Edexcel International AS Maths: Pure 2

Your notes

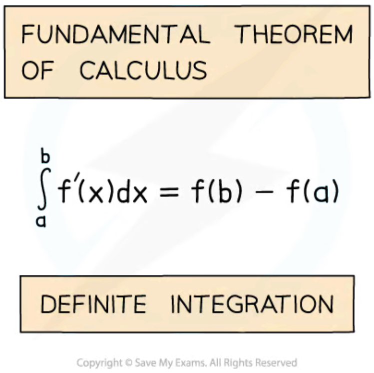
Integration

Contents

Definite integration

What is definite integration?

- **Definite Integration** occurs in an alternative version of the Fundamental Theorem of Calculus
- This version of the Theorem is the one referred to by most AS/A level textbooks/websites



- **a** and **b** are called limits
  - **a** is the lower limit
  - **b** is the upper limit
- **f'(x)** is the **derivative** of **f(x)**

Why is there no constant of integration in definite integration?

e.g. FIND $\int_2^5 9x^2\,dx$

$f'(x) = 9x^2$

$\therefore\ f(x) = 3x^3 + c$ ← INDEFINITE INTEGRATION

$f(2) = 3(2)^3 + c = 24 + c$

$f(5) = 3(5)^3 + c = 375 + c$

$\therefore\ f(5) - f(2) = (375 + c) - (24 + c)$

$= 375 - 24 + c - c$

$= 351$

DEFINITE INTEGRATION

c's CANCEL

- "**+c**" would appear in both **f(a)** and **f(b)**
  - Since we then calculate **f(b)** – **f(a)** they cancel each other out
  - There would be a "**+c**" from **f(b)** and a **–**"**+c**" from **f(a)**
- So "**+c**" is not included with definite integration

How do I find a definite integral on my calculator?

- STEP 1: If not given a name, call the integral
  - This saves you having to rewrite the whole integral every time!
- STEP 2: If necessary rewrite the integral into a more easily **integrable** form
  - Not all functions can be integrated directly
- STEP 3: Integrate without applying the limits
  - Notation: use square brackets [ ] with limits placed after the end bracket
- STEP 4: Substitute the limits into the function and calculate the answer

Your notes

e.g. EVALUATE $\int_1^9 \left(2x^2 + \frac{3}{x^2} - 4\sqrt{x}\right) dx$

STEP 1 NAME THE INTEGRAL, I

LET $I = \int_1^9 \left(2x^2 + \frac{3}{x^2} - 4\sqrt{x}\right) dx$

STEP 2 REWRITE IN A MORE EASILY INTEGRABLE FORM

$I = \int_1^9 \left(2x^2 + 3x^{-2} - 4x^{\frac{1}{2}}\right) dx$

STEP 3 INTEGRATE

- USE SQUARE BRACKETS
- LIMITS AT END

$I = \left[\frac{2}{3}x^3 - 3x^{-1} - \frac{8}{3}x^{\frac{3}{2}}\right]_1^9$

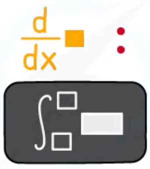
STEP 4 SUBSTITUTE LIMITS AND CALCULATE

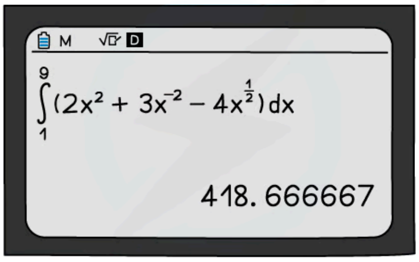
$I = \left(486 - \frac{1}{3} - 72\right) - \left(\frac{2}{3} - 3 - \frac{8}{3}\right)$

$I = \frac{1256}{3}$

Using a calculator

- Advanced scientific calculators can work out the values of definite integrals
- The button will look similar to:





Your notes

- (Note how the calculator did not return the exact value $\left(\dfrac{1256}{3}\right)$ of the integral)

Examiner Tips and Tricks

- Look out for questions that ask you to find an **indefinite** integral in one part (so "**+c**" needed), then in a later part use the same integral as a **definite** integral (where "**+c**" is not needed).

Worked Example

Find the value of

$$\int_2^4 3x(x^2 - 2)\ \mathrm{d}x$$

Answer:

Start by expanding the brackets inside the integral

$$\int_2^4 (3x^3 - 6x)\ \mathrm{d}x$$

Integrate as usual (here it's a 'powers of X' integration)

Write the answer in square brackets with the integration limits outside

$$\int_2^4 (3x^3 - 6x)\ \mathrm{d}x = \left[3\left(\frac{x^{3+1}}{3+1}\right) - 6\left(\frac{x^{1+1}}{1+1}\right)\right]_2^4$$

$$= \left[\frac{3}{4}x^4 - 3x^2\right]_2^4$$

Now substitute 4 into that function
And subtract from it the function with 2 substituted in

$$\left[\frac{3}{4}x^4 - 3x^2\right]_2^4 = \left(\frac{3}{4}(4)^4 - 3(4)^2\right) - \left(\frac{3}{4}(2)^4 - 3(2^2)\right)$$
$$= (192 - 48) - (12 - 12)$$
$$= 144 - 0$$
$$= 144$$

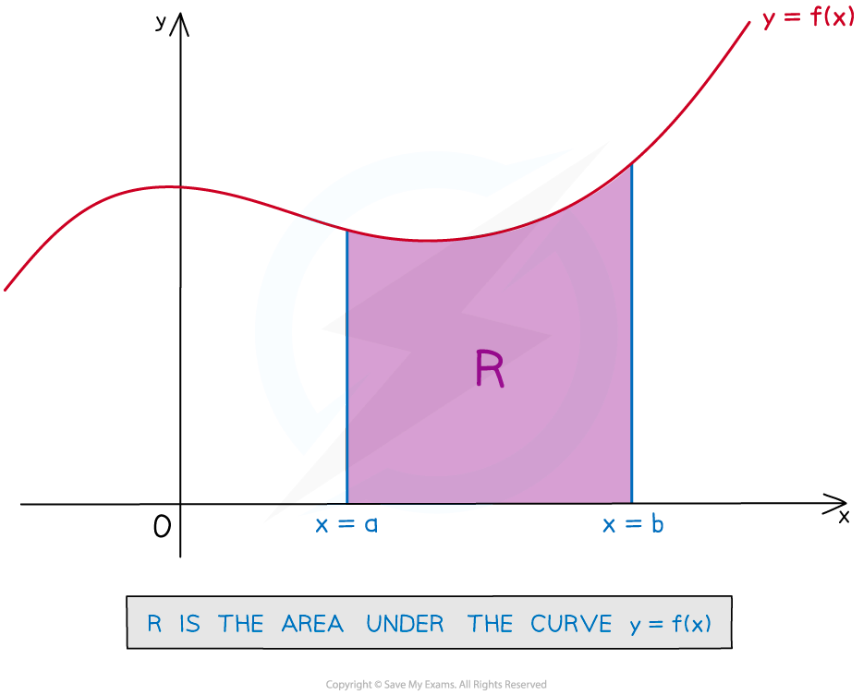
$$\int_2^4 3x(x^2 - 2)\ \mathrm{d}x = 144$$

Your notes

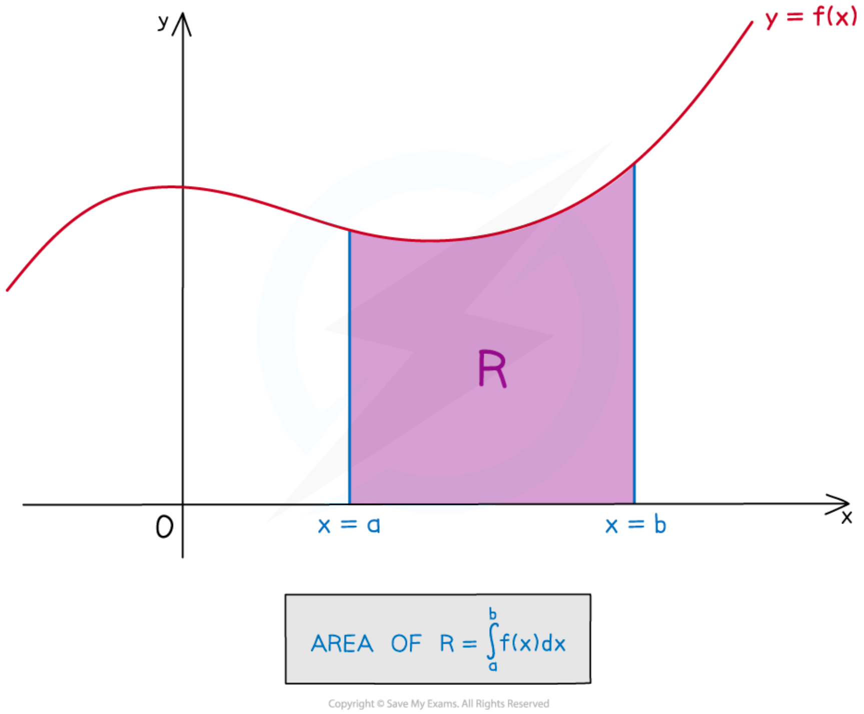
Area under a curve

What is the area under a curve?

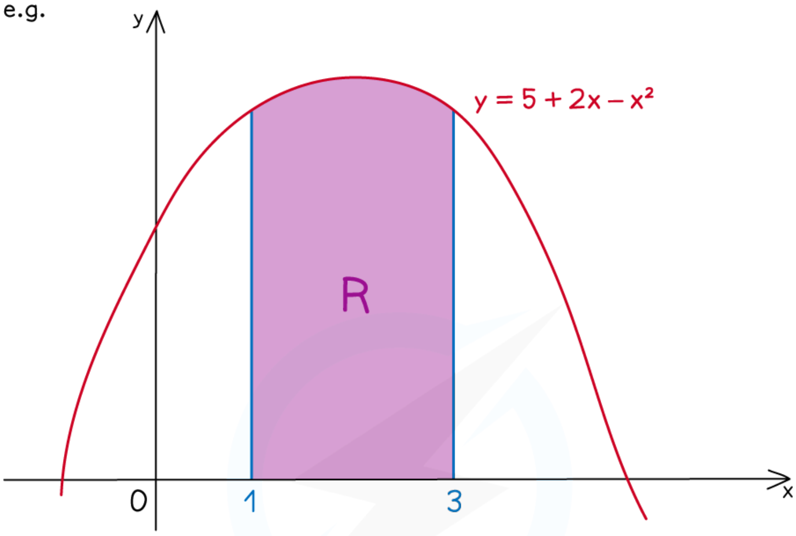
- The phrase "area under a curve" refers to the area bounded by ...
  - ... the **x**-axis
  - ... the graph of **y = f(x)**
  - ... the vertical line **x = a**
  - ... the vertical line **x = b**



How do I find the area under a curve using definite integration?



- The value from **definite integration** is equal to the "area under a curve" e.g.



FIND THE AREA OF THE SHADED REGION.

$$\text{AREA} = \int_1^3 (5 + 2x - x^2)\,dx$$

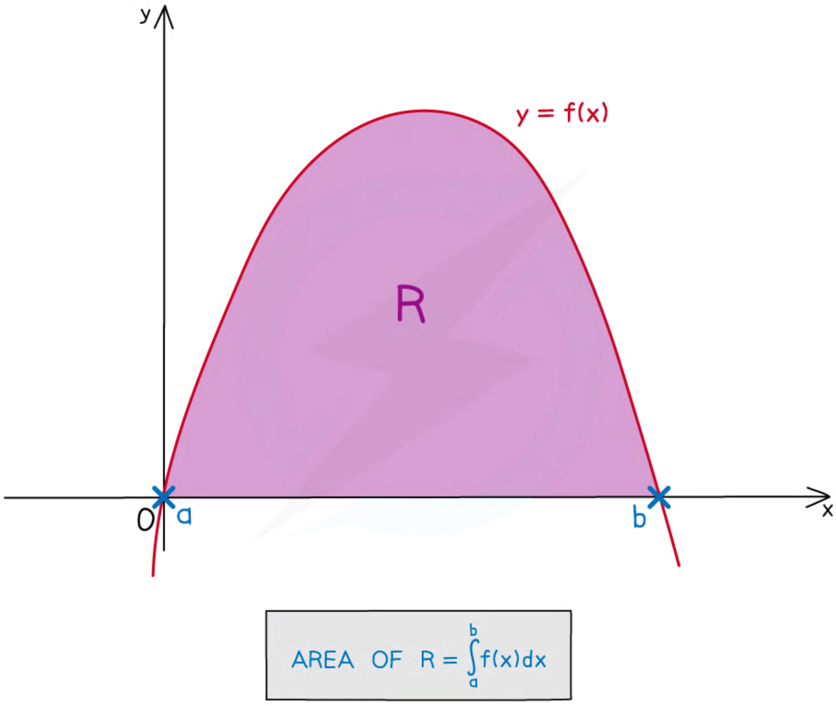
$$= \left[5x + x^2 - \frac{1}{3}x^3\right]_1^3$$

$$= (15 + 9 - 9) - (5 + 1 - \frac{1}{3})$$

$$\text{AREA} = \frac{28}{3} \text{ SQUARE UNITS}$$

How do I find the limits of integration?

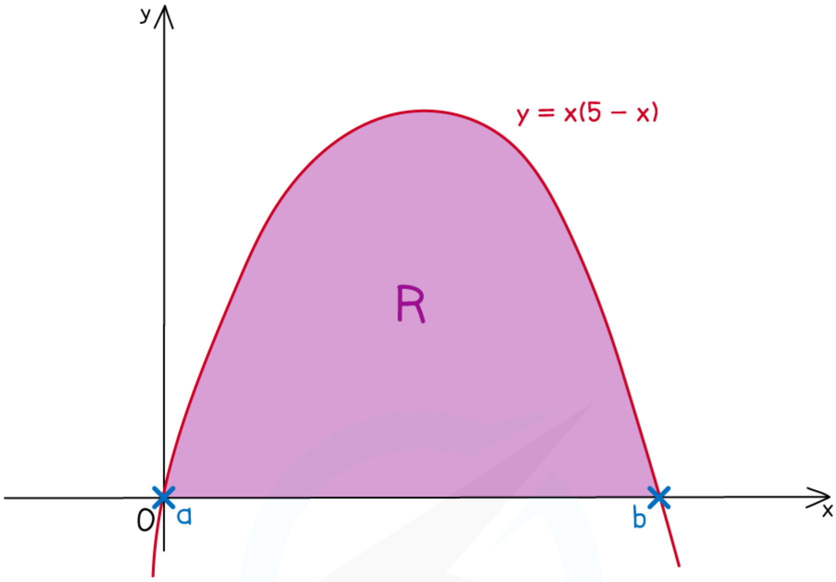
Your notes



- If limits are not provided they will be the **x**-axis intercepts
  - Set **y = 0** and solve the equation to find the **x**-axis intercepts

Your notes

e.g. FIND THE SHADED AREA MARKED R IN THE DIAGRAM BELOW



$x(5-x)=0$ SET $y=0$ FOR x-AXIS INTERCEPTS

$x=0 \qquad x=5$

$\therefore \quad a=0 \qquad b=5$

EXPAND SO "INTEGRATE-ABLE"

AREA OF R $= \int_0^5 (5x-x^2)\,dx$

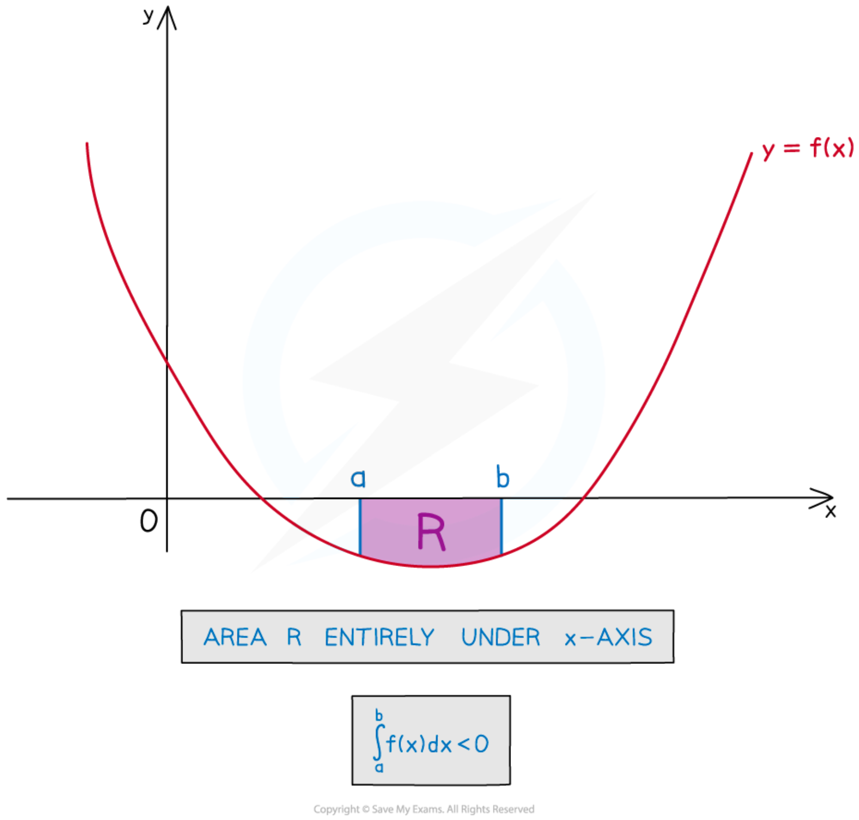
$= \left[\frac{5x^2}{2}-\frac{x^3}{3}\right]_0^5$

$= \frac{125}{2}-\frac{125}{3}-(0-0)$

AREA OF R $= \frac{125}{6}$ SQUARE UNITS

How do I find areas that are below the x-axis (negative areas)?

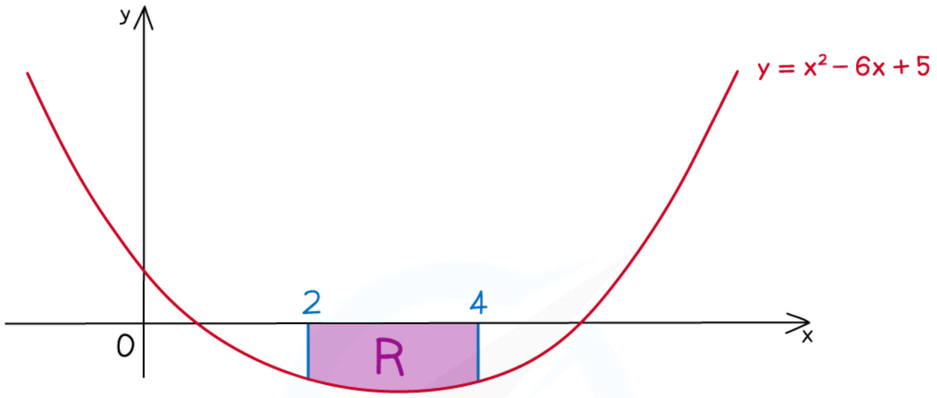
Your notes



- If the area lies underneath the **x**-axis the value of the integral will be negative
- An **area** cannot be negative, so take the **modulus** of the integral

Your notes

e.g. FIND THE SHADED AREA R IN THE DIAGRAM BELOW



$$\int_2^4 (x^2 - 6x + 5)\,dx = \left[\frac{x^3}{3} - 3x^2 + 5x\right]_2^4$$

WRITE AS AN INTEGRAL

$$= \left(\frac{64}{3} - 48 + 20\right) - \left(\frac{8}{3} - 12 + 10\right)$$

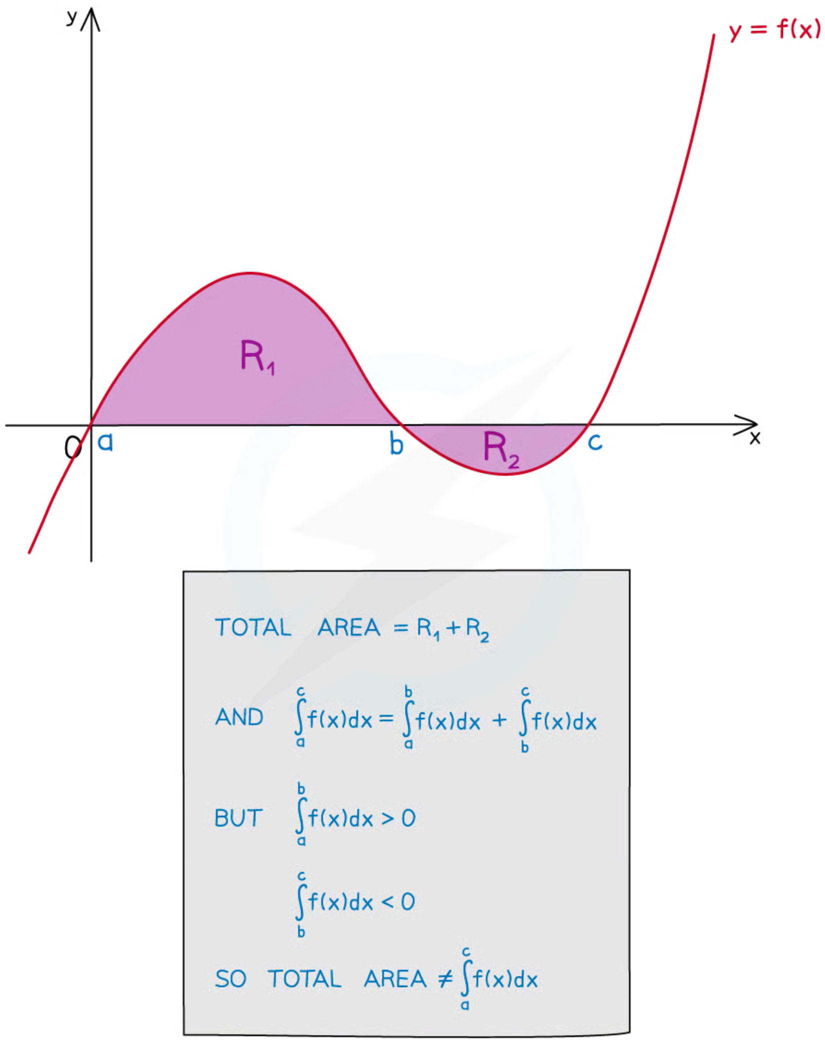
$$= -\frac{22}{3}$$

$\therefore$ AREA OF $R = \frac{22}{3}$ SQUARE UNITS

$$\left|-\frac{22}{3}\right| = \frac{22}{3}$$

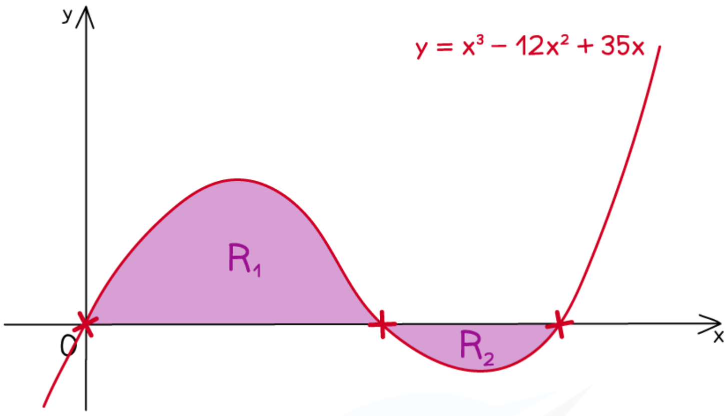
How do I find total areas without areas cancelling each other out?

Your notes



- Be careful when one section is above and one section is below the **x**-axis
  - You will need a separate integral for each section, BUT...
  - ...One section's integral will be **negative**
  - ...One section's integral will be **positive**
  - So you'll need to take the **modulus** before adding to find the total area

e.g. FIND THE TOTAL OF THE SHADED AREAS



FACTORISE TO SOLVE $f(x) = 0$

$$x^3 - 12x^2 + 35x = x(x^2 - 12x + 35)$$
$$= x(x-7)(x-5) = 0$$

$\therefore$ x−AXIS INTERCEPTS ARE 0, 5 AND 7

$$\int_0^5 (x^3 - 12x^2 + 35x)dx = \frac{375}{4}$$

$$\int_5^7 (x^3 - 12x^2 + 35x)dx = -8$$

FIND INTEGRALS SEPARATELY

USE CALCULATOR ?

$\therefore$ AREA OF $R_1 = \frac{375}{4}$

AREA OF $R_2 = 8$

MODULUS NEEDED

TOTAL SHADED AREA $= \frac{375}{4} + 8 = \frac{407}{4}$ SQUARE UNITS

Examiner Tips and Tricks

- Add information to any diagram provided in the question, as well as axes intercepts and values of limits.
- Mark and shade the area you're trying to find, and if no diagram is provided, **sketch** one!

Your notes

Worked Example

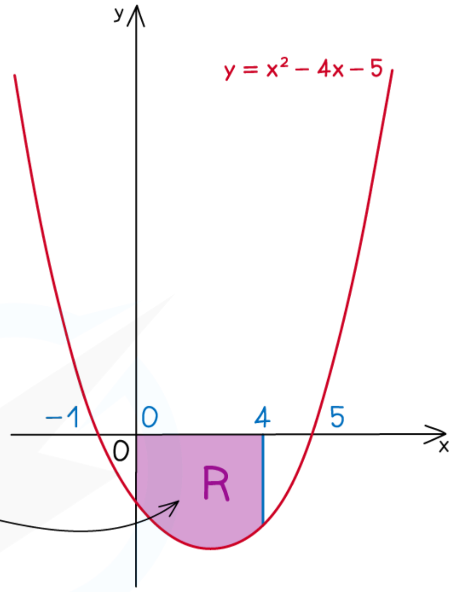
Your notes

Find the area bounded by the curve with equation $y = x^2 - 4x - 5$, the x-axis and the lines $x = 0$ and $x = 4$.

SKETCH AS NO DIAGRAM GIVEN

$x^2 - 4x - 5 = (x-5)(x+1) = 0$

$\therefore$ x–AXIS INTERCEPTS ARE -1 AND 5



EXPECT NEGATIVE AS BELOW x–AXIS

$$\int_0^4 (x^2 - 4x - 5)\,dx = \left[\frac{x^3}{3} - 2x^2 - 5x\right]_0^4$$

WRITE AS AN INTEGRAL

$$= \left(\frac{64}{3} - 32 - 20\right) - (0 - 0 - 0)$$

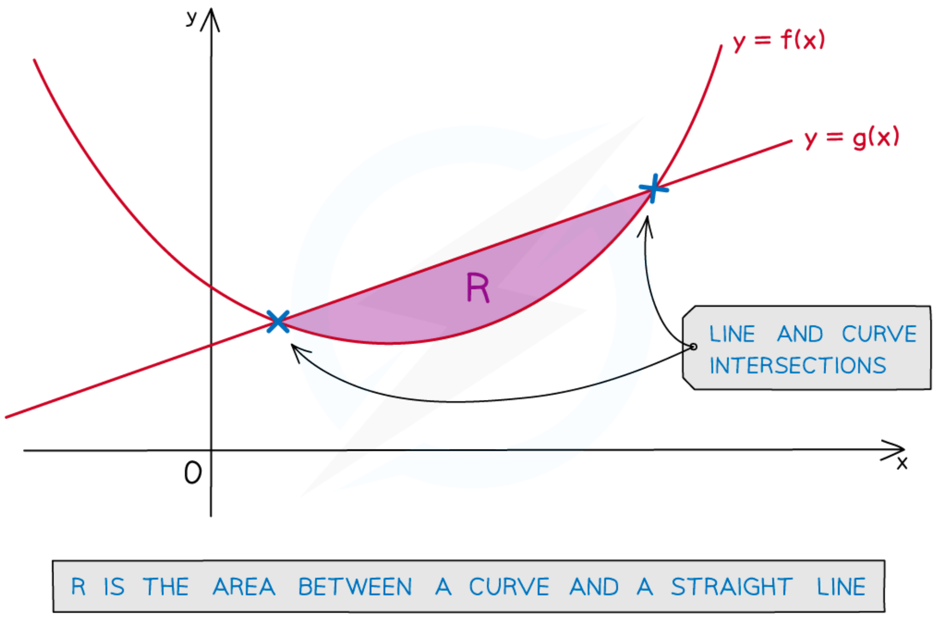
$$= -\frac{92}{3}$$

$\therefore$ AREA REQUIRED $= \frac{92}{3}$ SQUARE UNITS

MODULUS

Area between a curve and a line

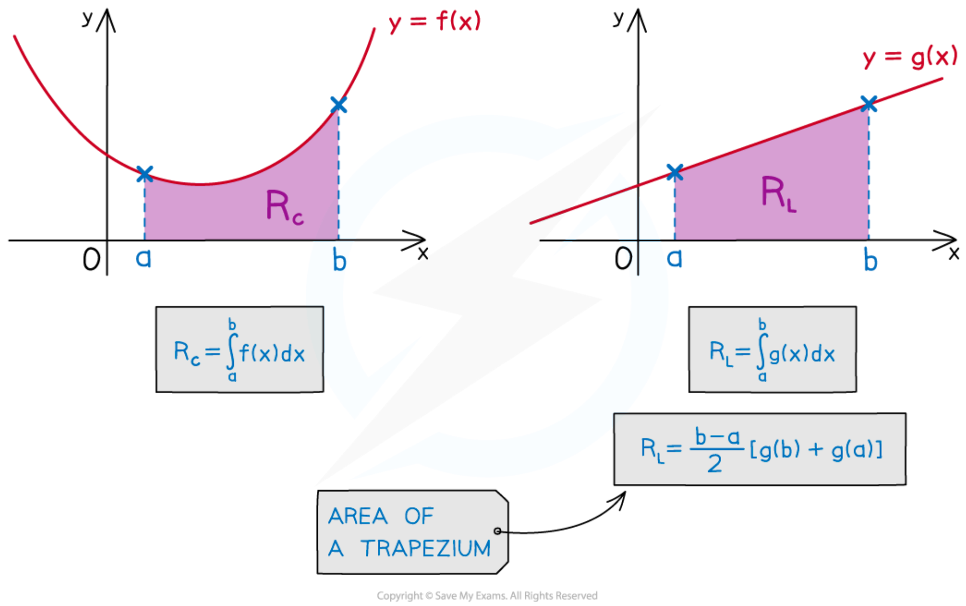
What is the area between a curve and a straight line?



- This is the area enclosed by a curve and a line between their **intersections**
- The intersections may need to be found using simultaneous equations

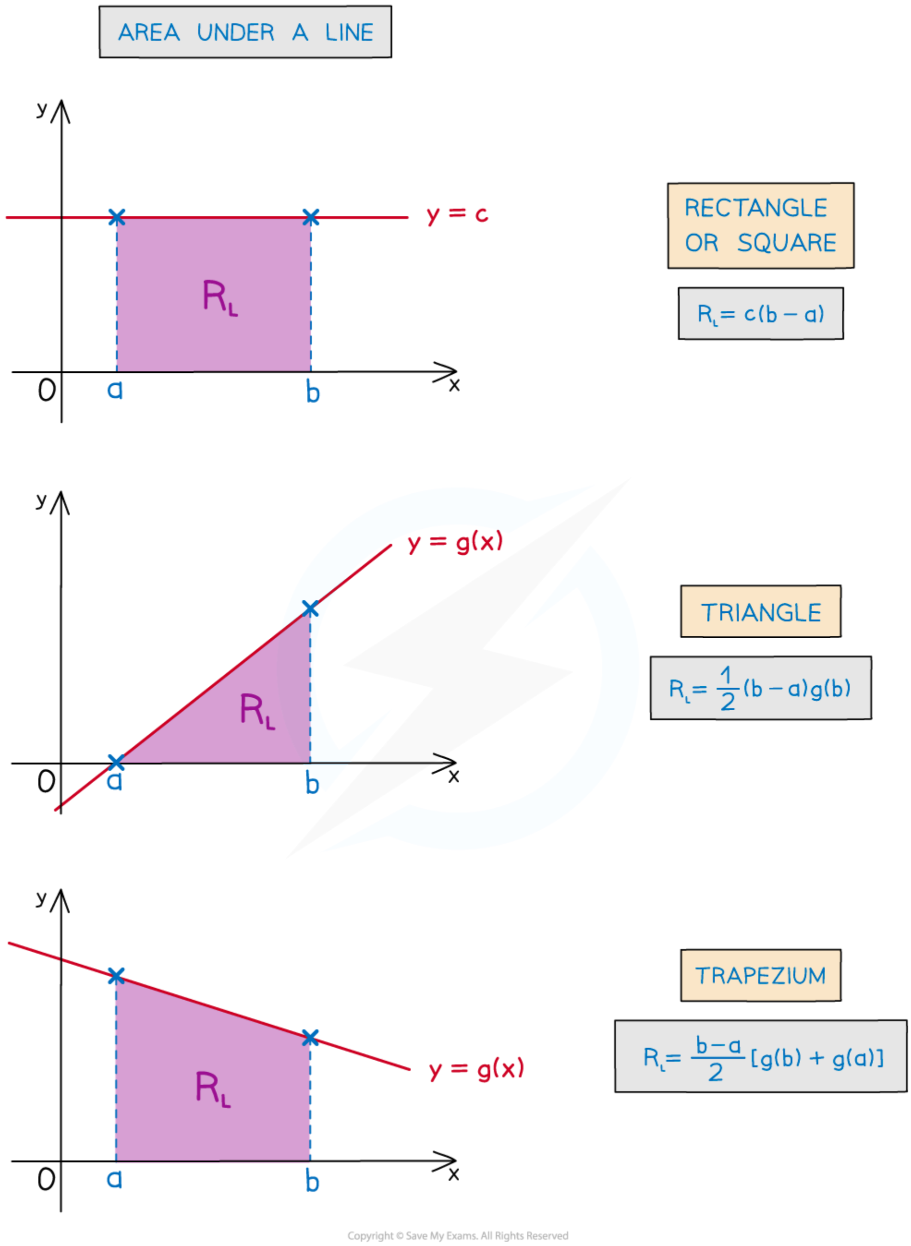
How do I find the area between a curve and a line by separating into two areas?

- The area enclosed will be the **difference** between ...
  - the **area under the curve** and
  - the **area under** the line
- These can be found separately

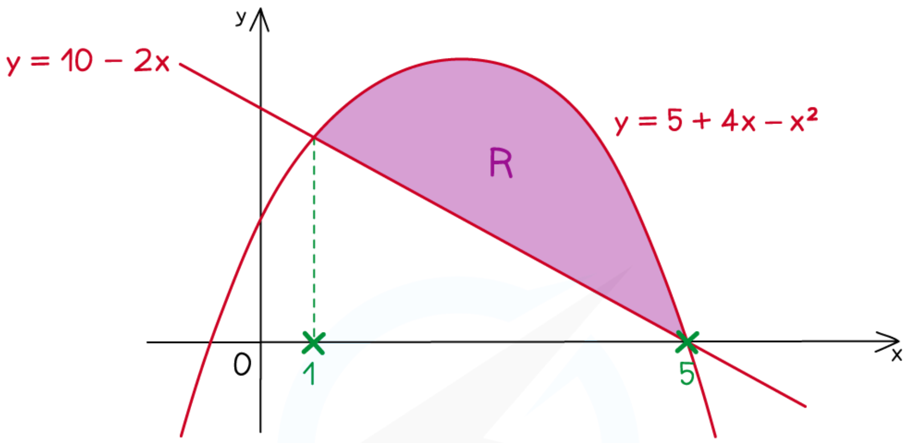


- There may be easier ways to find the area under a line

Your notes



- Once the areas are found a question can be answered in full e.g. FIND THE AREA MARKED R IN THE DIAGRAM BELOW.



STEP 1 FIND THE INTERSECTIONS OF CURVE AND LINE

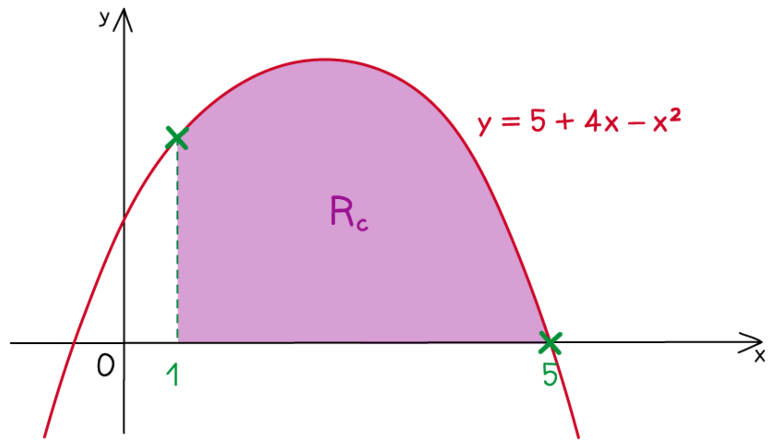
$10 - 2x = 5 + 4x - x^2$ ← SOLVE SIMULTANEOUSLY

$x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0$

$x = 5, \quad x = 1$

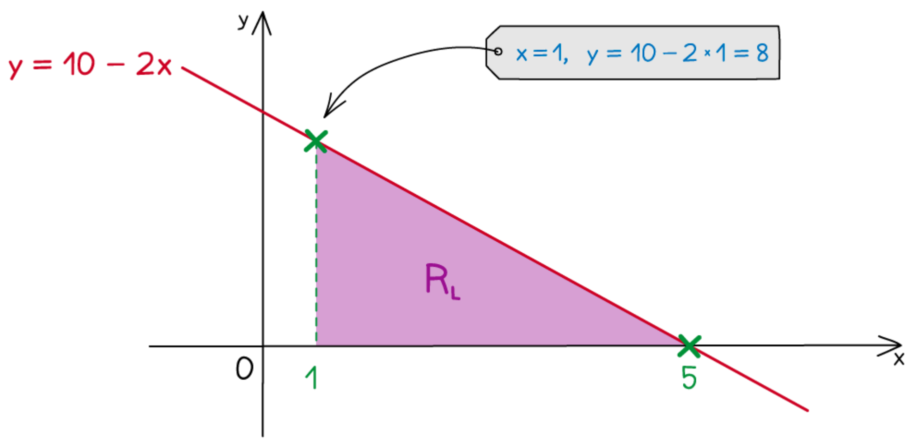
STEP 2 FIND AREA UNDER THE CURVE



$$R_c = \int_1^5 (5 + 4x - x^2)\,dx$$ ← DEFINITE INTEGRATION

$$R_c = \frac{80}{3}$$ ← USE CALCULATOR

STEP 3 FIND AREA UNDER THE LINE



$R_L = \frac{1}{2} \times 4 \times 8$ ← AREA OF TRIANGLE $\frac{1}{2} \times b \times h$

$R_L = 16$

STEP 4 SUBTRACT SMALLER AREA FROM LARGER AREA

$R = R_C - R_L$ $\quad \boxed{R_C > R_L}$

$R = \frac{80}{3} - 16$

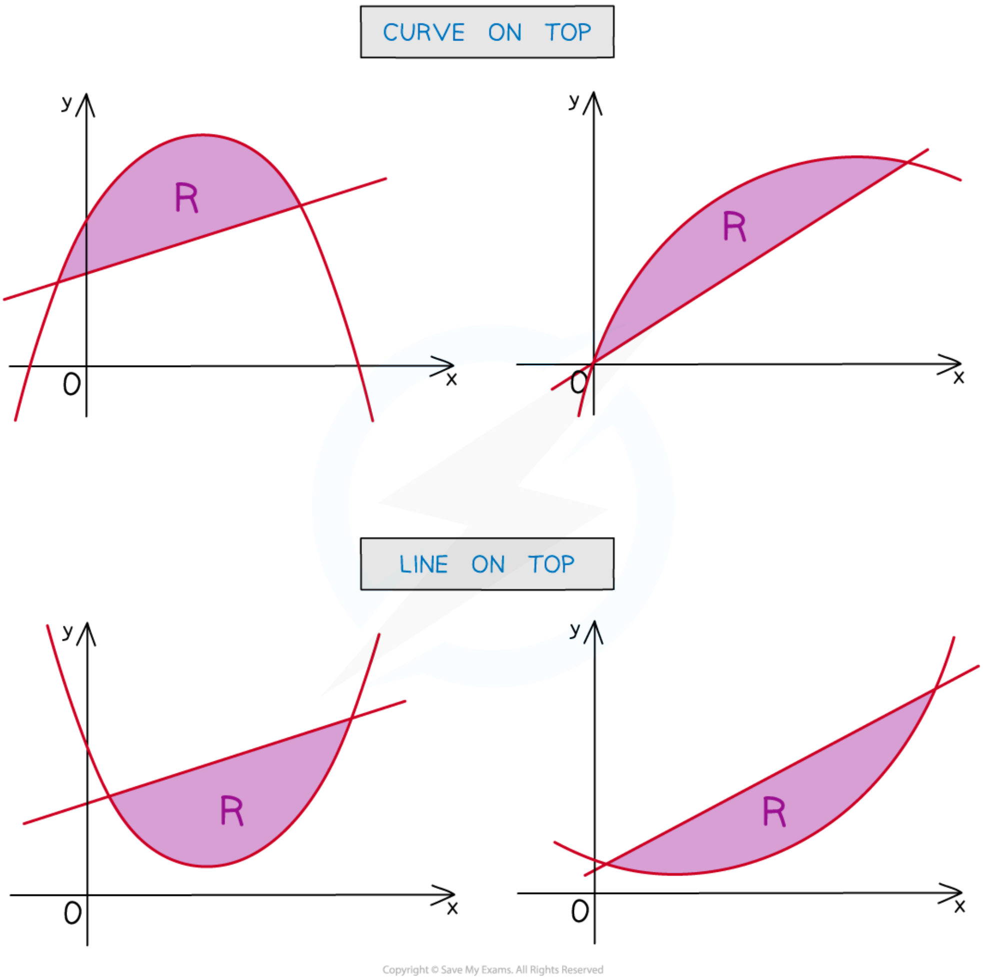
$R = \frac{32}{3}$ SQUARE UNITS

- STEP 1: Find the intersections of the line and the curve
- STEP 2: Find the area under a curve, R_C, using definite integration
- STEP 3: Find the area under a line, R_L, either using definite integration **or** the area formulae for basic shapes
- STEP 4: To find the area, **R**, between the curve and the line **subtract** the **smaller** area **from** the **larger** area
  - If curve on top this will be $R_C - R_L$
  - If line on top this will be $R_L - R_C$

How do I find the area between a curve and a line by subtracting their equations?

- An alternative method allows subtraction of the two functions first
- But be extremely careful as to whether the curve or the line is on top

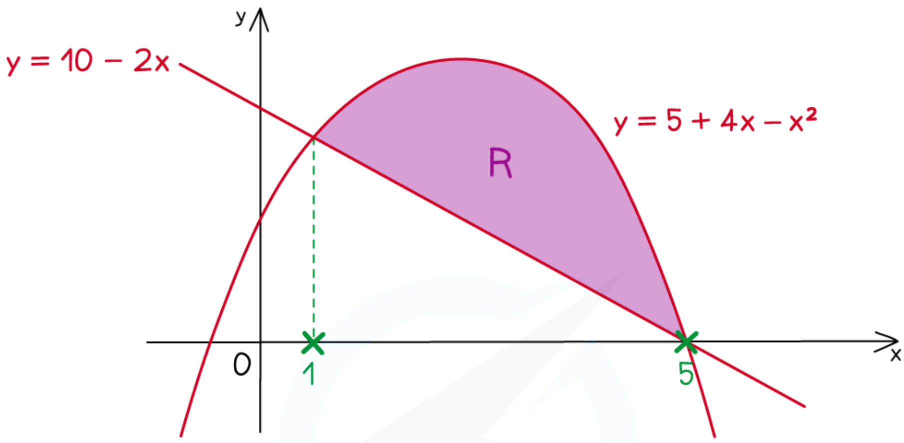
Your notes



- STEP 1: Find the intersections of the line and the curve
- STEP 2: Subtract the functions algebraically
  - Ensure this is done the correct way round!
- STEP 3: Find the area, **R**, between the curve and the line using **definite integration**

Your notes

e.g. FIND THE AREA MARKED R IN THE DIAGRAM BELOW.



STEP 1 FIND THE INTERSECTIONS OF CURVE AND LINE

$10 - 2x = 5 + 4x - x^2$

$x^2 - 6x + 5 = 0$

$(x - 5)(x - 1) = 0$

$x = 5, \quad x = 1$

SOLVE SIMULTANEOUSLY

STEP 2 SUBTRACT THE FUNCTIONS

$5 + 4x - x^2 - (10 - 2x) = -x^2 + 6x - 5$

CURVE ON TOP

SUBTRACT "ALL" OF LINE

STEP 3 DEFINITE INTEGRATION FINDS R

$R = \int_1^5 (-x^2 + 6x - 5)\,dx$

$R = \frac{32}{3}$ SQUARE UNITS

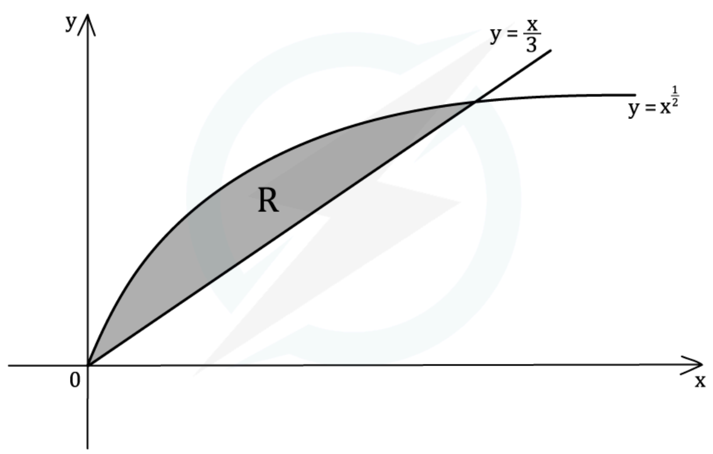
Your notes

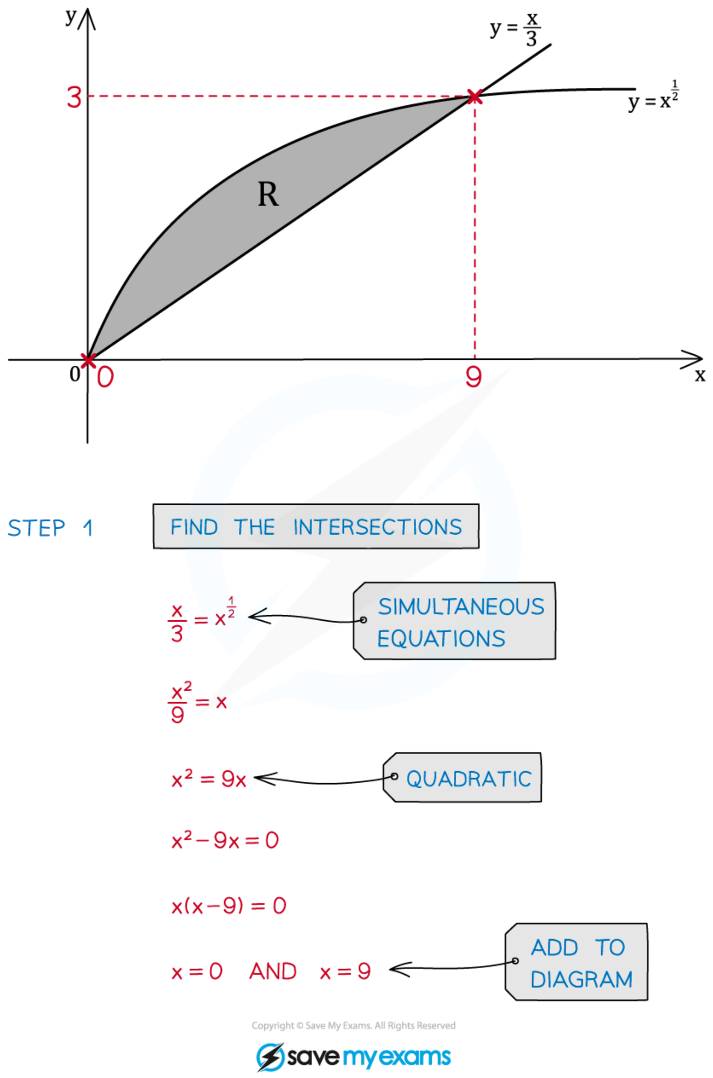
Examiner Tips and Tricks

- Add information to any diagram provided
- Add axes intercepts, as well as intercepts between lines and curves
- Mark and shade the area you're trying to find
- If no diagram provided, **sketch** one!

Worked Example

Find the area indicated by R in the diagram below.





y
y = x/3
y = x^(1/2)
3
R
0
0
9
x
STEP 1
FIND THE INTERSECTIONS
$\frac{x}{3} = x^{\frac{1}{2}}$
SIMULTANEOUS EQUATIONS
$\frac{x^2}{9} = x$
$x^2 = 9x$
QUADRATIC
$x^2 - 9x = 0$
$x(x-9) = 0$
$x = 0$ AND $x = 9$
ADD TO DIAGRAM
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Your notes

STEP 2 FIND THE AREA UNDER THE CURVE

$$R_c = \int_0^9 x^{\frac{1}{2}}\,dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^9$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^9$$

$$= \left(\frac{2}{3}(9)^{\frac{3}{2}}\right) - (0)$$

$$R_c = 18$$

STEP 3 FIND THE AREA UNDER THE LINE

$$R_L = \frac{1}{2} \times 9 \times 3$$

TRIANGLE BASE = 9
HEIGHT = 3

$$R_L = \frac{27}{2}$$

STEP 4 SUBTRACT SMALL FROM LARGE

$$R = R_c - R_L$$

CURVE IS ON TOP

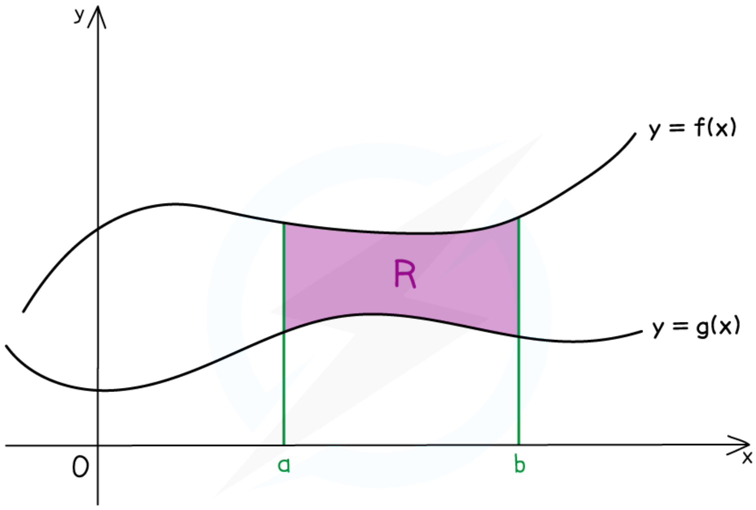
$$R = 18 - \frac{27}{2}$$

$R = \frac{9}{2}$ SQUARE UNITS

Area between 2 curves

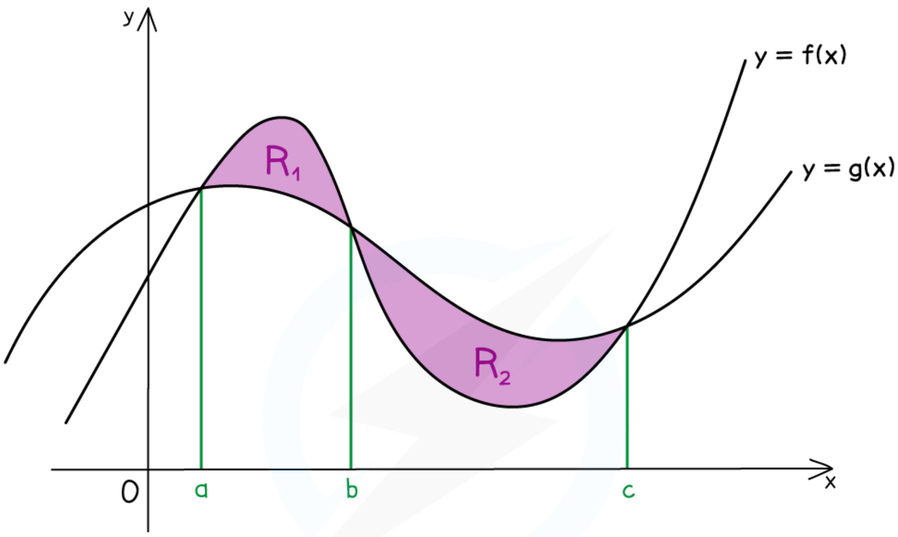
What is the area between two curves?

- Ensure you are familiar with ...
  - Area under a curve
  - Area between a line and a curve



$$R = \int_a^b [f(x) - g(x)]dx$$

- In general find the definite integral of “upper curve” – “lower curve”
- However this does depend on ...
  - ... the area being found
  - ... if the curves intersect (and cross over)
- The area may have to be split into separate integrals



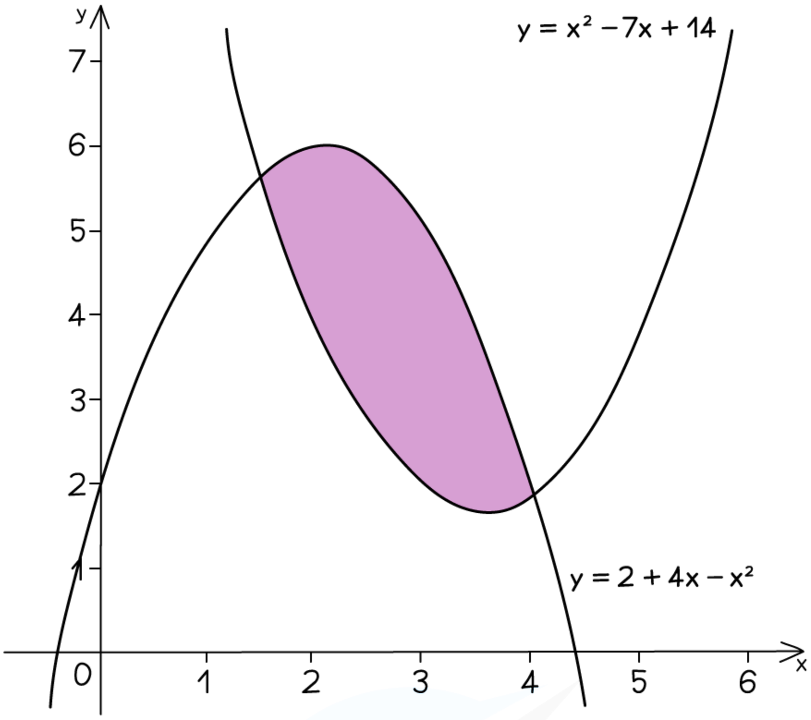
$$R_1 = \int_a^b [f(x) - g(x)]dx$$

$$R_2 = \int_b^c [g(x) - f(x)]dx$$

- The points at which curves intersect may need to be calculated

How do I find the area between two curves?

e.g. FIND THE SHADED AREA IN THE DIAGRAM BELOW



STEP 1: FIND THE INTERSECTIONS

$x^2 - 7x + 14 = 2 + 4x - x^2$

$2x^2 - 11x + 12 = 0$

$(2x - 3)(x - 4) = 0$

$x = \frac{3}{2}, \qquad x = 4$

COULD USE CALCULATOR

STEP 2: FORM INTEGRAL AND FIND ITS VALUE

$$\text{AREA} = \int_{\frac{3}{2}}^{4} \left[(2 + 4x - x^2) - (x^2 - 7x + 14)\right] dx$$

UPPER CURVE

LOWER CURVE

$$\text{AREA} = \int_{\frac{3}{2}}^{4} (-2x^2 + 11x - 12)\, dx$$

USE CALCULATOR

$$\text{AREA} = \frac{125}{24} \text{ SQUARE UNITS}$$

Your notes

Your notes

- **STEP 1**
  Find the intersections of the curves if needed
- **STEP 2**
  Form the integral ...
  - ... using the intersections as limits
  - ... "upper curve" – "lower curve" ...
  - ... and find the value of the integral
- **STEP 3**
  Repeat STEP 2 if more than one area needed
- **STEP 4**
  Add areas together

Examiner Tips and Tricks

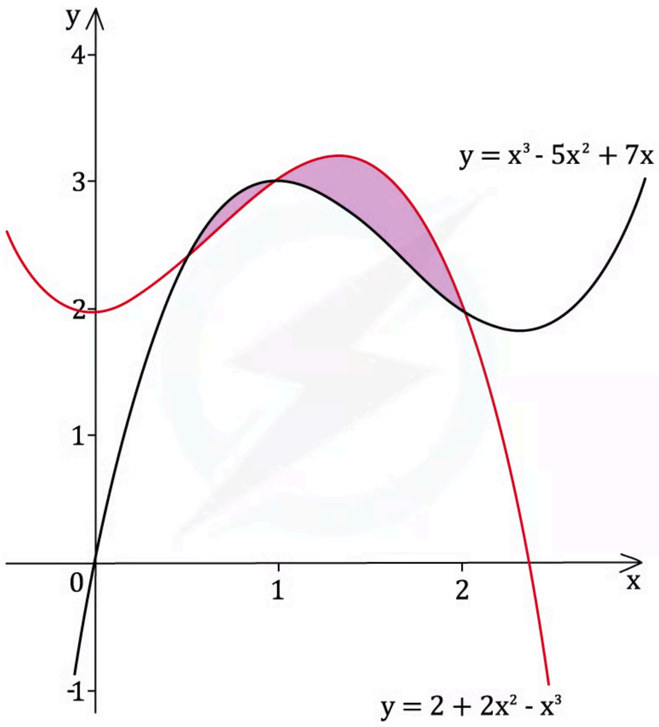
- If no diagram is provided sketch one, even if the curves are not accurate.
- Add information to any given diagram as you work through a question.
- Maximise use of your calculator to save time and maintain accuracy:
  - Solving equations, especially **cubic** equations
  - Finding definite integrals

Worked Example

Your notes

The diagram below shows a sketch of part of the graphs with equations

$y = x^3 - 5x^2 + 7x$ and $y = 2 + 2x^2 - x^3$



Find the shaded area.

Your notes

STEP 1: FIND THE INTERSECTIONS

$x^3 - 5x^2 + 7x = 2 + 2x^2 - x^3$

$2x^3 - 7x^2 + 7x - 2 = 0$ SOLVE USING CALCULATOR

$x = 2, \quad x = 1, \quad x = \frac{1}{2}$

STEP 2: FORM AND FIND INTEGRAL FOR FIRST AREA

$$R_1 = \int_{0.5}^{1} \left[(x^3 - 5x^2 + 7x) - (2 + 2x^2 - x^3)\right] dx$$

$$R_1 = \int_{0.5}^{1} (2x^3 - 7x^2 + 7x - 2) dx$$

USE CALCULATOR

$$R_1 = \frac{5}{96}$$

STEP 3: FORM AND FIND INTEGRAL FOR SECOND AREA

$$R_2 = \int_{1}^{2} \left[(2 + 2x^2 - x^3) - (x^3 - 5x^2 + 7x)\right] dx$$

$$R_2 = \int_{1}^{2} (-2x^3 + 7x^2 - 7x + 2) dx$$

$$R_2 = \frac{1}{3}$$

STEP 4: ADD AREAS TOGETHER

$$\text{SHADED AREAS} = R_1 + R_2 = \frac{5}{96} + \frac{1}{3}$$

$$\text{SHADED AREA} = \frac{37}{96} \text{ SQUARE UNITS}$$

Your notes

Trapezium rule (numerical integration)

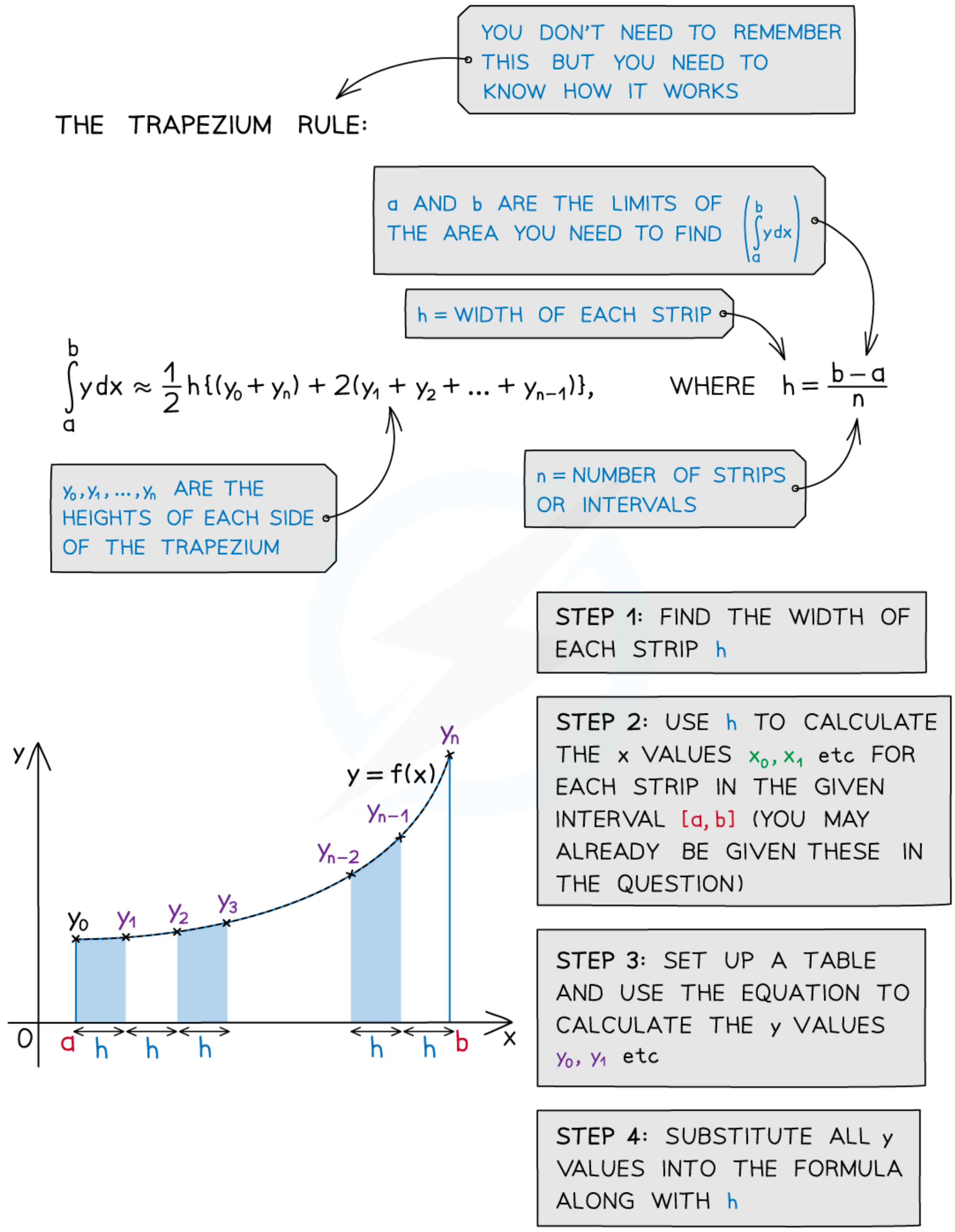
What is the trapezium rule?

- The **trapezium rule** is a numerical method of integration used to find the **approximate area** under a curve
- It can be used when **analytical methods** of **integration** won't work
- It finds an **approximation** of the area by finding the **sum of the areas** of trapeziums beneath the curve
- The **formula** for an **estimate** of the area when splitting it into n strips (trapezium) is

$$\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\} \text{ where } h = \frac{b-a}{n}$$

- If you have n **strips** then you need $n + 1$ y**-values**

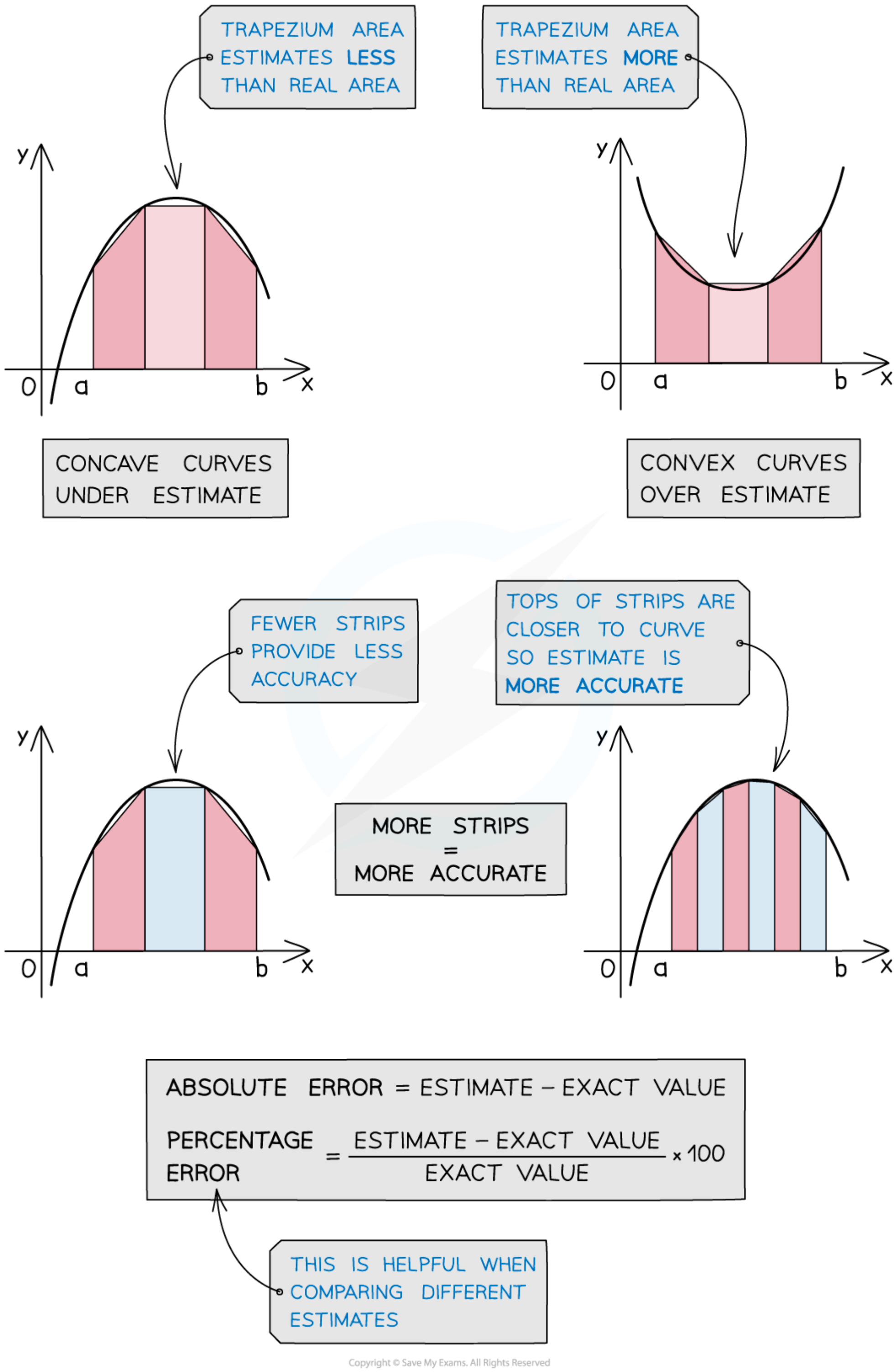
Your notes



How do I know if it is an underestimate or an overestimate?

- Depending on the shape of the curve, the trapezium rule could be an **underestimate** or **overestimate**
- The **more trapezium strips** used the **more accurate** the estimate
- Calculating the **percentage error** can tell you how **accurate** your estimate is

Your notes



Examiner Tips and Tricks

The formula for the trapezium rule is given in the formula booklet.

Your notes

Worked Example

Your notes

a) Using the trapezium rule, find an approximate value for $\int_0^4 \frac{6x^2}{x^3 + 2} dx$, to 3 decimal places, using n = 4.

b) Given that the exact value of $\int_0^4 \frac{6x^2}{x^3 + 2} dx =$ 6.993 to 3 decimal places, calculate the percentage error for a) to 2 decimal places.

a) $h = \frac{4-0}{4} = 1$

STEP 1: FIND $h = \frac{b-a}{n}$ USING $\int_{a=0}^{b=4}$ AND $n = 4$

STEP 2: USING WIDTH $h = 1$, VALUES ARE 0, 1, 2, 3, 4

STEP 3: SETUP TABLE AND USE EQUATION TO CALCULATE y VALUES

x	$y = \frac{6x^2}{x^3+2}$
$x_0 = 0$	$y_0 = 0$
$x_1 = 1$	$y_1 = 2$
$x_2 = 2$	$y_2 = 2.4$
$x_3 = 3$	$y_3 = 1.862...$
$x_4 = 4$	$y_4 = 1.454...$

STEP 4: USING FORMULA FROM FORMULA BOOKLET

$$\int_a^b y\,dx \approx \frac{1}{2}h(y_0 + y_n + 2(y_1 + y_2 + ... + y_{n-1}))$$

SUBSTITUTE ALL VALUES

$$\frac{1}{2} \times 1 \times (0 + 1.454... + 2(2 + 2.4 + 1.862...))$$

$$= \frac{1}{2}(1.454... + 12.524...)$$

$$= 6.9893... = 6.989 \quad (3\text{ dp})$$

USING $\%\text{ ERROR} = \frac{\text{ESTIMATE} - \text{EXACT}}{\text{EXACT}} \times 100$

b) $\frac{6.989 - 6.993}{6.993} \times 100$

$= -0.\dot{0}5720\dot{0}$

IGNORE MINUS, YOU COULD DO EXACT − ESTIMATE TO GET RID OFF THIS

$= 0.06\%$ (2 dp)