Edexcel International AS Maths: Pure 2

Your notes

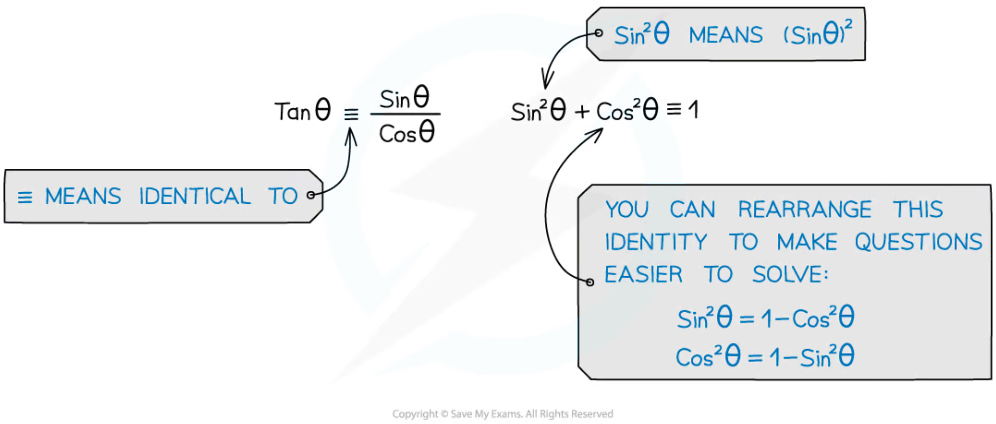
Trigonometric Equations

Contents

Simple trigonometric identities

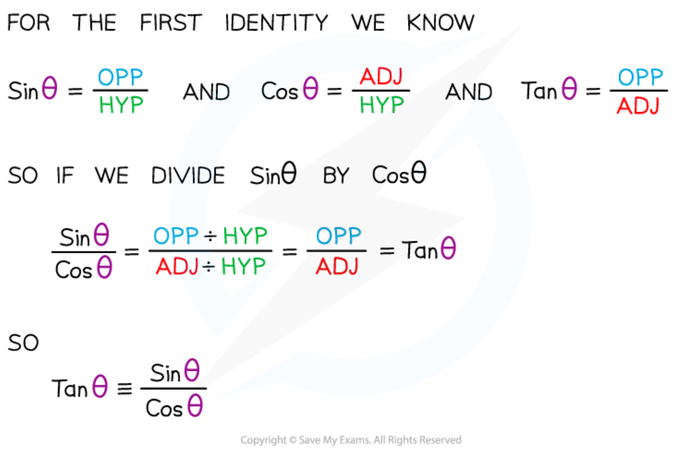
What is a trigonometric identity?

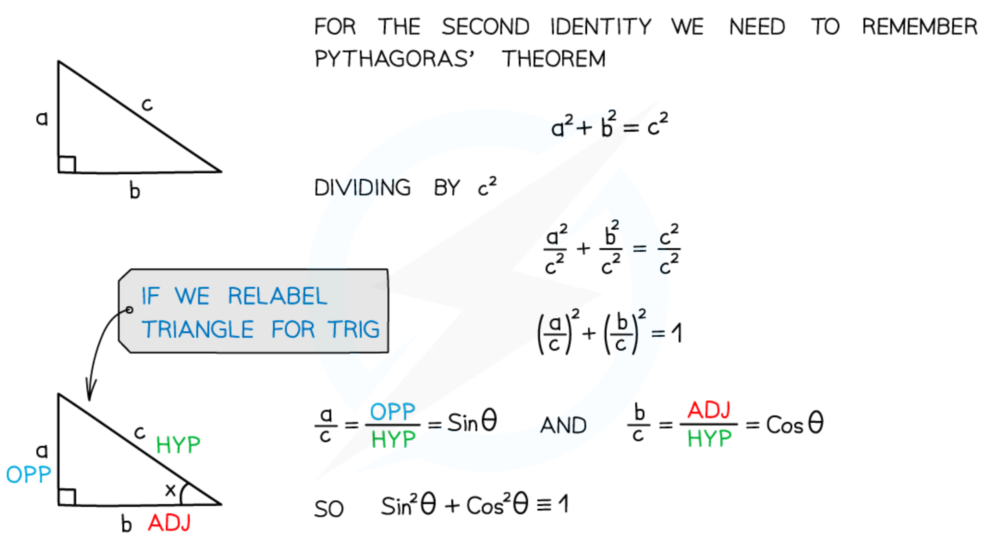
- Trigonometric identities are statements that are true for all values of ***x* or theta (θ)**
- They are used to help simplify trig equations before solving them
- The first two identities you must know are:



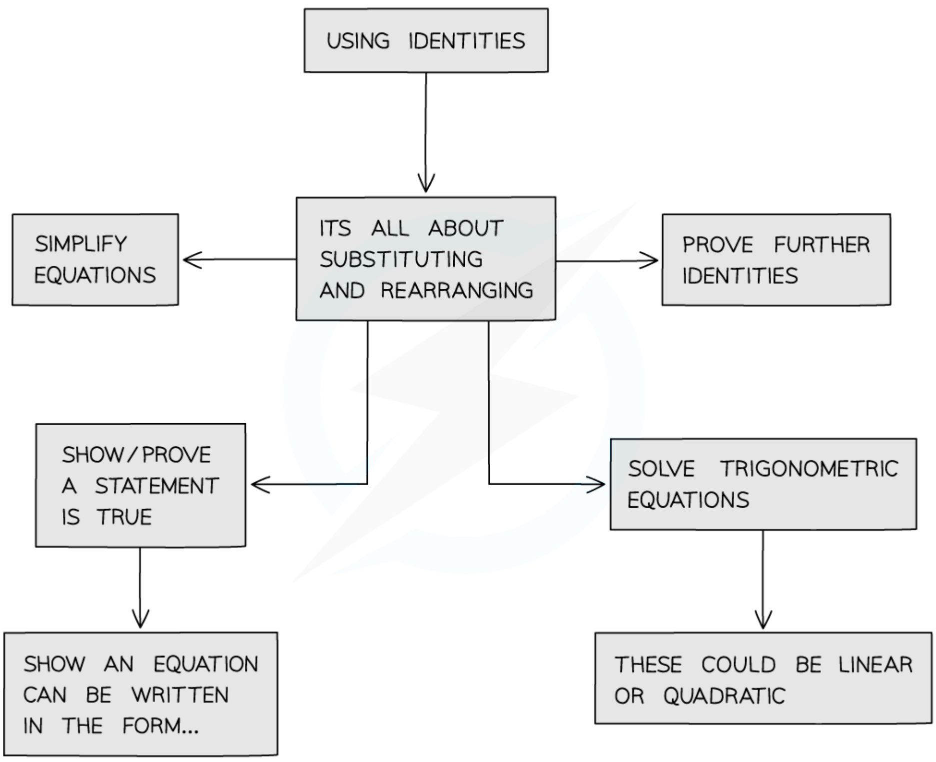
Where do trigonometric identities come from?

- Although you don't need to know the proof for these identities it is important to understand where they come from





How do I use trigonometric identities?



Your notes

Examiner Tips and Tricks

- You'll need to remember these identities as they aren't in the formula booklet.
- If asked to show one thing is identical ($\equiv$) another look at what parts are missing – for example, if $\tan x$ has gone it must have been substituted.

Your notes

Worked Example

a) For an angle x, $5\cos x = 2\sin x$. Find $\tan x$.

b) Show that the equation $3\cos x = 2(1 + \sin^2 x)$ can be written in the form $2\cos^2 x + 3\cos x - 4 = 0$.

a) $5\cos x = 2\sin x$

USING $\tan x \equiv \dfrac{\sin x}{\cos x}$

DIVIDE BY $\cos x$

$5 = 2\dfrac{\sin x}{\cos x}$

SUBSTITUTE $\tan x$ IN FOR $\dfrac{\sin x}{\cos x}$

$5 = 2\tan x$

DIVIDE BY TWO

$\tan x = \dfrac{5}{2} = 2.5$

USING $\sin^2 x + \cos^2 x \equiv 1$ REARRANGED TO $\sin^2 x = 1 - \cos^2 x$

b) $3\cos x = 2(1 + \sin^2 x)$

EXPAND BRACKETS

$3\cos x = 2 + 2\underline{\sin^2 x}$

SUBSTITUTE $1 - \cos^2 x$ IN FOR $\sin^2 x$

$3\cos x = 2 + 2\underline{(1 - \cos^2 x)}$

REARRANGE

$3\cos x = 2 + 2 - 2\cos^2 x$

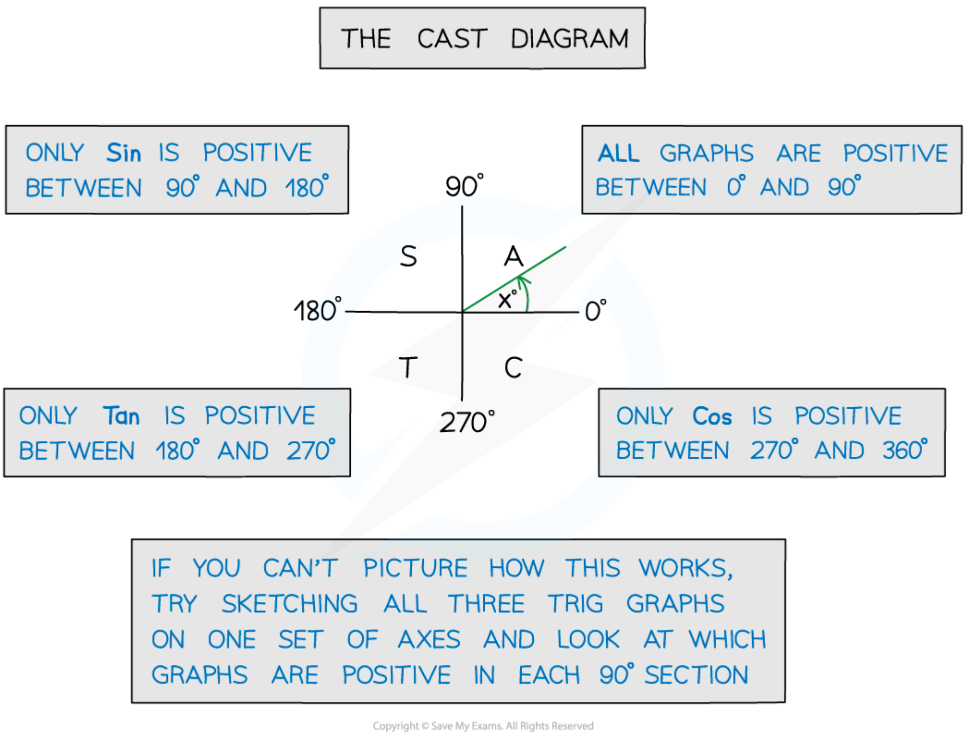
$2\cos^2 x + 3\cos x - 4 = 0$

YOU NEED TO SHOW ALL YOUR STEPS TO GAIN MAXIMUM MARKS

Linear trigonometric equations

How do I solve linear trigonometric equations?

- You should have already seen two ways to solve linear trig equations
  - By **sketching the graph** (see Graphs of Trigonometric Functions) you can read off all the solutions in a given range (or interval)
  - By using **trigonometric identities** you can simplify harder equations
- Another way to find solutions is by using the CAST diagram which shows where each function has positive solutions
- You may be asked to use **degrees or radians** to solve trigonometric equations
  - Make sure your calculator is in the **correct mode**
  - Remember common angles
    - 90° is ½π radians
    - 180° is π radians
    - 270° is 3π/2 radians
    - 360° is 2π radians



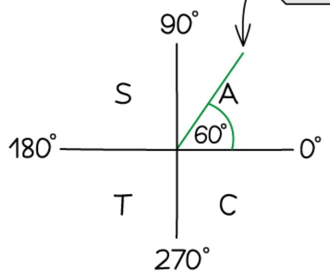
How do I use the CAST diagram?

Your notes

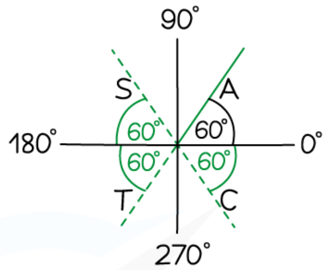
e.g. FIND ALL THE VALUES OF x IN THE INTERVAL $-360^\circ \leq x \leq 720^\circ$ WHERE $\cos x = \frac{1}{2}$

STEP 1: CALCULATE THE **PRINCIPLE VALUE** AND **DRAW IT** INTO THE CAST DIAGRAM STARTING FROM 0°

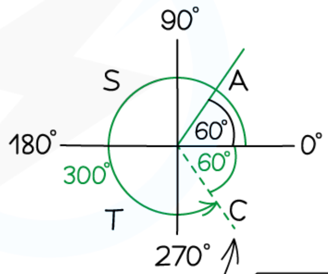
PRINCIPAL VALUE $\cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$



STEP 2: CONTINUE THE LINE ACROSS THE CAST DIAGRAM MAKING THE **SAME ANGLE** IN ALL **4 QUADRANTS**



STEP 3: FOCUSING ON THE LINES IN THE **VALID QUADRANTS** WORK OUT ALL POSITIVE SOLUTIONS FOR $0^\circ \leq x \leq 360^\circ$



SOLUTIONS FOR $\cos x = \frac{1}{2}$ ARE POSITIVE IN THE 1st AND 4th QUADRANTS ONLY
$x = 60^\circ, 300^\circ$

STEP 4: CHECK YOUR QUESTION FOR THE GIVEN **INTERVAL,** IF THIS IS BEYOND $0^\circ \leq x \leq 360^\circ$ YOU MUST ADD AND SUBTRACT FROM 360° TO FIND **ALL** SOLUTIONS

FIND ALL SOLUTIONS BETWEEN $-360^\circ \leq x \leq 720^\circ$

$x = 60$
$60 + 360 = 420$
$60 - 360 = -300$

$x = 300$
$300 + 360 = 660$
$300 - 360 = -60$

ADDING OR SUBTRACTING ANOTHER 360° FROM ANY OF THESE SOLUTIONS GIVES AN ANSWER THAT IS **TOO BIG OR TOO SMALL**

$x = -300^\circ, -60^\circ, 60^\circ, 300^\circ, 420^\circ, 660^\circ$

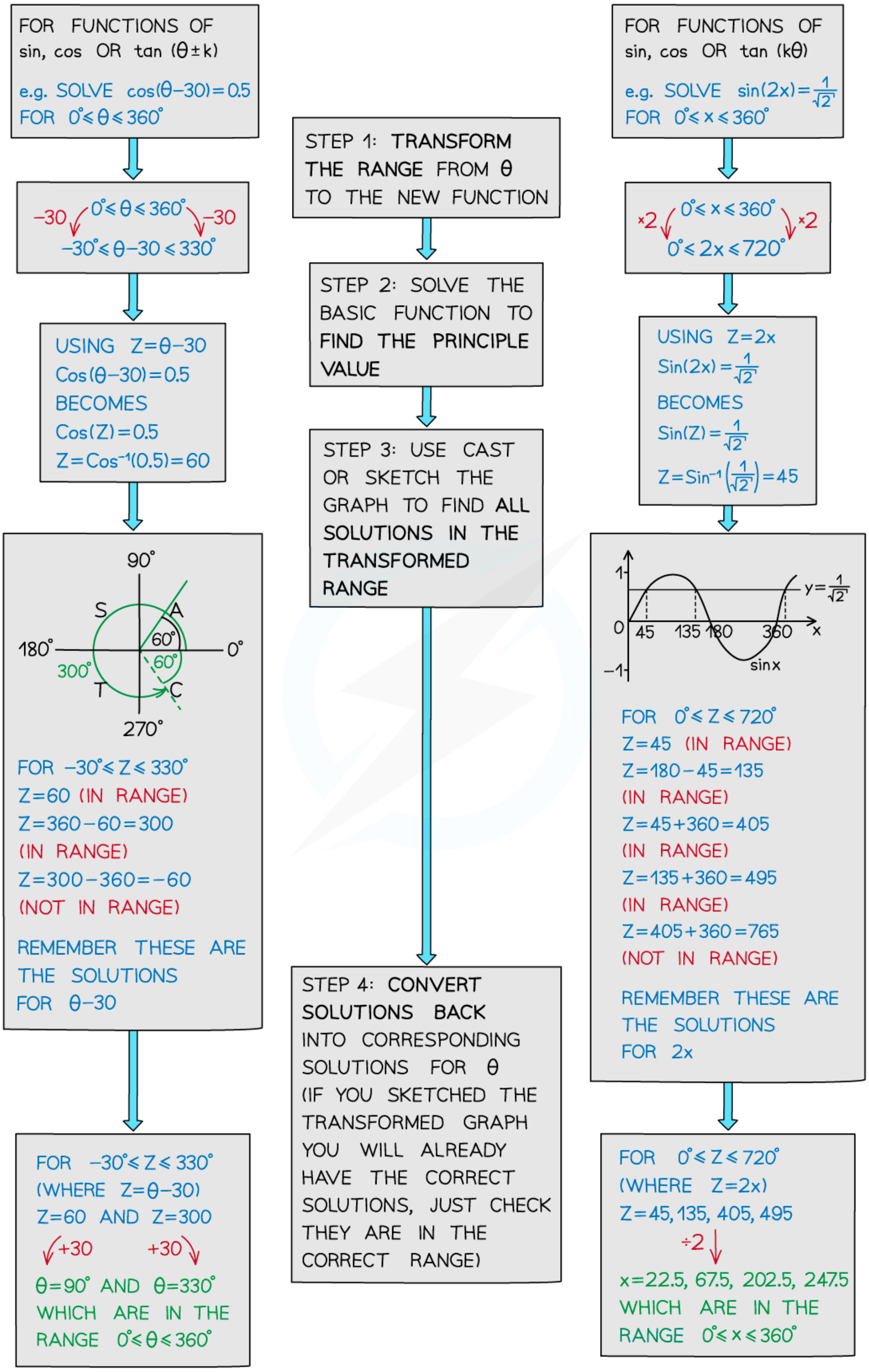
Your notes

How do I solve more complicated trigonometric equations?

Your notes

- Some trig equations could involve a **function** of *x* or θ (see Transformations of Trigonometric Functions)
- Functions could be in two forms, either y = **sin(θ ± k)** or y = **sin kθ**
- An easy way to solve an equation involving sin, cos or tan of (θ± k) or kθ is by transforming the range of the question

Your notes



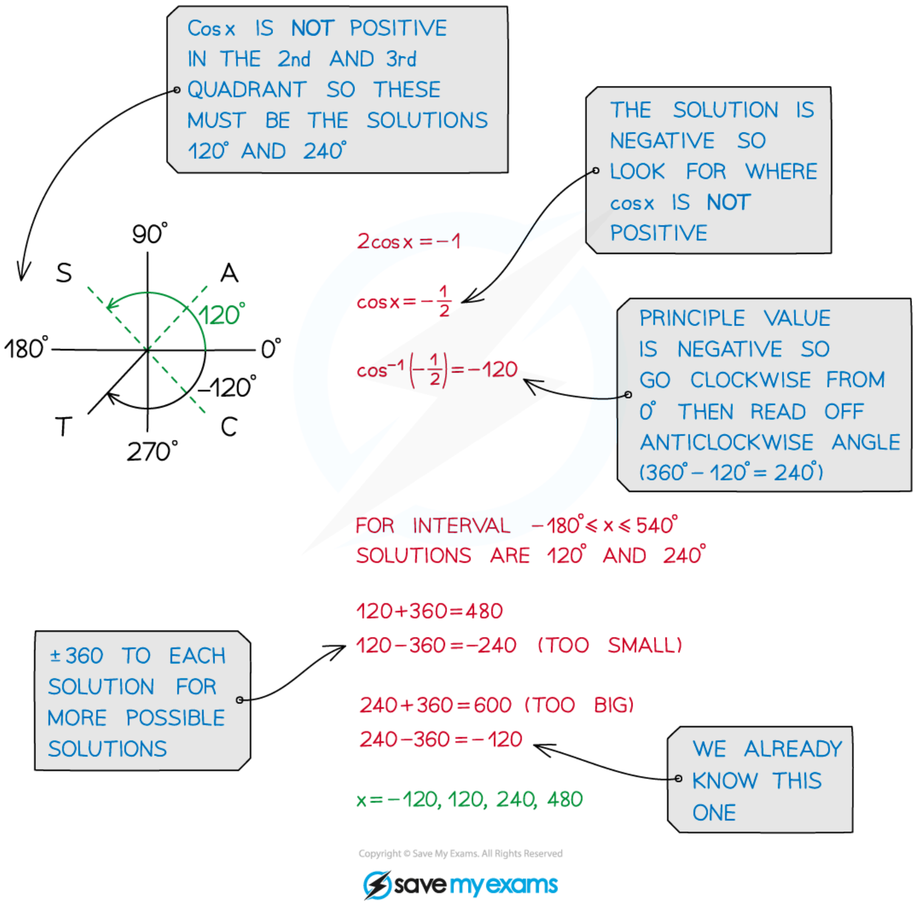
FOR FUNCTIONS OF sin, cos OR tan (θ±k)
e.g. SOLVE cos(θ−30)=0.5 FOR 0°≤θ≤360°
−30 0°≤θ≤360° −30
−30°≤θ−30≤330°
USING Z=θ−30
Cos(θ−30)=0.5
BECOMES
Cos(Z)=0.5
Z=Cos⁻¹(0.5)=60
FOR −30°≤Z≤330°
Z=60 (IN RANGE)
Z=360−60=300
(IN RANGE)
Z=300−360=−60
(NOT IN RANGE)
REMEMBER THESE ARE THE SOLUTIONS FOR θ−30
FOR −30°≤Z≤330°
(WHERE Z=θ−30)
Z=60 AND Z=300
+30 +30
θ=90° AND θ=330°
WHICH ARE IN THE RANGE 0°≤θ≤360°
STEP 1: TRANSFORM THE RANGE FROM θ TO THE NEW FUNCTION
STEP 2: SOLVE THE BASIC FUNCTION TO FIND THE PRINCIPLE VALUE
STEP 3: USE CAST OR SKETCH THE GRAPH TO FIND ALL SOLUTIONS IN THE TRANSFORMED RANGE
STEP 4: CONVERT SOLUTIONS BACK INTO CORRESPONDING SOLUTIONS FOR θ (IF YOU SKETCHED THE TRANSFORMED GRAPH YOU WILL ALREADY HAVE THE CORRECT SOLUTIONS, JUST CHECK THEY ARE IN THE CORRECT RANGE)
FOR FUNCTIONS OF sin, cos OR tan (kθ)
e.g. SOLVE sin(2x)=1/√2 FOR 0°≤x≤360°
×2 0°≤x≤360° ×2
0°≤2x≤720°
USING Z=2x
Sin(2x)=1/√2
BECOMES
Sin(Z)=1/√2
Z=Sin⁻¹(1/√2)=45
FOR 0°≤Z≤720°
Z=45 (IN RANGE)
Z=180−45=135
(IN RANGE)
Z=45+360=405
(IN RANGE)
Z=135+360=495
(IN RANGE)
Z=405+360=765
(NOT IN RANGE)
REMEMBER THESE ARE THE SOLUTIONS FOR 2x
FOR 0°≤Z≤720°
(WHERE Z=2x)
Z=45, 135, 405, 495
÷2
x=22.5, 67.5, 202.5, 247.5
WHICH ARE IN THE RANGE 0°≤x≤360°

Examiner Tips and Tricks

- Your calculator will only give you the principal value and you need to find all other solutions for the given interval.
- Also, remember the CAST diagram only gives you some solutions, so again you may need to find more depending on the given interval.
- It is entirely up to you how you solve a trig equation, but some ways are more helpful than others depending on the type of equation you are trying to solve

Worked Example

Solve the equation 2cos x = -1, in the range $-180° \leq x \leq 540°$.



Your notes

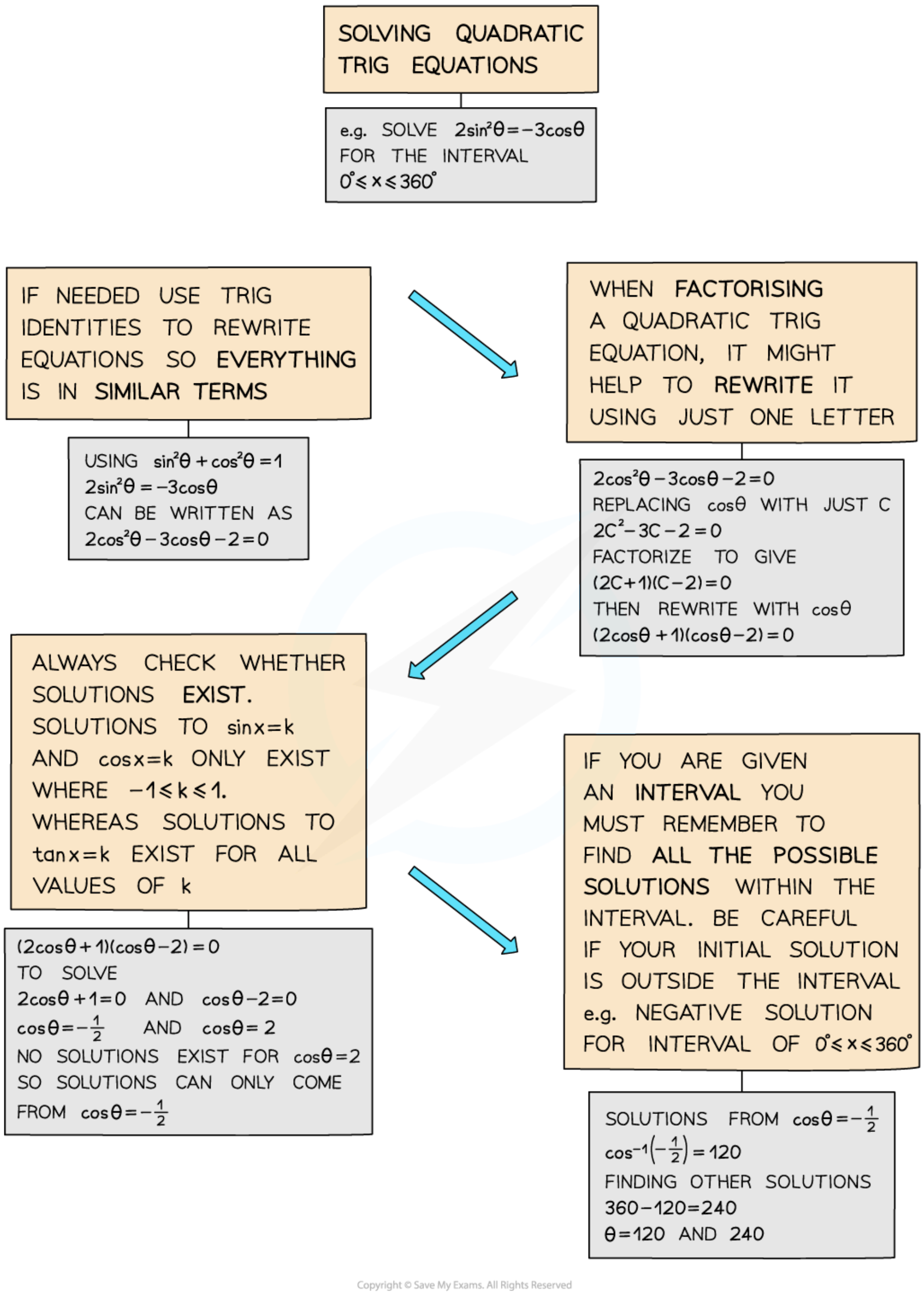
Your notes

Quadratic trigonometric equations

How do I solve quadratic trigonometric equations?

- If an equation involves $\sin^2\theta$ or $\cos^2\theta$ then it is a quadratic trigonometric equation
- These can be solved by factorising and/or using **trigonometric identities** (see **Trigonometry – Simple Identities**)
- As a quadratic can result in two solutions, you will need to consider whether each solution exists and then find all solutions within a given interval for each
- You may be asked to use **degrees or radians** to solve trigonometric equations
  - Make sure your calculator is in the **correct mode**
  - Remember common angles
    - 90° is $\frac{1}{2}\pi$ radians
    - 180° is π radians
    - 270° is $3\pi/2$ radians
    - 360° is 2π radians

Your notes



Examiner Tips and Tricks

- Sketch the appropriate sin, cos, tan graph to ensure you find ALL solutions **within** the given interval, and be super-careful if you get a negative solution but have a positive interval.

- For example, for an equation, in the interval $0° \leq x \leq 360°$, with solution $\sin x = -\frac{1}{4}$ then $\sin^{-1}(-\frac{1}{4}) = -14.5$ (to 1d.p.), which is not between 0 and 360 – by sketching the graph you'll be able to spot the two solutions will be 180 + 14.5 and 360 - 14.5.

Your notes

Worked Example

Your notes

a) Solve $\tan^2 x - 2\tan x = 0$, in the interval $0° \le x \le 360°$.
Giving each solution correct to 1 decimal place.

b) i) Show that $11 \sin x = 5\cos^2 x + 7$ can be written in the form $5\sin^2 x + 11\sin x - 12 = 0$.

ii) Find all the solutions of the equation $11 \sin x = 5\cos^2 x + 7$, in the interval $0° \le x \le 360°$.
Giving each solution correct to 1 decimal place.

a)

$\tan^2 x - 2\tan x = 0$

$\tan x(\tan x - 2) = 0$

FACTORISE

TWO POSSIBLE SOLUTIONS

$\tan x = 0$

$\tan^{-1}(0) = 0$

$\tan x - 2 = 0$

$\tan x = 2$

$\tan^{-1}(2) = 63.43$

$63.4 + 180 = 243.4$

Tan REPEATS EVERY 180° SO ADD 180° TO FIND ALL SOLUTIONS

$x = 0, 180, 360$ AND $x = 63.4, 243.4$

b) i)

$11\sin x = 5\cos^2 x + 7$

$11\sin x = 5(1 - \sin^2 x) + 7$

$11\sin x = 5 - 5\sin^2 x + 7$

$5\sin^2 x + 11\sin x - 12 = 0$

USE $\cos^2 x = 1 - \sin^2 x$ TO GET EVERYTHING IN TERMS OF $\sin x$

EXPAND THEN REARRANGE

ii)

$5\sin^2 x + 11\sin x - 12 = 0$

$(5\sin x - 4)(\sin x + 3) = 0$

USING $11\sin x = 5\cos^2 x + 7$ WRITTEN IN THE FORM $5\sin^2 x + 11\sin x - 12 = 0$ FROM PART i)

FACTORISE USING $\sin x = s$ TO MAKE IT EASIER $(5s - 4)(s + 3)$

$5\sin x - 4 = 0$

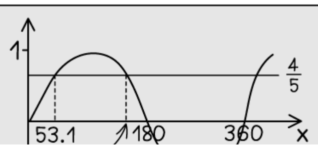
$5\sin x = 4$

$\sin x = \frac{4}{5}$

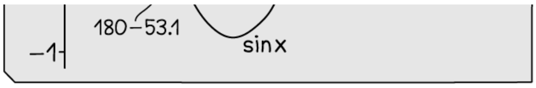
$\sin x + 3 = 0$

$\cancel{\sin x = -3}$

$\sin x = -3$ HAS NO SOLUTION



SKETCH $\sin x$



ONLY TWO VALID SOLUTIONS FOR GIVEN INTERVAL

$\sin^{-1}\left(\frac{4}{5}\right) = 53.1$

$180 - 53.1 = 126.9$

TWO SOLUTIONS $x = 53.1,\ 126.9$

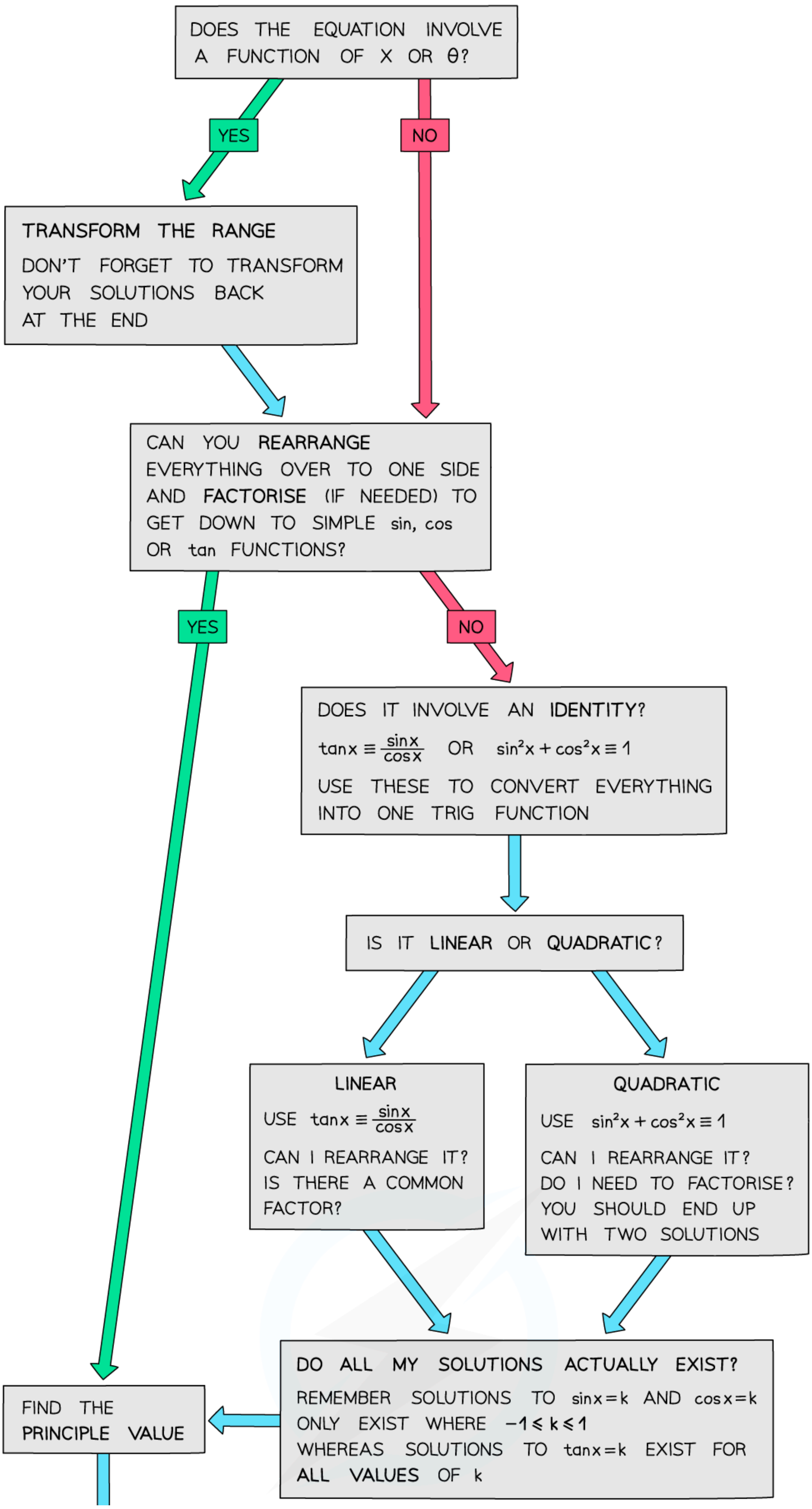
Your notes

Your notes

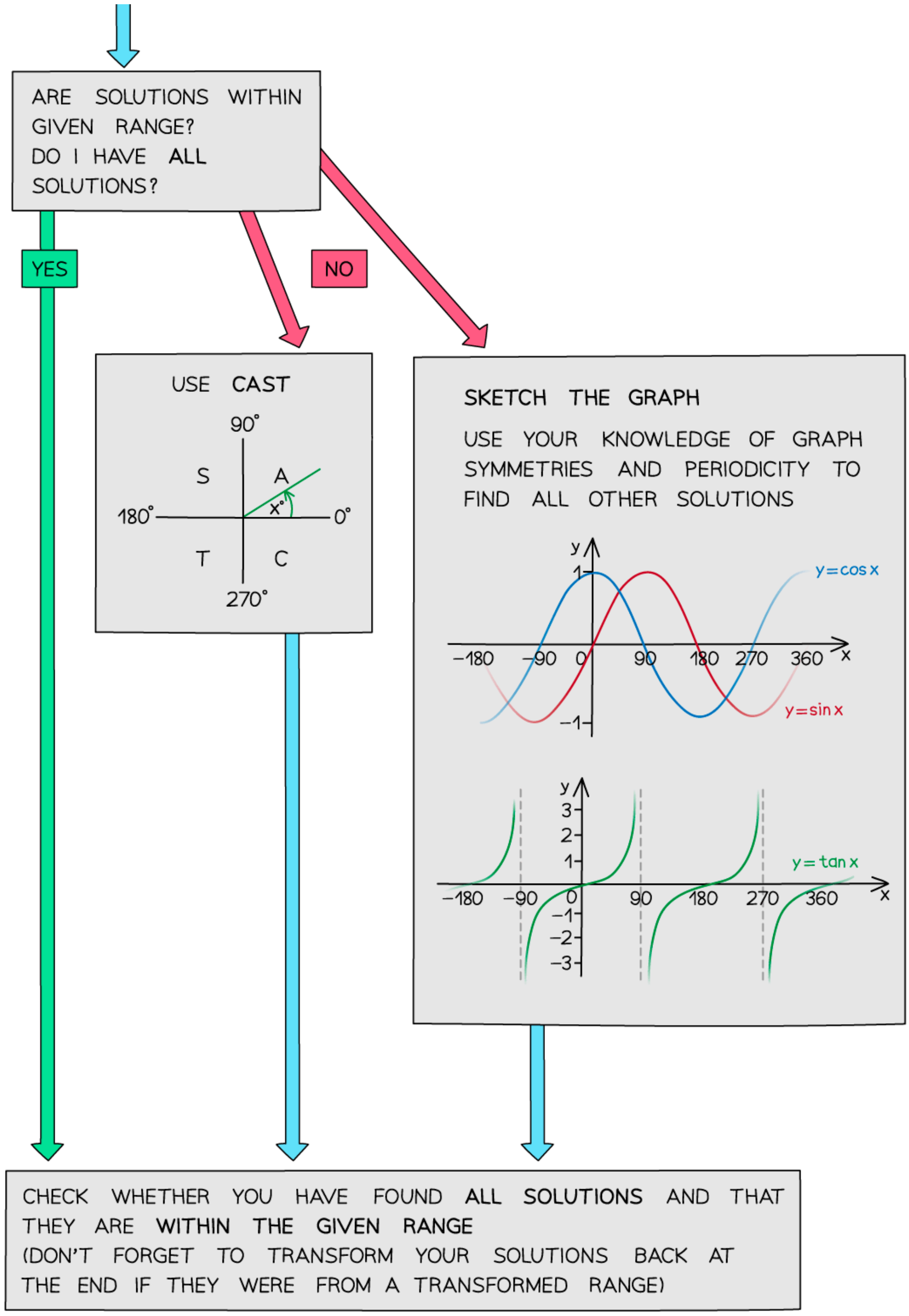
Strategy for trigonometric equations

What strategies can I use for solving trigonometric equations?

- You can solve trig equations in a variety of different ways
  - **Sketching a graph** (see **Graphs of Trigonometric Functions**)
  - Using **trigonometric identities** (see **Trigonometry – Simple Identities**)
  - Using the **CAST** diagram (see **Linear Trigonometric Equations**)
  - Factorising **quadratic** trig equations (see **Quadratic Trigonometric Equations**)
- You may be asked to use **degrees or radians** to solve trigonometric equations
  - Make sure your calculator is in the **correct mode**
  - Remember common angles
    - 90° is ½π radians
    - 180° is π radians
    - 270° is 3π/2 radians
    - 360° is 2π radians
- If you're having trouble solving a trig equation, this flowchart might help:


DOES THE EQUATION INVOLVE A FUNCTION OF X OR θ?
YES
NO
TRANSFORM THE RANGE
DON'T FORGET TO TRANSFORM YOUR SOLUTIONS BACK AT THE END
CAN YOU REARRANGE EVERYTHING OVER TO ONE SIDE AND FACTORISE (IF NEEDED) TO GET DOWN TO SIMPLE sin, cos OR tan FUNCTIONS?
YES
NO
DOES IT INVOLVE AN IDENTITY?
$\tan x \equiv \frac{\sin x}{\cos x}$ OR $\sin^2 x + \cos^2 x \equiv 1$
USE THESE TO CONVERT EVERYTHING INTO ONE TRIG FUNCTION
IS IT LINEAR OR QUADRATIC?
LINEAR
USE $\tan x \equiv \frac{\sin x}{\cos x}$
CAN I REARRANGE IT?
IS THERE A COMMON FACTOR?
QUADRATIC
USE $\sin^2 x + \cos^2 x \equiv 1$
CAN I REARRANGE IT?
DO I NEED TO FACTORISE?
YOU SHOULD END UP WITH TWO SOLUTIONS
DO ALL MY SOLUTIONS ACTUALLY EXIST?
REMEMBER SOLUTIONS TO $\sin x = k$ AND $\cos x = k$ ONLY EXIST WHERE $-1 \leqslant k \leqslant 1$
WHEREAS SOLUTIONS TO $\tan x = k$ EXIST FOR ALL VALUES OF k
FIND THE PRINCIPLE VALUE

Your notes



Your notes

Examiner Tips and Tricks

- Don't forget to check the function range and ensure you have included all possible solutions.
- If the question involves a function of x or θ, make sure you transform the range first (and ensure you transform your solutions back again at the end!).

Your notes

Worked Example

Your notes

a) Solve $3 + 5\cos 2x = 1$, for $-180° < x < 180°$.

b) Solve $6\cos^2 x + \sin x - 5 = 0$, for $0 \le x < 360°$.

a)

$3 + 5\cos 2x = 1$

FOR $-180° < x < 180°$

$-360° < \overset{z}{2x} < 360°$

TRANSFORM THE RANGE AND REPLACE $2x = z$

$3 + 5\cos z = 1$

(-3) (-3)

REARRANGE

$5\cos z = -2$

$(\div 5)$ $(\div 5)$

$\cos z = -\frac{2}{5}$

FIND PRINCIPLE VALUE

$z = \cos^{-1}\left(-\frac{2}{5}\right) = 113.58$

FOR $-360° < z < 360°$

USE CAST OR GRAPH SYMMETRIES FOR OTHER VALUES IN RANGE

$(360 - 113.58)$

$z = 113.58,\ 246.42,\ -113.58,\ -246.42$

$\div 2$

$x = 56.79,\ 123.21,\ -56.79,\ -123.21$

TRANSFORM BACK TO GIVE SOLUTIONS IN ORIGINAL INTERVAL $-180° < x < 180°$

b)

$6\cos^2 x + \sin x - 5 = 0$

FOR $0° \leqslant x < 360°$

USING $\sin^2 x + \cos^2 x \equiv 1$ SUBSTITUTE $\cos^2 x \equiv 1 - \sin^2 x$

$6(1 - \sin^2 x) + \sin x - 5 = 0$

REARRANGE MAKE $\sin^2 x$ POSITIVE

$6 - 6\sin^2 x + \sin x - 5 = 0$

$6\sin^2 x - \sin x - 1 = 0$

FACTORISE USING $\sin x = s$ $6s^2 - s - 1 = 0$ $(3s+1)(2s-1) = 0$

$(3\sin x + 1)(2\sin x - 1) = 0$

REARRANGE TWO SEPERATE SOLUTIONS

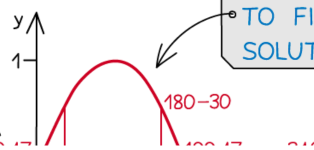
$3\sin x + 1 = 0$ $\qquad$ $2\sin x - 1 = 0$

$\sin x = -\frac{1}{3}$ $\qquad$ $\sin x = \frac{1}{2}$

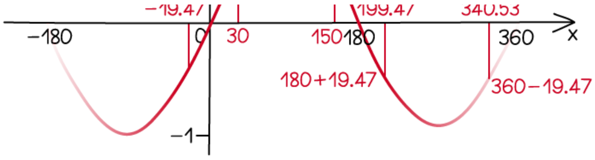
$\sin^{-1}\left(-\frac{1}{3}\right) = -19.47$ $\qquad$ $\sin^{-1}\left(\frac{1}{2}\right) = 30$

-19.47 IS OUTSIDE $0° \leqslant x < 360°$ RANGE BUT CAN BE USED TO FIND OTHER

SKETCH GRAPH TO FIND ALL SOLUTIONS



SOLUTIONS



x = 30, 150, 199.47, 340.53

Your notes