Edexcel International A Level (IAL) Maths: Pure 1

Your notes

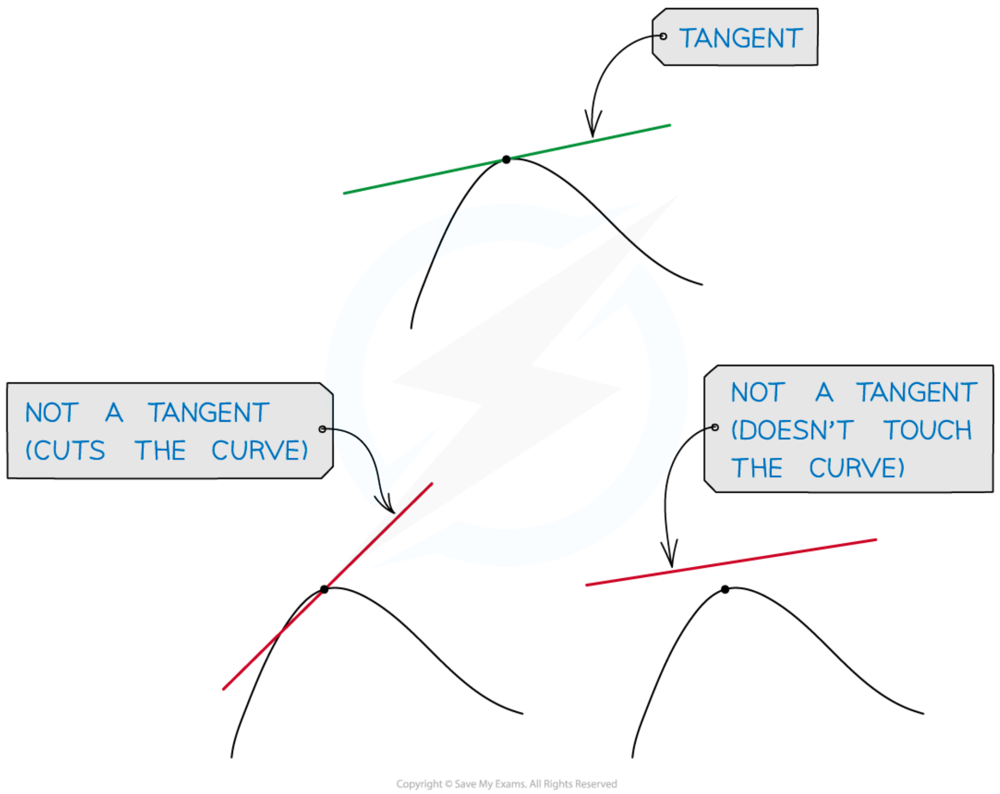
Differentiation

Contents

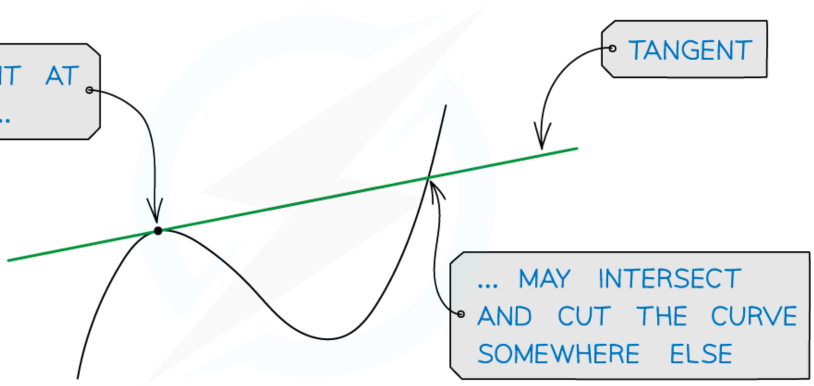
Definition of gradient

What is the gradient of a curve?

- At a given point the **gradient** of a curve is defined as the gradient of the **tangent** to the curve at that point
- A **tangent** to a curve is a line that just **touches** the curve at one point but doesn't cut the curve at that point



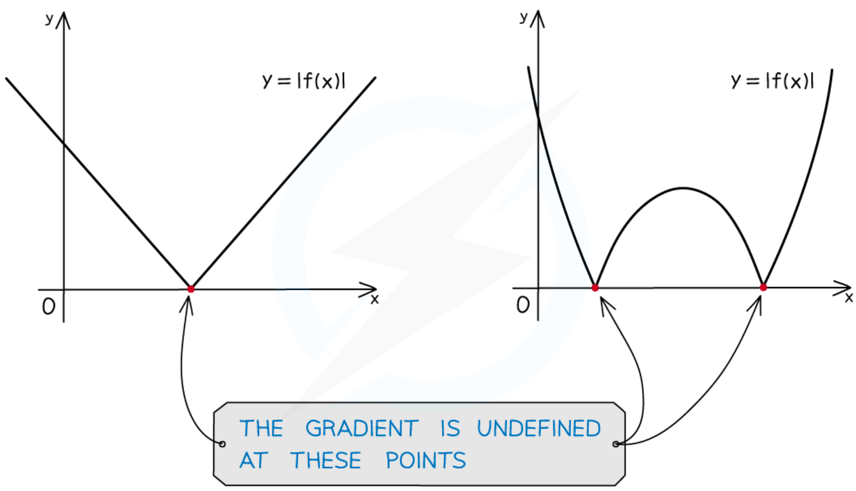
- A tangent *may* cut the curve somewhere else on the curve



- It is only possible to draw **one** tangent to a curve at any given point
- Note that unlike the gradient of a straight line, the gradient of a curve is constantly changing

Examiner Tips and Tricks

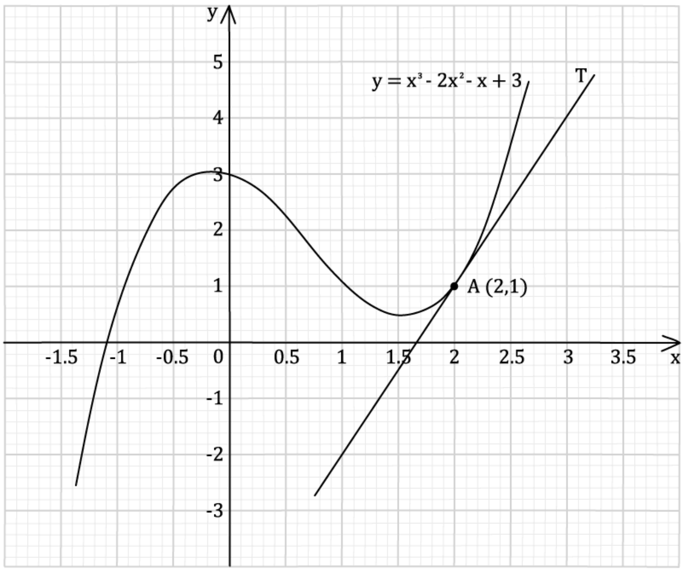
- A tangent only exists at points where a curve is smooth.
- For example, there is no tangent (or gradient) at one of the 'corners' in a modulus function.



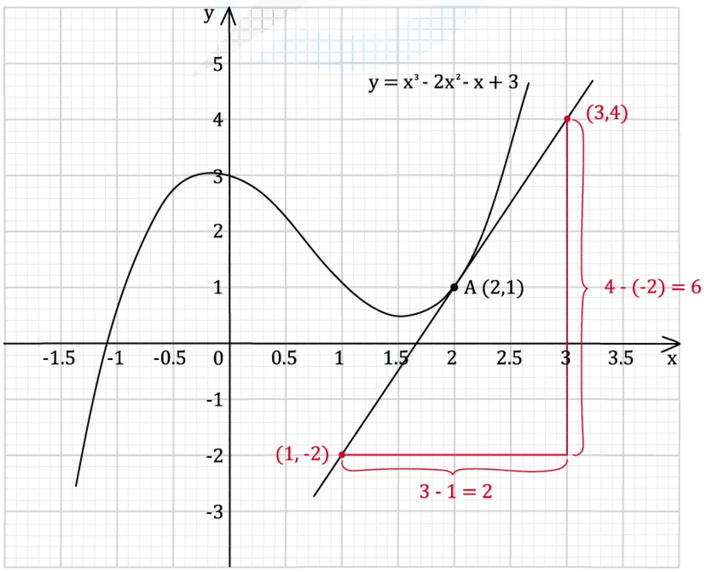
Worked Example

Your notes

The diagram shows the curve with equation $y = x^3 - 2x^2 - x + 3$. The tangent, T, to the curve at point $A(2, 1)$ is also shown.



Using the diagram, calculate the gradient of the curve at point A.



THE GRADIENT OF THE CURVE AT POINT A IS THE SAME AS THE GRADIENT OF THE TANGENT LINE.

SO

$$\text{GRADIENT} = \frac{\text{CHANGE IN } y}{\text{CHANGE IN } x} = \frac{4-(-2)}{3-1} = \frac{6}{2} = 3$$

Your notes

Definition of derivatives

What is the derivative of a function?

- **Differentiation** is an operation that calculates the **rate of change** of a **function** with respect to a variable
  - This means how much the function varies when the variable increases by one unit
- To **differentiate** a function (f(*x*)) with respect to the variable *x* we use the **notation**
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{f}(x))$
- The result is called the **derivative**

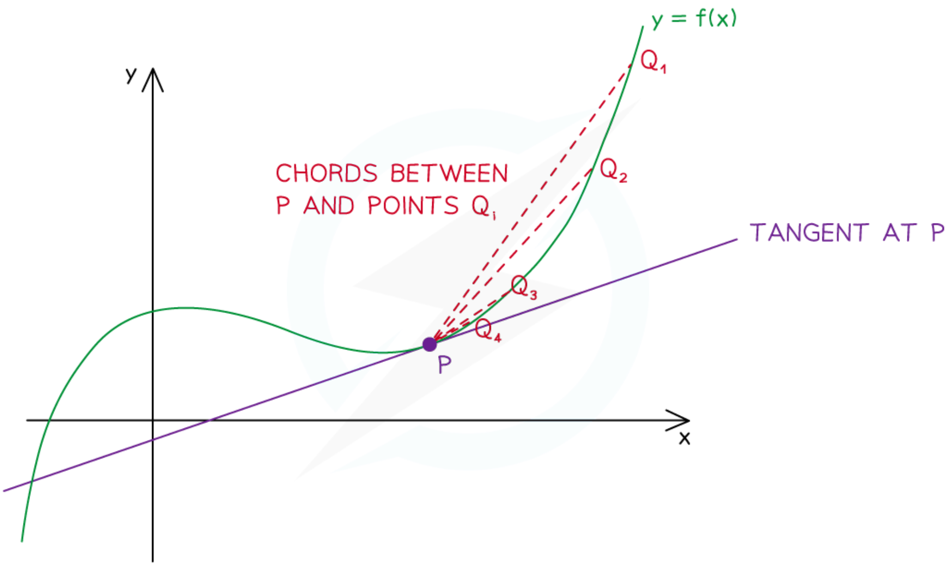
What is the link between derivatives and gradients?

- The rate of change of a function f(*x*) with respect to *x* can be thought of as the **gradient function** of the graph *y* = f(*x*)
  - We can write the gradient function (or derivative) as
    - $\mathrm{f}'(x)$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$
- The rate of change of a function (or the gradient of its graph) varies for different values of *x*
  - For a **linear function** f(*x*) = *mx* + c the gradient is constant
    - We can write this as $\mathrm{f}'(x) = m$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = m$
  - For the **quadratic function** f(*x*) = x^2 the gradient varies
    - Near the origin the gradient is close to 0
    - As *x* increases the gradient of the graph increases

How can I find the derivative of a function at a point?

- The **derivative** of a function (or gradient of its graph) at a point is equal to the **gradient of the tangent** to the graph at that point
- To **estimate** the gradient you could **draw the tangent** and calculate its **gradient**
- To find the **actual gradient** at a point *x*
  - Pick a **second point** on the curve close to the first point (call it *x* + *h*)
  - Calculate the **gradient of the chord** joining the two points

- $$\frac{f(x+h)-f(x)}{(x+h)-x}=\frac{f(x+h)-f(x)}{h}$$
- Move the **second point closer to the first point** (make h get close to zero)
- Examine what **happens to the gradient** of the chord
- The gradient of the tangent will be the **limit of the gradients of the chords**
  - $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
  - You do **not** need to remember this formula



Your notes

Worked Example

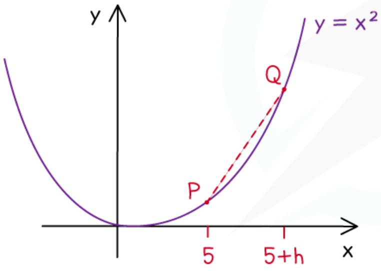
Your notes

The curve C is given by the equation $y = x^2$. The points P and Q lie on C and have x coordinates 5 and $(5 + h)$ respectively, where $h > 0$.

(a) Find, in terms of h, the gradient of the chord joining P and Q.

(b) Hence explain why the gradient of the tangent to C at P is 10.

a) QUICK SKETCH



FIND THE y-COORDINATES

$x = 5: \quad y = 5^2 = 25$

$x = 5 + h: \quad y = (5 + h)^2 = 25 + 10h + h^2$

FIND THE GRADIENT BETWEEN P AND Q

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(\cancel{25} + 10h + h^2) - \cancel{25}}{(\cancel{5} + h) - \cancel{5}} = \frac{10h + h^2}{h} = \frac{\cancel{h}(10+h)}{\cancel{h}}$$

GRADIENT $= 10 + h$

b) THE GRADIENT OF THE TANGENT AT P IS EQUAL TO THE LIMIT OF THE GRADIENT OF THE CHORD JOINING P AND Q AS Q GETS CLOSER TO P.

AS Q GETS CLOSER TO P, $(5 + h)$ GETS CLOSER TO 5 SO h GETS CLOSER TO 0.

AS h GETS CLOSE TO 0, $10 + h$ LIMITS 10.

Examiner Tips and Tricks

- Deriving a derivative from scratch is **not examinable**
- This revision note is intended to give you an understanding of what derivatives do

Your notes

Differentiating powers of x

How do I differentiate expressions with powers of x?

- Powers of ***x*** are differentiated according to the following formula
  - $\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$
    - Where n is any real constant
  - This can be written as:
    - If $\mathrm{f}(x) = x^n$ then $\mathrm{f}'(x) = nx^{n-1}$
    - If $y = x^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$

Examiner Tips and Tricks

$\mathrm{f}'(x)$ is usually used when a function, $\mathrm{f}(x)$, has been defined. $\frac{\mathrm{d}y}{\mathrm{d}x}$ is used when a graph of a function is given in the form $y = \ldots$ They both represent the derivative of an expression.

- If the term involves a **scalar multiple**, then you can differentiate the power of *x* and then multiply by the scalar
  - $\frac{\mathrm{d}}{\mathrm{d}x}(ax^n) = anx^{n-1}$
    - Where n and a are any real constants
- Differentiating terms with positive integer powers of ***x*** is straightforward

$$f(x) = x^2 \longrightarrow f'(x) = 2x^{2-1} = 2x$$

$$f(x) = 5x^3 \longrightarrow f'(x) = 5 \times 3x^{3-1} = 15x^2$$

$$y = -4x^5 \longrightarrow \frac{dy}{dx} = -4 \times 5x^{5-1} = -20x^4$$

$$y = \frac{2}{3}x^7 \longrightarrow \frac{dy}{dx} = \frac{2}{3} \times 7x^{7-1} = \frac{14}{3}x^6$$

- But negative and fractional powers of ***x*** can also be differentiated this way

$$f(x) = 2x^{-1} \longrightarrow f'(x) = 2 \times (-1)x^{-1-1} = -2x^{-2}$$

$$f(x) = 5\sqrt{x} = 5x^{\frac{1}{2}} \longrightarrow f'(x) = 5 \times \frac{1}{2}x^{\frac{1}{2}-1} = \frac{5}{2}x^{-\frac{1}{2}} = \frac{5}{2\sqrt{x}}$$

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$y = \frac{4}{5}x^{\frac{2}{3}} \longrightarrow \frac{dy}{dx} = \frac{4}{5} \times \frac{2}{3}x^{\frac{2}{3}-1} = \frac{8}{15}x^{-\frac{1}{3}}$$

$$y = \frac{1}{x^4} = x^{-4} \longrightarrow \frac{dy}{dx} = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

$$\frac{1}{x^p} = x^{-p}$$

- Don't forget these two special cases:
  - The derivative of a **linear function** is a **constant**
    - $\frac{\mathrm{d}}{\mathrm{d}x}(ax) = a$
    - Where a is any real constant
  - The derivative of a **constant function** is **zero**
    - $\frac{\mathrm{d}}{\mathrm{d}x}(a) = 0$
    - Where a is any real constant
- These allow you to differentiate constants and linear terms in ***x***

Your notes

$$f(x) = 3 \longrightarrow f'(x) = 0$$

$$f(x) = -5x \longrightarrow f'(x) = -5$$

$$y = \sqrt{7} \longrightarrow \frac{dy}{dx} = 0$$

$$y = x \longrightarrow \frac{dy}{dx} = 1$$

- If the expression involves a sum or difference of terms just differentiate one term at a time

$$f(x) = 3x^2 - 5x + 3$$

$$f'(x) = 6x - 5$$

THE DERIVATIVE OF 3 (A CONSTANT) IS ZERO

$$y = x^4 - \sqrt{x} + \frac{4}{x}$$

$$\frac{dy}{dx} = 4x^3 - \frac{1}{2\sqrt{x}} - \frac{4}{x^2}$$

IN GENERAL IF $y = f(x) + g(x) + h(x) + \ldots$

THEN $\frac{dy}{dx} = f'(x) + g'(x) + h'(x) + \ldots$

Examiner Tips and Tricks

- Take extra care when differentiating negative and fractional powers of ***x*** as mishandling negative signs and fractions is a common way to lose marks in these sorts of questions.

Worked Example

Your notes

$\mathrm{f}(x) = \dfrac{8}{q\sqrt{x}} + \sqrt{x}$, where q is a real constant and $x > 0$.

Given that $\mathrm{f}'(3) = -\dfrac{\sqrt{3}}{18}$, find the value of q.

$$f(x) = \frac{8}{q}x^{-\frac{1}{2}} + x^{\frac{1}{2}}$$

$\sqrt{x} = x^{\frac{1}{2}}$ $\quad \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$$f'(x) = -\frac{1}{2} \times \frac{8}{q}x^{-\frac{3}{2}} + \frac{1}{2} \times x^{-\frac{1}{2}}$$

$$= \frac{1}{2}x^{-\frac{1}{2}} - \frac{4}{q}x^{-\frac{3}{2}}$$

$$= \frac{1}{2\sqrt{x}} - \frac{4}{qx\sqrt{x}}$$

$x^{\frac{3}{2}} = x\sqrt{x}$

$$f'(3) = \frac{1}{2\sqrt{3}} - \frac{4}{3q\sqrt{3}} = \frac{3q-8}{6q\sqrt{3}}$$

REWRITE AS SINGLE FRACTION WITH COMMON DENOMINATOR

SO $\quad \dfrac{3q-8}{6q\sqrt{3}} = -\dfrac{\sqrt{3}}{18}$

$$3q-8 = -\frac{\sqrt{3}}{18}\left(6q\sqrt{3}\right) = -q$$

$$4q = 8$$

$$q = 2$$

Gradients, tangents & normals

How do I find the gradient at a point on a curve?

- To find the gradient of a curve ***y*= f(*x*)** at any point on the curve, substitute the ***x*-coordinate** of the point into the derivative **f'(*x*)**

e.g. FIND THE GRADIENT OF THE CURVE $y = x - \frac{1}{x}$ AT THE POINT $(2, \frac{3}{2})$

$y = x - x^{-1}$

$\frac{dy}{dx} = 1 - (-1)x^{-1-1} = 1 + x^{-2} = 1 + \frac{1}{x^2}$ FIND DERIVATIVE

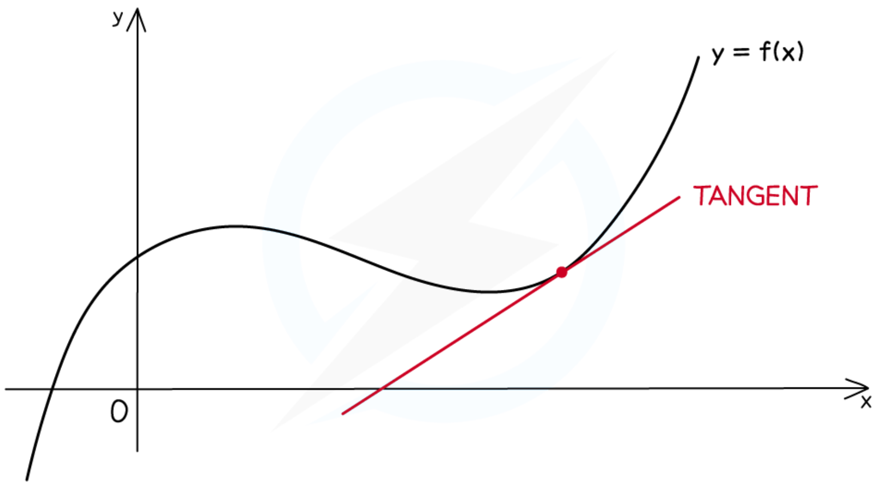
WHEN $x = 2$,

$\frac{dy}{dx} = 1 + \frac{1}{2^2} = \frac{5}{4}$ PUT x-COORDINATE INTO DERIVATIVE

THE GRADIENT AT $(2, \frac{3}{2})$ IS $\frac{5}{4}$

How do I find the equation of a tangent to a curve?

- At any point on a curve, the **tangent** is the line that goes through the point and has the same gradient as the curve at that point

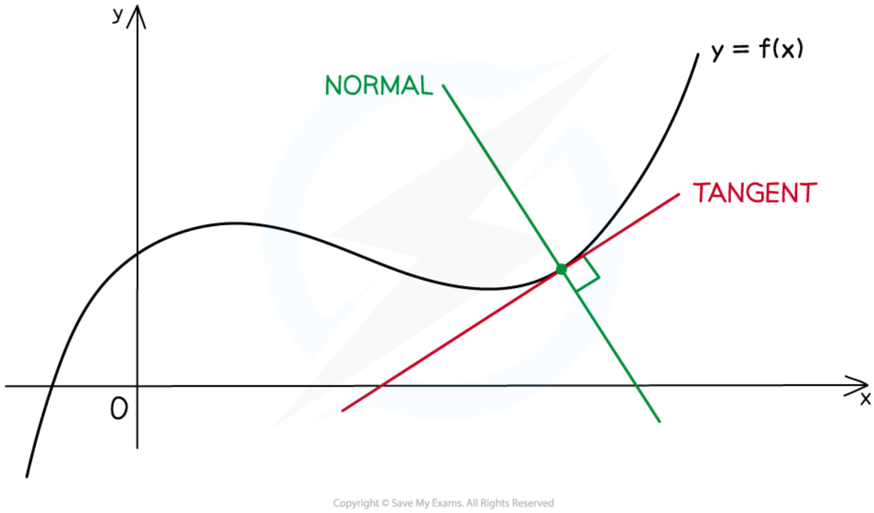


- For the curve y = f(x), you can find the equation of the **tangent** at the point (a, f(a)) using

$$y - f(a) = f'(a)(x - a)$$

How do I find the equation of a normal to a curve?

- At any point on a curve, the **normal** is the line that goes through the point and is **perpendicular** to the tangent at that point



- For the curve y = f(x), you can find the equation of the **normal** at the point (a, f(a)) using

$$y - f(a) = -\frac{1}{f'(a)}(x - a)$$

Examiner Tips and Tricks

- The formulae above are not in the exam formulae booklet, but if you understand what tangents and normals are, then the formulae follow from the equation of a straight line combined with parallel and perpendicular gradients (see Worked Example below).

Worked Example

Your notes

Your notes

The curve C has the equation $y = 7 - 12\sqrt{x}$.

a) Find the equation of the tangent to C at the point where $x = 9$.

b) Find the equation of the normal to C at the same point.

a) WHEN $x = 9$, $y = 7 - 12\sqrt{9} = -29$ ← FIND y−COORDINATE

$y = 7 - 12x^{\frac{1}{2}}$ ← $\sqrt{x} = x^{\frac{1}{2}}$

$\frac{dy}{dx} = -12\left(\frac{1}{2}\right)x^{\frac{1}{2}-1} = -6x^{-\frac{1}{2}} = -\frac{6}{\sqrt{x}}$ ← FIND DERIVATIVE

WHEN $x = 9$,

$\frac{dy}{dx} = -\frac{6}{\sqrt{9}} = -2$ ← USE DERIVATIVE TO FIND GRADIENT OF TANGENT

SO THE TANGENT AT POINT $(9, -29)$ HAS EQUATION

$y - (-29) = -2(x - 9)$ ← $y - y_1 = m(x - x_1)$

$y = -2x - 11$

b) THE NORMAL IS PERPENDICULAR TO THE TANGENT, SO THE GRADIENT OF THE NORMAL IS $-\frac{1}{(-2)} = \frac{1}{2}$

$m_2 = -\frac{1}{m_1}$ FOR PERPENDICULAR LINES

AT $(9, -29)$ THE NORMAL HAS EQUATION

$y - (-29) = \frac{1}{2}(x - 9)$

$y = \frac{1}{2}x - \frac{67}{2}$

Your notes

Second order derivatives

What is the second order derivative of a function?

- If you differentiate the derivative of a function (ie differentiate the function a second time) you get the **second order derivative** of the function
- For a function $y = f(x)$, there are two forms of notation for the second derivative (or second order derivative)

$$f''(x) \text{ or } \frac{d^2y}{dx^2}$$

- Note the positions of the power of 2's in the second version

e.g. $f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

$y = 7x^2 - 4x + 19$

$\frac{dy}{dx} = 14x - 4$

$\frac{d^2y}{dx^2} = 14$

- The second order derivative can also be referred to simply as the **second derivative**
  - Similarly, the 'regular' derivative can also be referred to as either the **first order derivative** or the **first derivative**
- The second order derivative gives the rate of change of the gradient function (ie of the first derivative) – this will be important for identifying the nature of **stationary points**

Examiner Tips and Tricks

- When finding second derivatives be especially careful with functions that have negative or fractional powers of ***x*** (see Worked Example below).

- Mistakes made with fractions or negative signs can build up as you calculate the derivative more than once.

Your notes

Worked Example

Given that $f(x) = 4 - \sqrt{x} + \dfrac{3}{\sqrt{x}}$

a) Find $f'(x)$ and $f''(x)$.

b) Evaluate $f''(3)$. Give your answer in the form $a\sqrt{b}$,

where b is an integer and a is a rational number.

a) $f(x) = 4 - x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$ REWRITE AS POWERS OF x

$f'(x) = -(\frac{1}{2})x^{\frac{1}{2}-1} + 3(-\frac{1}{2})x^{-\frac{1}{2}-1}$

$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$ DIFFERENTIATE ONCE TO FIND $f'(x)$

$f''(x) = -\frac{1}{2}(-\frac{1}{2})x^{-\frac{1}{2}-1} - \frac{3}{2}(-\frac{3}{2})x^{-\frac{3}{2}-1}$

$f''(x) = \frac{1}{4}x^{-\frac{3}{2}} + \frac{9}{4}x^{-\frac{5}{2}}$ DIFFERENTIATE A SECOND TIME TO FIND $f''(x)$

b) $f''(x) = \dfrac{1}{4x\sqrt{x}} + \dfrac{9}{4x^2\sqrt{x}}$ $x^{\frac{3}{2}} = x\sqrt{x}$ $x^{\frac{5}{2}} = x^2\sqrt{x}$

$f''(3) = \dfrac{1}{12\sqrt{3}} + \dfrac{9}{36\sqrt{3}}$

$= \dfrac{12}{36\sqrt{3}} = \dfrac{1}{3\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{9}$

RATIONALISE DENOMINATOR

$f''(3) = \frac{1}{9}\sqrt{3}$