Edexcel International AS Maths: Pure 2

Your notes

Applications of Differentiation

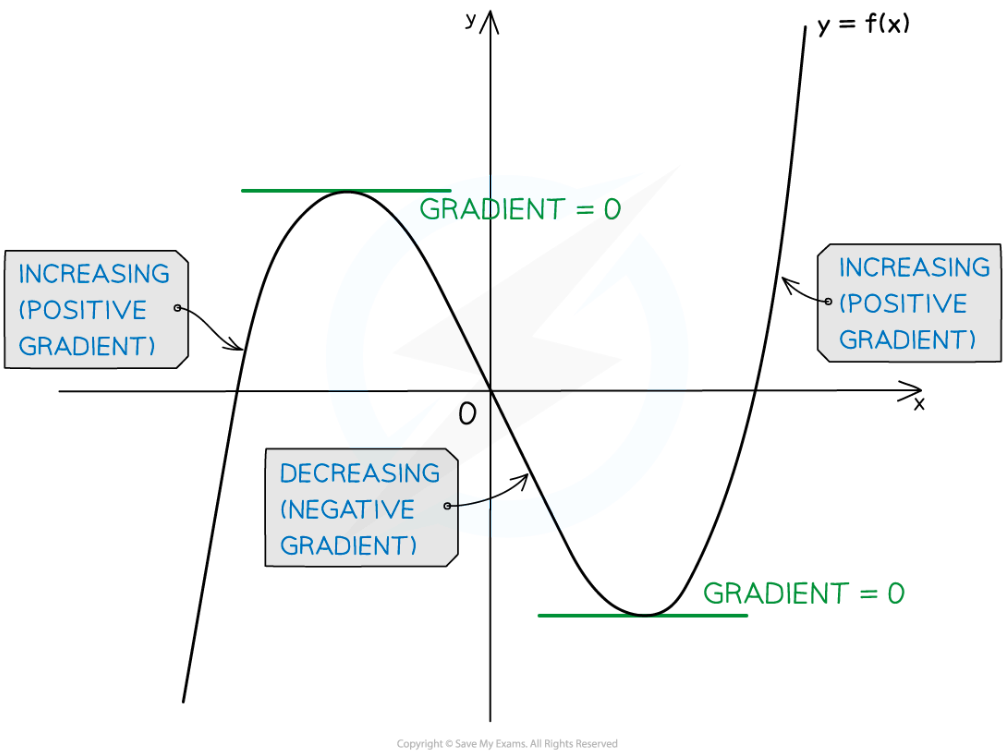
Contents

Your notes

Increasing & decreasing functions

What are increasing and decreasing functions?

- A function **f(x)** is **increasing** on an interval **[a, b]** if **f'(x) ≥ 0** for all values of x such that $a < x < b$.
    - If **f'(x) > 0** for all x values in the interval then the function is said to be **strictly increasing**
    - In most cases, on an increasing interval the graph of a function goes **up** as x increases
- A function **f(x)** is **decreasing** on an interval **[a, b]** if **f'(x) ≤ 0** for all values of x such that $a < x < b$
    - If **f'(x) < 0** for all x values in the interval then the function is said to be **strictly decreasing**
    - In most cases, on a decreasing interval the graph of a function goes **down** as x increases



- To identify the intervals on which a function is increasing or decreasing you need to:

1. Find the derivative **f'(x)**

2. Solve the inequalities **f'(x) ≥ 0** (for increasing intervals) and/or **f'(x) ≤ 0** (for decreasing intervals)

Your notes

Examiner Tips and Tricks

- On an exam, if you need to show a function is increasing or decreasing you can use either strict (<, >) or non-strict (≤, ≥) inequalities

Worked Example

Your notes

$f(x) = x^3 + x^2 - x + 2$

a) Find the values of x for which $f(x)$ is an increasing function.

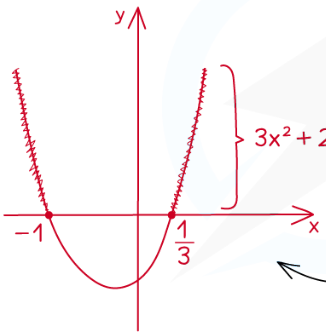
b) Find the values of x for which $f(x)$ is a decreasing function.

a) $f'(x) = 3x^2 + 2x - 1$ ← FIND THE DERIVATIVE

$3x^2 + 2x - 1 = 0$

$(3x-1)(x+1) = 0$

$x = \frac{1}{3}$ OR $x = -1$ ← SOLVE THE QUADRATIC



USE SKETCH TO IDENTIFY INTERVALS SATISFYING THE INEQUALITY

f(x) IS INCREASING WHERE $f'(x) \geqslant 0$

SO f(x) IS INCREASING ON THE INTERVALS $(-\infty, -1]$ AND $[\frac{1}{3}, +\infty)$

b) f(x) IS DECREASING WHERE $f'(x) \leqslant 0$

SO f(x) IS DECREASING ON THE INTERVAL $[-1, \frac{1}{3}]$

Your notes

Stationary points & turning points

What are stationary points?

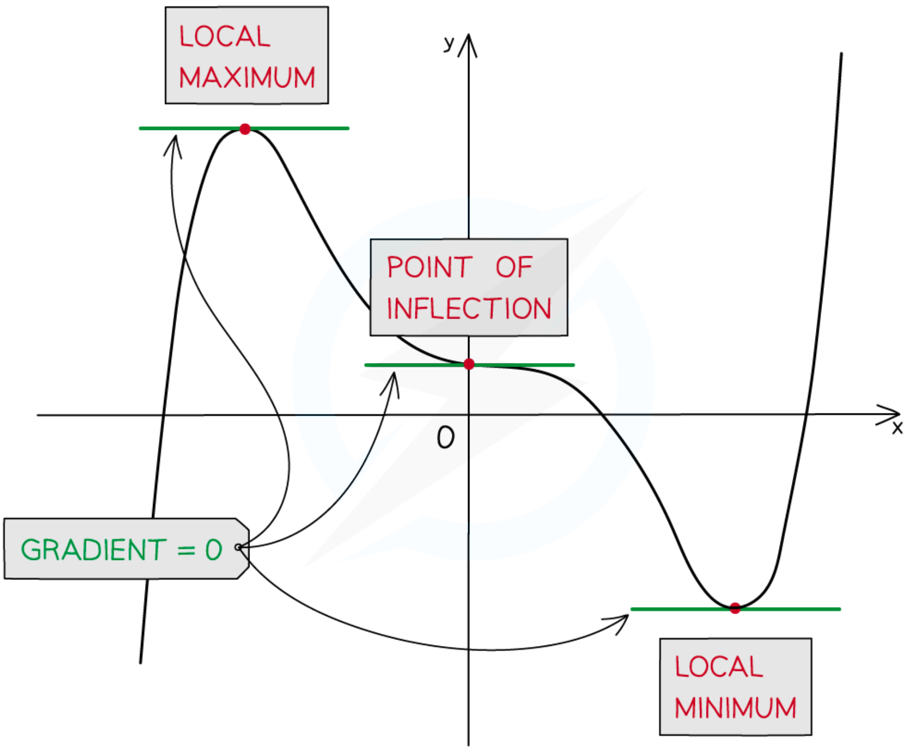
- A stationary point is any point on a curve where the gradient is zero
- To find stationary points of a function **f(*x*)**

Step 1: Find the first derivative **f'(*x*)**

Step 2: Solve **f'(*x*) = 0** to find the *x*-coordinates of the stationary points

Step 3: Substitute those *x*-coordinates into **f(*x*)** to find the corresponding *y*-coordinates

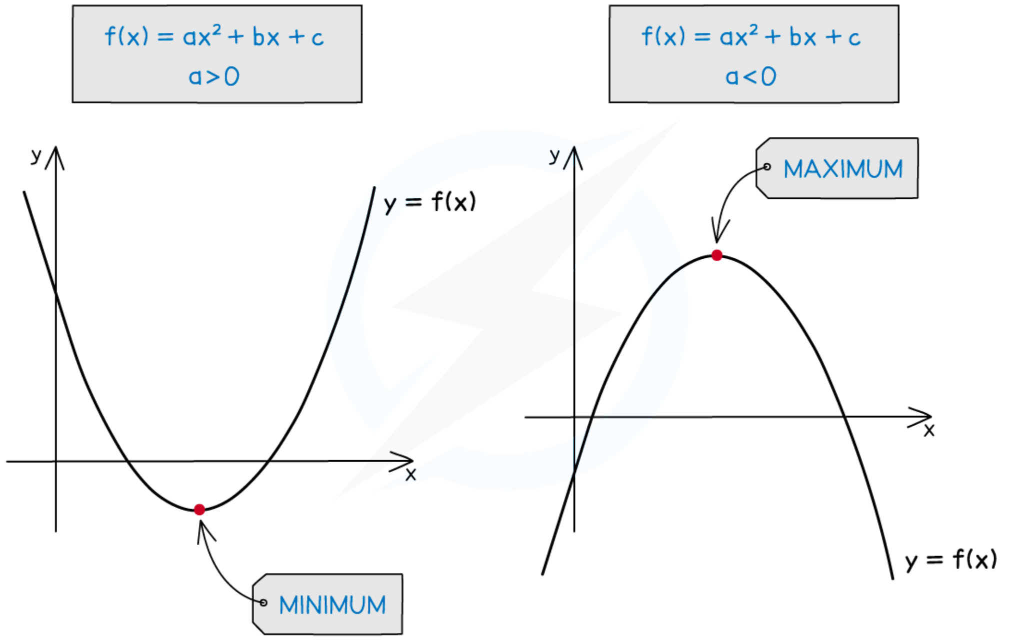
- A stationary point may be either a **local minimum**, a **local maximum**, or a **point of inflection**
- **Turning points** are maximum points or minimum points only
    - The graph turns from going up to down (or down to up)



How do I determine the type of stationary point on a quadratic?

Your notes

- The graph of a quadratic function (ie a parabola) only has a single stationary point
- For an 'up' parabola this is the **minimum**; for a 'down' parabola it is the **maximum** (no need to talk about 'local' here)

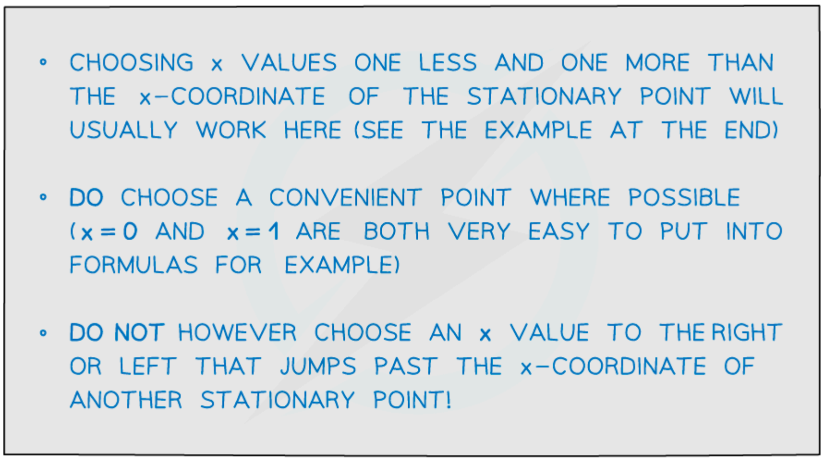


- The ***y*** value of the stationary point is thus the minimum or maximum value of the quadratic function
- Quadratics can only have minimum and maximum points
  - so quadratics have **turning points** (also called a **vertex**)

How do I determine the type of stationary point on a curve?

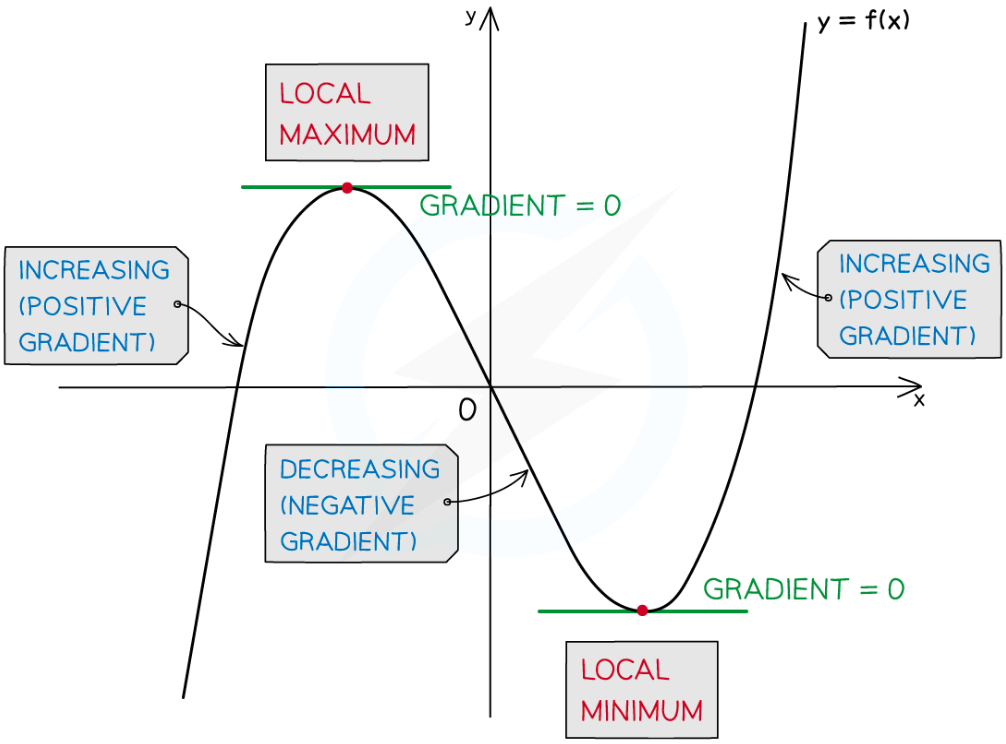
- For a function **f(*x*)** there are two ways to determine the nature of its stationary points
- **Method A:** Compare the signs of the **first derivative** (positive or negative) a little bit **to either side of the stationary point**
  - (After completing **Steps 1 – 3 above** to find the stationary points)

Step 4: For each stationary point find the values of the first derivative a little bit 'to the left' (ie slightly smaller *x* value) and a little bit 'to the right' (slightly larger *x* value) of the stationary point

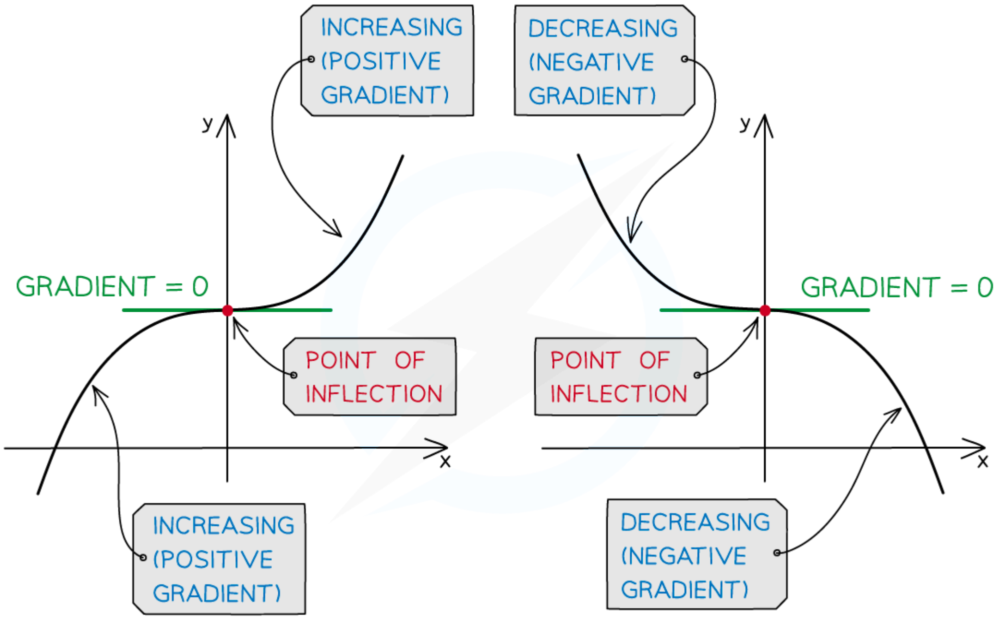


Step 5: Compare the **signs** (positive or negative) of the derivatives on the left and right of the stationary point

- If the derivatives are **negative** on the **left** and **positive** on the **right**, the point is a **local minimum**
- If the derivatives are **positive** on the **left** and **negative** on the **right**, the point is a **local maximum**
- If the signs of the derivatives are the **same** on both sides (both positive or both negative) then the point is a **point of inflection**



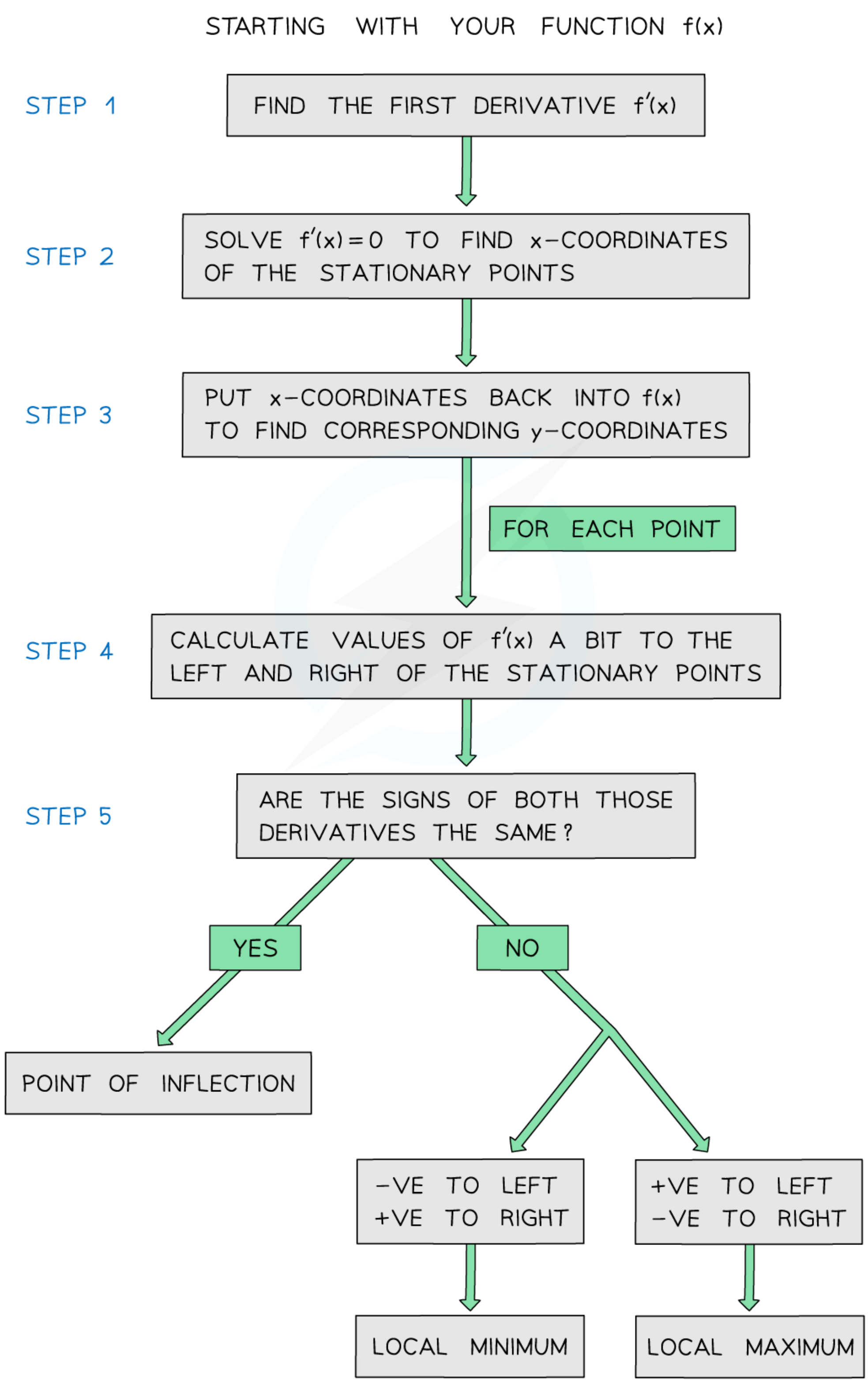
Your notes


INCREASING (POSITIVE GRADIENT)
DECREASING (NEGATIVE GRADIENT)
y
GRADIENT = 0
POINT OF INFLECTION
x
INCREASING (POSITIVE GRADIENT)
y
GRADIENT = 0
POINT OF INFLECTION
x
DECREASING (NEGATIVE GRADIENT)

Your notes

Your notes

METHOD A: DETERMINING NATURE OF STATIONARY POINTS USING FIRST DERIVATIVE



Method B: Look at the sign of the **second derivative** (positive or negative) **at the stationary point**

(After completing **Steps 1 – 3 above** to find the stationary points)

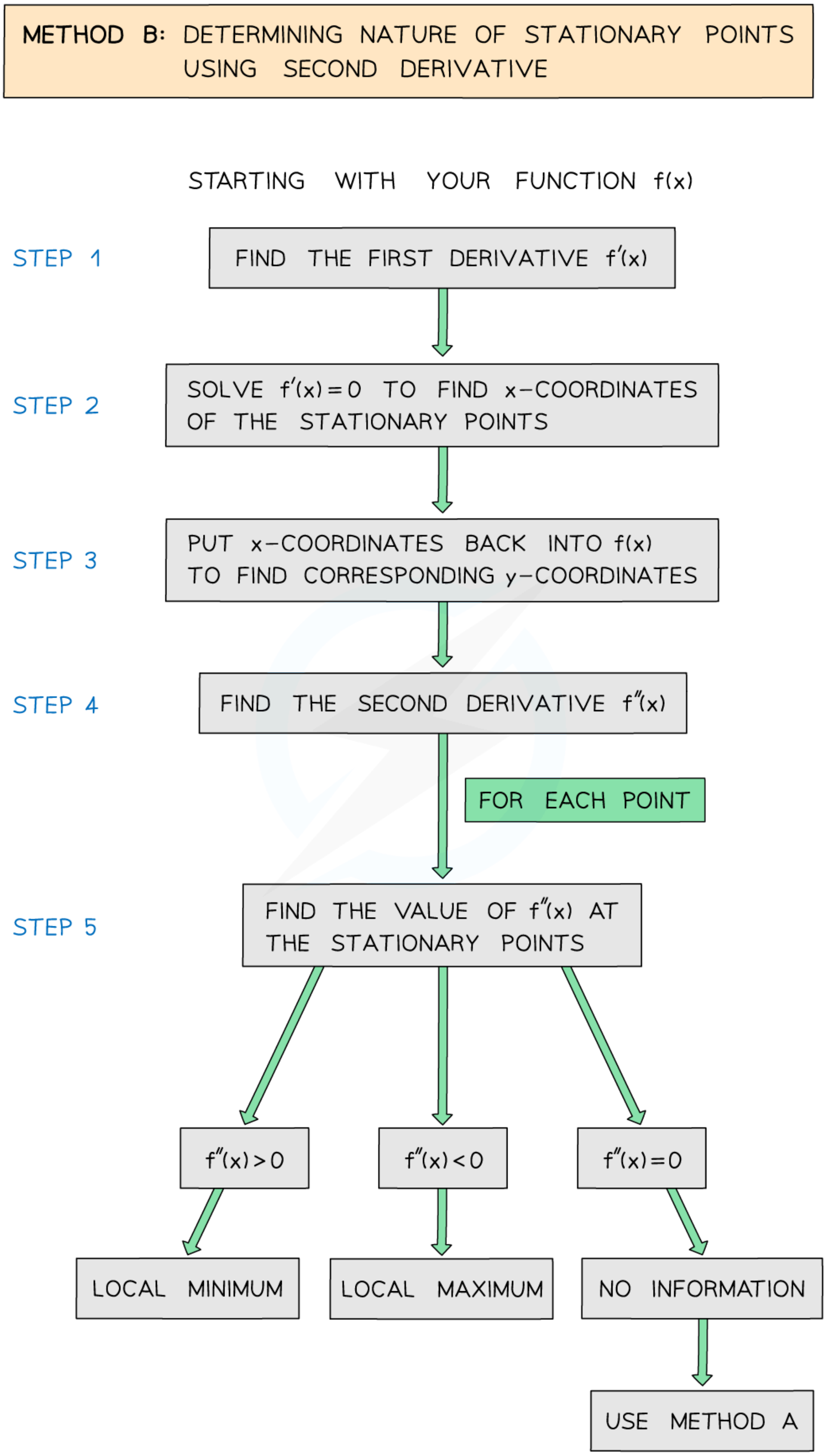
Step 4: Find the second derivative **f''(*x*)**

Step 5: For each stationary point find the value of **f''(*x*)** at the stationary point (ie substitute the *x*-coordinate of the stationary point into **f''(*x*)**)

- If **f''(*x*)** is **positive** then the point is a **local minimum**
- If **f''(*x*)** is **negative** then the point is a **local maximum**
- If **f''(*x*)** is **zero** then the point could be a **local minimum**, a **local maximum** OR a **point of inflection** (use Method A to determine which)

Your notes

Your notes


METHOD B: DETERMINING NATURE OF STATIONARY POINTS USING SECOND DERIVATIVE
STARTING WITH YOUR FUNCTION f(x)
STEP 1
FIND THE FIRST DERIVATIVE f′(x)
STEP 2
SOLVE f′(x) = 0 TO FIND x−COORDINATES OF THE STATIONARY POINTS
STEP 3
PUT x−COORDINATES BACK INTO f(x) TO FIND CORRESPONDING y−COORDINATES
STEP 4
FIND THE SECOND DERIVATIVE f″(x)
FOR EACH POINT
STEP 5
FIND THE VALUE OF f″(x) AT THE STATIONARY POINTS
f″(x) > 0
f″(x) < 0
f″(x) = 0
LOCAL MINIMUM
LOCAL MAXIMUM
NO INFORMATION
USE METHOD A

Your notes

Examiner Tips and Tricks

- Usually using the second derivative (Method B above) is a much quicker way of determining the nature of a stationary point.
- However, if the second derivative is zero it tells you nothing about the point.
    - In that case you will have to use Method A (which **always** works – see the Worked Example).

Worked Example

Your notes

Find the stationary points of $f(x) = x^3(3x^2 - 20)$,

and determine the nature of each.

$f(x) = 3x^5 - 20x^3$ ← EXPAND THE BRACKETS

$f'(x) = 15x^4 - 60x^2$ ← STEP 1: FIND FIRST DERIVATIVE

$15x^4 - 60x^2 = 0$ ← STEP 2: SOLVE $f'(x) = 0$

$x^4 - 4x^2 = 0$

$x^2(x^2 - 4) = 0$

$x^2(x+2)(x-2) = 0$

$x = 0, -2, 2$ ← x-COORDINATES OF THE TURNING POINTS

IF $x = 0$, $y = (0)^3(3(0)^2 - 20) = 0$

IF $x = -2$, $y = (-2)^3(3(-2)^2 - 20) = 64$

IF $x = 2$, $y = (2)^3(3(2)^2 - 20) = -64$

STEP 3: FIND CORRESPONDING y-COORDINATES

THE STATIONARY POINTS ARE
(−2, 64), (0, 0) AND (2, −64)

Your notes

START WITH METHOD B

$f''(x) = 60x^3 - 120x$

STEP 4: FIND SECOND DERIVATIVE

$f''(-2) = 60(-2)^3 - 120(-2) = -240 < 0$

$f''(0) = 60(0)^3 - 120(0) = 0$

$f''(2) = 60(2)^3 - 120(2) = 240 > 0$

STEP 5: TEST SECOND DERIVATIVE AT STATIONARY POINTS

(−2,64) IS A LOCAL MAXIMUM

(2,−64) IS A LOCAL MINIMUM

NOTE THAT AT (0,0) THE SECOND DERIVATIVE IS ZERO, AND SO DOESN'T TELL US WHAT THE POINT IS!

SWITCH TO METHOD A

TEST $f'(x)$ AT $x = -1$ AND $x = 1$

STEP 4: COMPARE FIRST DERIVATES TO LEFT AND RIGHT OF STATIONARY POINT

$f'(-1) = 15(-1)^4 - 60(-1)^2 = -45 < 0$

$f'(1) = 15(1)^4 - 60(1)^2 = -45 < 0$

STEP 5: INTERPRET RESULT

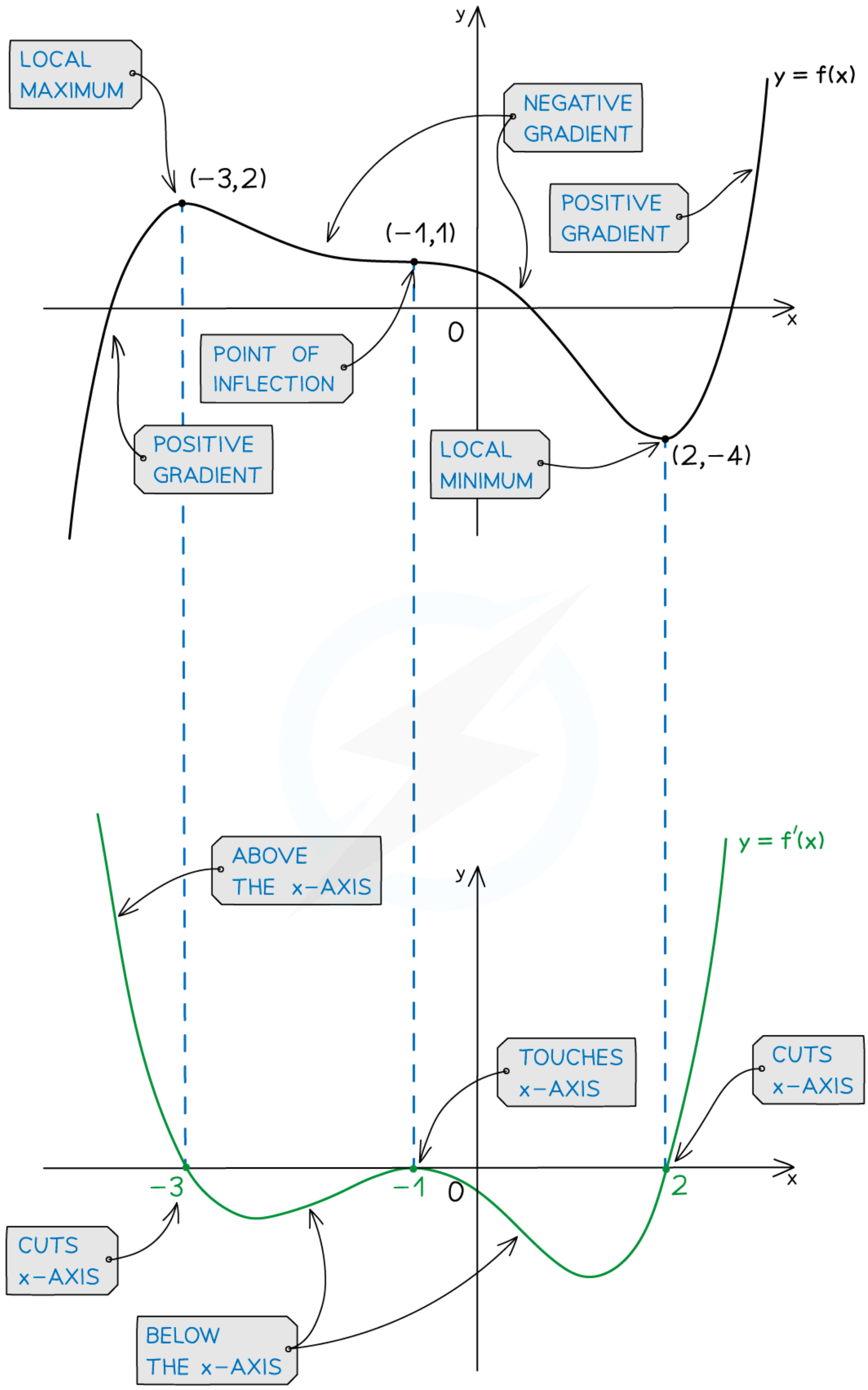
$f'(x)$ HAS SAME SIGN ON BOTH SIDES OF (0,0)

(0,0) IS A POINT OF INFLECTION

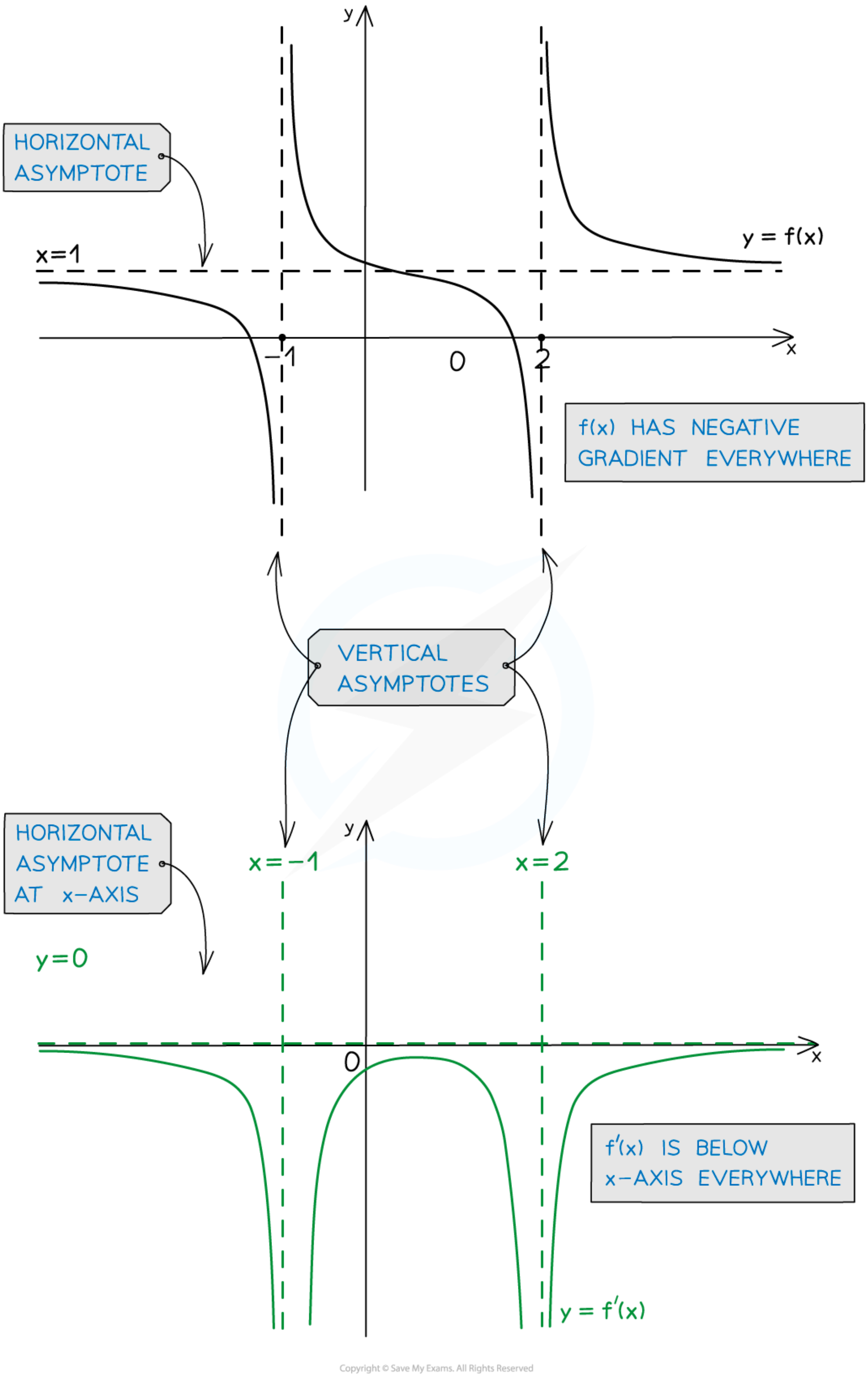
Sketching gradient functions

How do I sketch the gradient function from the original function?

- Using your knowledge of gradients and derivatives you can use the graph of a function to sketch the corresponding gradient function
- The behaviour of a function tells you about the behaviour of its gradient function



y
y = f(x)
LOCAL
MAXIMUM
(−3,2)
NEGATIVE
GRADIENT
POSITIVE
GRADIENT
(−1,1)
0
x
POINT OF
INFLECTION
POSITIVE
GRADIENT
LOCAL
MINIMUM
(2,−4)
y = f′(x)
ABOVE
THE x−AXIS
y
TOUCHES
x−AXIS
CUTS
x−AXIS
−3
−1
0
2
x
CUTS
x−AXIS
BELOW
THE x−AXIS



Your notes

Examiner Tips and Tricks

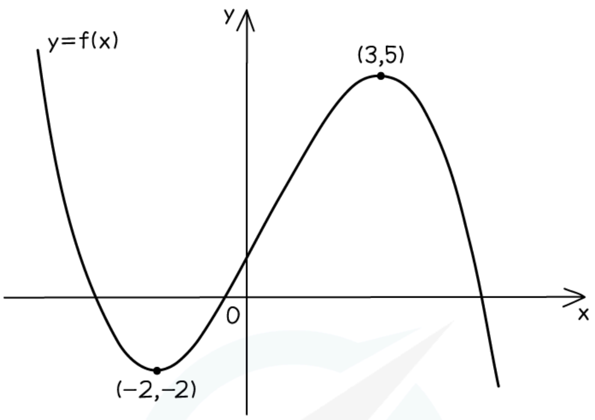
- If **f(x)** is a smooth curve then **f'(x)** will also be a smooth curve.
- Take what you know about **f'(x)** (based on the table above) and then 'fill in the blanks' in between.
- If all you have is the graph of **f(x)** you will not be able to specify the coordinates of the y-intercept or any stationary points of **f'(x).**
- Be careful – points where **f(x)** cuts the x-axis don't tell you anything about the graph of **f'(x)**!

Your notes

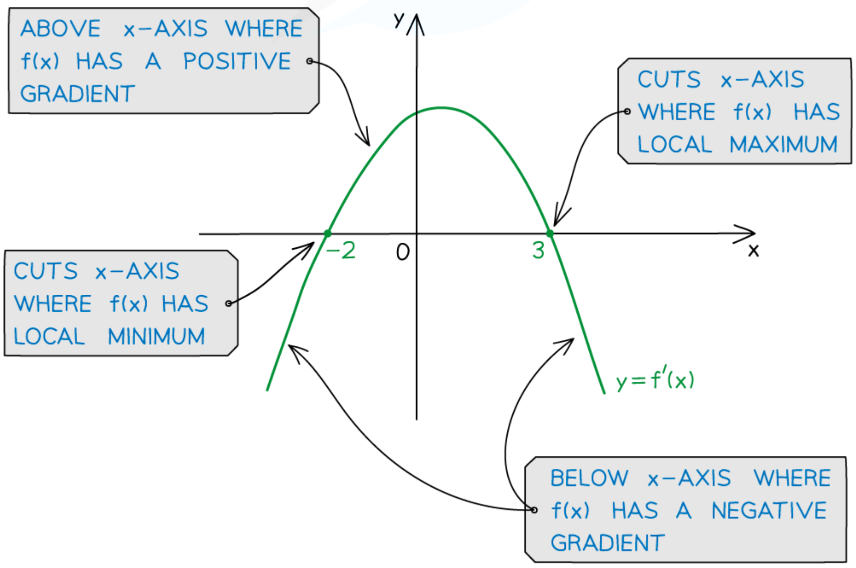
Worked Example

Your notes

The following diagram shows the graph of $y = \mathrm{f}(x)$



Sketch the graph of $y = \mathrm{f}'(x)$.

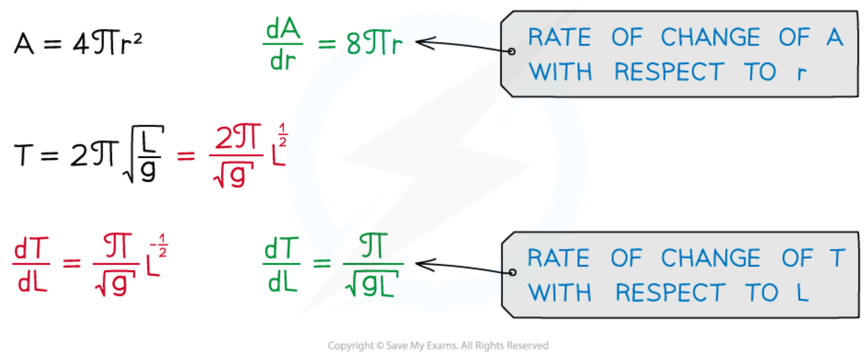


Your notes

Modelling with differentiation (optimisation)

How do I find the maximum or minimum values of models?

- Derivatives can be calculated for any variables – not just ***y*** and ***x***
- In every case the derivative is a formula giving the rate of change of one variable with respect to the other variable



- Differentiation can be used to find maximum and minimum points of a function (see Stationary Points)
- Therefore it can be used to solve maximisation and minimisation problems in modelling questions

Your notes

e.g. AT TIME t SECONDS THE HEIGHT OF A BALL ABOVE THE GROUND IN METRES IS GIVEN BY FORMULA

$$h = 20t - 5t^2$$

USE DIFFERENTIATION TO FIND THE MAXIMUM HEIGHT REACHED BY THE BALL.

$\frac{dh}{dt} = 20 - 10t$

$20 - 10t = 0$

$t = 2$

THE GRAPH OF $20t - 5t^2$ IS A "DOWN" PARABOLA. THE MAXIMUM VALUE OCCURS WHEN $\frac{dh}{dt} = 0$

WHEN $t = 2$,

$h = 20(2) - 5(2)^2 = 20$ METRES

Examiner Tips and Tricks

Exam questions on this topic will often be divided into two parts:

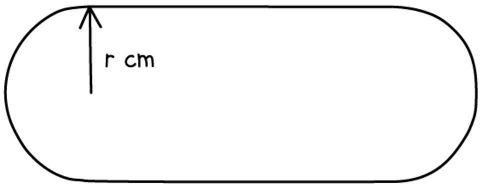
- First a 'Show that...' part where you derive a given formula from the information in the question
- And then a 'Find...' part where you use differentiation to answer a question about the formula

Even if you can't answer the first part you can still use the formula to answer the second part.

Worked Example

Your notes

A large allotment bed is being designed as a rectangle with a semicircle on each end, as shown in the diagram below.



The total area of the bed is to be $100\pi \text{ m}^2$.

a) Show that the perimeter of the bed is given by the formula $P = \pi\left(r + \dfrac{100}{r}\right)$

b) For what value of r is the rate of change of P with respect to r equal to $-\dfrac{9\pi}{16}$ metres per metre?

c) Find the perimeter of the bed for the value of r found in part (b).

a)

THE WIDTH OF THE RECTANGLE IS 2r AND THE LENGTH IS L. THE AREA OF THE BED IS

$$A = \tfrac{1}{2}\pi r^2 + 2rL + \tfrac{1}{2}\pi r^2 = \pi r^2 + 2rL$$

SEMICIRCLE

RECTANGLE

SEMICIRCLE

THE AREA IS 100π, SO

$$\pi r^2 + 2rL = 100\pi$$

$$2rL = 100\pi - \pi r^2$$

$$L = \frac{50\pi}{r} - \frac{\pi}{2}r$$

USE GIVEN AREA TO WRITE L IN TERMS OF r

Your notes

THE PERIMETER OF THE BED IS

$P = \pi r + \pi r + 2L = 2\pi r + 2L$

SEMICIRCLES

$P = 2\pi r + 2\left(\frac{50\pi}{r} - \frac{\pi}{2}r\right)$

$= 2\pi r + \frac{100\pi}{r} - \pi r$

$= \pi r + \frac{100\pi}{r}$

$P = \pi\left(r + \frac{100}{r}\right)$

SUBSTITUTE VALUE FOR L INTO PERIMETER FORMULA

b) $P = \pi r + 100\pi r^{-1}$

$\frac{dP}{dr} = \pi - 100\pi r^{-2} = \pi\left(1 - \frac{100}{r^2}\right)$

DIFFERENTIATE P WITH RESPECT TO r

$\pi\left(1 - \frac{100}{r^2}\right) = -\frac{9\pi}{16}$

$1 - \frac{100}{r^2} = -\frac{9}{16}$

$\frac{100}{r^2} = \frac{25}{16}$

$r^2 = 64 \quad \Rightarrow \quad r = \pm 8$

BECAUSE r IS A LENGTH IT CANNOT BE NEGATIVE

$r = 8$ METRES

c) WHEN $r = 8$

$P = \pi\left(8 + \frac{100}{8}\right)$

$P = \frac{41\pi}{2}$ METRES $\quad (\approx 64.4 \text{ m})$