 14$

$f(1) \neq 0$, SO TRY $x=-1$ NEXT

$= -1 + 6 + 9 - 14 = 0$

SO $(x+1)$ IS ANOTHER FACTOR OF $f(x)$

$f(-1) = 0$, SO BY THE FACTOR THEOREM $(x-(-1)) = (x+1)$ IS A FACTOR OF $f(x)$

Remainder theorem

What is the remainder theorem?

- The **factor theorem** is actually a special case of the more general **remainder theorem**
- The **remainder theorem** states that when the polynomial f(x) is divided by (x - a) the remainder is f(a)
    - You may see this written formally as f(x) = (x - a)Q(x) + f(a)
    - In **polynomial division**
        - Q(x) would be the **result** (at the top) of the division (the **quotient**)
        - f(a) would be the **remainder** (at the bottom)
        - (x - a) is called the **divisor**
    - In the case when f(a) = 0, f(x) = (x - a)Q(x) and hence (x - a) is a factor of f(x)– the **factor theorem**!

How do I apply the remainder theorem?

- If it is the **remainder** that is of particular interest, the remainder theorem saves the need to carry out polynomial division in full
  - e.g. The remainder from $(x^2 - 2x) \div (x - 3)$ is $3^2 - 2 \times 3 = 3$
  - This is because if f(x) = x^2- 2x and a = 3
- If the remainder from a polynomial division is known, the **remainder theorem** can be used to find unknown coefficients in polynomials
  - g. The remainder from $(x^2 + px) \div (x - 2)$ is 8 so the value of *p* can be found by solving $2^2 + p(2) = 8$, leading to $p = 2$
  - In harder problems there may be more than one unknown in which case simultaneous equations would need setting up and solving
- The more general version of remainder theorem is if f(x) is divided by (ax - b) then the remainder is $\mathrm{f}\left(\frac{b}{a}\right)$
  - The shortcut is still to evaluate the polynomial at the value of x that makes the divisor (ax - b) zero but it is not necessarily an integer

Your notes

Worked Example

Your notes

The polynomial $p(x)$ is given by $8x^4 + ax^2 + bx - 1$,

where a and b are integer constants.

When $p(x)$ is divided by $(x - 1)$ the remainder is 9.

When $p(x)$ is divided by $(2x - 1)$ the remainder is 1.

Find the values of a and b.

REMAINDER THEOREM: "f(a) IS THE REMAINDER WHEN f(x) IS DIVIDED BY (x − a)"

$x - 1 = 0$ WHEN $x = 1$

$\therefore p(1) = 9$

$8 + a + b - 1 = 9$

$a + b = 2$

AND

$2x - 1 = 0$ WHEN $x = \frac{1}{2}$

$p\left(\frac{1}{2}\right) = 1$

$8\left(\frac{1}{2}\right)^4 + a\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) - 1 = 1$

$\frac{1}{2} + \frac{1}{4}a + \frac{1}{2}b - 1 = 1$

$a + 2b = 6$

SOLVING SIMULTANEOUSLY:

$a + b = 2$

$\underline{a + 2b = 6}$

$b = 4$

$a = -2$

$a = -2, \ b = 4$

$p(x) = 8x^4 - 2x^2 + 4x - 1$

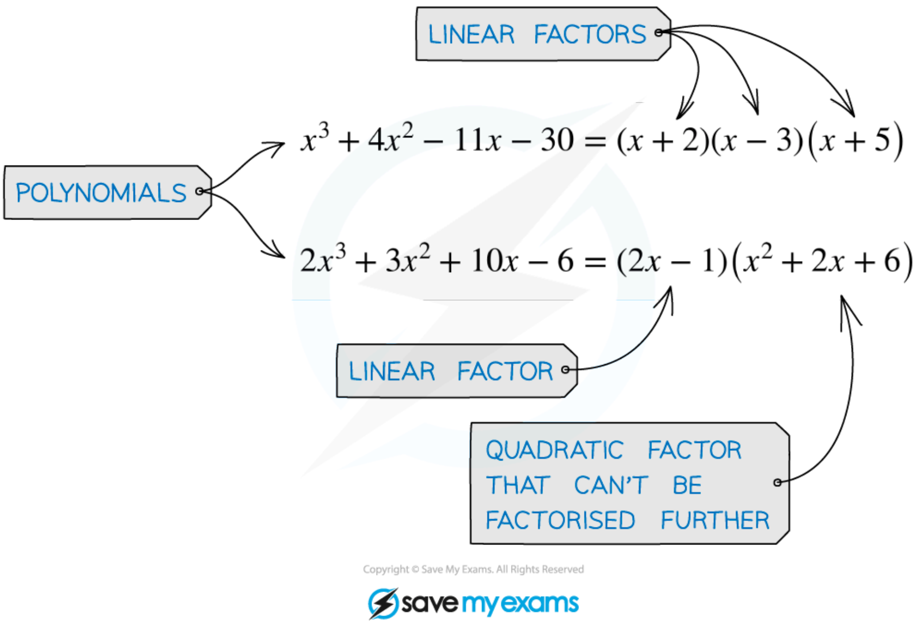
Examiner Tips and Tricks

- Exam questions will use formal mathematical language which can make factor and remainder theorem questions sound more complicated than they are.
- Ensure you are familiar with the various terms from these revision notes

Polynomial factorisation

What is polynomial factorisation?

- Factorising a polynomial combines the **factor theorem** with the method of **polynomial division**
- The goal is to break down a polynomial as far as possible into a product of linear factors



How do I factorise a polynomial?

- At A level you will usually be asked to factorise a **cubic** – i.e. a polynomial where the highest power of ***x*** is 3

THESE ARE ALL EXAMPLES OF CUBICS:

$$x^3 + 2x^2 - 5x + 7$$

$$3 - x^2 - 4x^3$$

$$x^3 + 2x$$

$$2x^3$$

- To factorise a cubic polynomial **f(x)** follow the following steps:

Fully factorise $x^3 + 4x^2 - 11x - 30$.

- Step 1. Find a value ***p*** that makes **f(*p*)** = 0

TRY x=1: $(1)^3 + 4(1)^2 - 11(1) - 30 = -36$

TRY x=−1: $(-1)^3 + 4(-1)^2 - 11(-1) - 30 = -16$

TRY x=2: $(2)^3 + 4(2)^2 - 11(2) - 30 = -28$

TRY x=−2: $(-2)^3 + 4(-2)^2 - 11(-2) - 30 = 0$ ✓

BY THE FACTOR THEOREM THIS TELLS US THAT $(x-(-2)) = (x+2)$ IS A FACTOR OF THE POLYNOMIAL

- Step 2. Use polynomial division to divide **f(x)** by **(*x* – *p*)**

$$
\begin{array}{r|l}
 & x^2 + 2x - 15 \\
\hline
x+2 & x^3 + 4x^2 - 11x - 30 \\
 & -(x^3 + 2x^2) \\
\hline
 & \quad 2x^2 - 11x - 30 \\
 & \quad -(2x^2 + 4x) \\
\hline
 & \qquad -15x - 30 \\
 & \qquad -(-15x - 30) \\
\hline
 & \qquad\qquad 0
\end{array}
$$

- Step 3. Use the result of your division to write

Your notes

$$f(x) = (x - p)(ax^2 + bx + c)$$

$$\text{SO} \quad x^3+4x^2-11x-30 = (x+2)(x^2+2x-15)$$

- Step 4. If the quadratic $(ax^2 + bx + c)$ is factorisable, factorise it and write $f(x)$ as a product of three linear factors (if the quadratic is not factorisable, then your result from Step 3 is the final factorisation)

$$x^2+2x-15 = (x+5)(x-3)$$

$$\text{SO} \quad x^3+4x^2-11x-30 = (x+2)(x+5)(x-3)$$

Examiner Tips and Tricks

- The method outlined above can be logically extended to factorise a polynomial of any degree.

Worked Example

Your notes

Your notes

Fully factorise $x^3 + 6x^2 - 9x - 14$.

STEP 1:

$f(1) = (1)^3 + 6(1)^2 - 9(1) - 14$

$= 1 + 6 - 9 - 14 = -16$

TRY $x=1$ FIRST

$f(-1) = (-1)^3 + 6(-1)^2 - 9(-1) - 14$

$= -1 + 6 + 9 - 14 = 0$

$f(1) \neq 0$, SO TRY $x=-1$ NEXT

SO $(x+1)$ IS A FACTOR OF $f(x)$

$f(-1)=0$, SO BY THE FACTOR THEOREM $(x-(-1)) = (x+1)$ IS A FACTOR OF $f(x)$

STEP 2:

$$
\begin{array}{r|l}
 & x^2 + 5x - 14 \\
\hline
x+1 & x^3 + 6x^2 - 9x - 14 \\
 & -(x^3 + x^2) \\
\hline
 & \quad 5x^2 - 9x - 14 \\
 & \quad -(5x^2 + 5x) \\
\hline
 & \qquad -14x - 14 \\
 & \qquad -(-14x - 14) \\
\hline
 & \qquad\qquad 0
\end{array}
$$

STEP 3:

SO $x^3 + 6x^2 - 9x - 14 = (x+1)(x^2 + 5x - 14)$

STEP 4:

$(x^2 + 5x - 14) = (x+7)(x-2)$

SO $x^3 + 6x^2 - 9x - 14 = (x+1)(x+7)(x-2)$