Edexcel International A Level (IAL) Maths: Mechanics 2

Your notes

Centres of Mass

Contents

Particles Along a Straight Line

What is/are centres of mass?

- The **centre of mass** of a **body** (or a **system** of **bodies**) is the **point** at which the **total mass** of the **body** (or **system**) can be considered to **act as one**
  - It is the **single point** at which the (force) **weight** (**W = mg**)of the **whole system** acts
- Bodies are modelled as **particles**

What is meant by particles along a straight line?

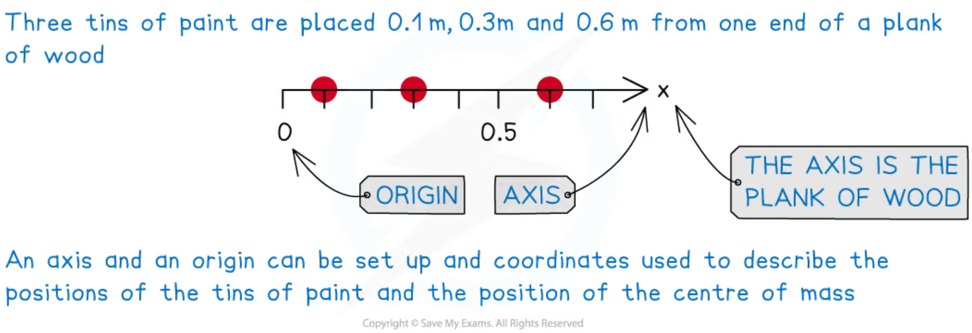
- This is when several **particles** making a **system** are arranged in a straight line
- This will be either horizontally or vertically – usually with reference to the *x*-axis or *y*-axis
  - $(\overline{x}, 0)$ would be the coordinates of the centre of mass along the x-axis
  - $(0, \overline{y})$ would be the coordinates of the centre of mass along the y-axis

So the particles do not have to be connected to each other?

- Not necessarily.
- Any **object** that is used to **connect** the **particles** can affect the centre of mass (see Revision Note 2.1.7 Non-uniform Objects & Other Problems) ***unless*** it is **modelled** as being **light**
  - e.g. Tins of paint (particles) placed at various points along a plank of wood (connecting object)
    - If the plank of wood is **modelled** as **light** it will have no influence on the position of the centre of mass – i.e. it can be ignored

How do I find the centre of mass of a system of particles along a straight line

- Firstly, if no **axis** or **origin** have been given in a question, you will need to **create your own**
  - For example, the left-hand edge of the plank of wood could be the origin with the x -axis increasing towards the right-hand edge of the plank of wood



Your notes

- In general for a system of *n* **particles** with **masses** $m_1, m_2, m_3, \ldots. m_n$ placed along the *x*-axis at the points with **coordinates** $(x_1, 0), (x_2, 0), (x_3, 0), \ldots. (x_n, 0)$, the **centre** of **mass**, point $(\bar{x}, 0)$ is found by solving

$$\sum_{i=1}^{n} m_i x_i = \bar{x} \sum_{i=1}^{n} m_i$$

- We're sure you can figure out the equivalent equation for particles placed along the y-axis!

STEP 1 Draw a diagram indicating the particles and their positions on the line.

This does not need to be a scale diagram but should indicate the axis and origin.

(Remember you may have to create your own axis and origin.)

If given a diagram, add anything necessary to it.

STEP 2 Set up an equation for the -coordinate for the centre of mass using

$$\sum m_i x_i = \bar{x} \sum m_i$$

(This is the same equation as above but you may sometimes see

it with the *n* and the *i* =1 removed, as we have done here.)

STEP 3 Solve the equation and answer the question.

This may be giving the centre of mass as coordinates or describing its position relative to an object or body.

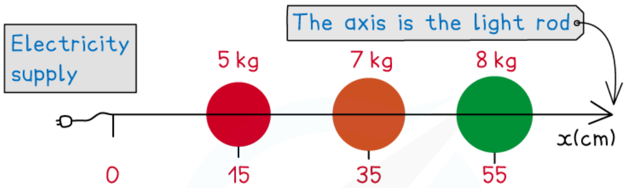
You can check your answer is sensible by comparing it to the location of the particles.

Worked Example

A set of 3 disco lights are modelled as being particles placed every 20 cm along a horizontal light rod. The first light is located 15 cm from the end of the rod that is connected to the electricity supply. Starting with the first light, in order, the lights have masses of 5 kg, 7 kg and 8 kg.

Describe the position of the centre of mass of the disco lights in terms of its distance from the end of the rod that is connected to the electricity supply.

Step 1: Draw a diagram, adding an axis and an origin



Step 2: Set up an equation using $\Sigma m_i x_i = \bar{x} \Sigma m_i$

$5 \times 15 + 7 \times 35 + 8 \times 55 = \bar{x}(5 + 7 + 8)$

Step 3: Solve the equation and answer in context required

$20\bar{x} = 760$

$\bar{x} = 38$

$\therefore$ The position of the centre of mass of the disco lights is 38 cm from the end of the rod that is connected to the electricity supply

Examiner Tips and Tricks

- Sketch diagram(s) or add to any given in a question.
- If not referenced in a question create a coordinate system of your own, making it clear on your diagram(s) where the origin is.

Particles in a (2D) plane

What is meant by particles in a (2D) plane?

- Particles in a (2D) **plane** refers to a system where **particles** are arranged on a **surface**
  - e.g. The playing pieces (particles) on a chess board (plane)
- The two dimensions are horizontal and vertical
  - **Cartesian** (x -y axes) **coordinates** are used to describe the positions of the particles with $(\bar{x}, \bar{y})$ being the coordinates of the **centre** of **mass** of the system
- **If** the particles are connected by an **object** – including the plane itself - the centre of mass may be affected (see Revision Note 2.1.7 Non-uniform Objects & Other Problems) - ***unless*** the connecting object is **modelled** as being **light**

Your notes

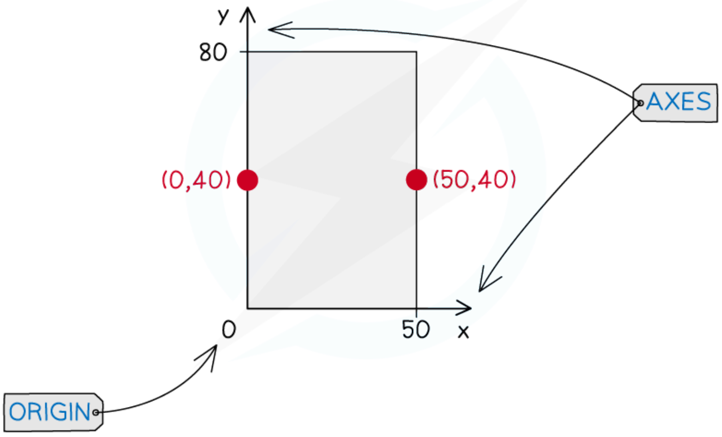
Your notes

How do I find the centre of mass of a system of particles in a (2D) plane?

- Firstly, if no **axis** or **origin** have been given in a question, you will need to **create your own**
  - For example, the bottom left corner of a sheet of metal edge could be the origin with the x-axis running along the bottom edge of the sheet and the y-axis running up the left hand side of the sheet.

Two weights are placed on a rectangular sheet of metal measuring 50 cm by 80 cm

The weights are placed at the midpoints on the two longer edges of the sheet of metal



An axis and an origin can be set up and coordinates used to describe the positions of the weights (and the position of the centre of mass)

- In general for a system of n **particles** with **masses** $m_1, m_2, m_3, \ldots.$ placed in a 2D plane at the points with **coordinates** $(x_1, y_1), (x_2, y_2), (x_3, y_3) \ldots.., (x_n, y_n)$, the **centre of mass**, point $(\bar{x}, \bar{y})$ is found by solving

$$\sum_{i=1}^{n} m_i r_i = \bar{r} \sum_{i=1}^{n} m_i$$

where $r_i = x_i \mathbf{i} + y_i \mathbf{j}$ are the **position vectors** of the particles and $\bar{r} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$ is the position vector of the centre of mass

- (Position) **vectors** can be given as **column vectors**; $\begin{pmatrix} x \\ y \end{pmatrix}$ or in **i-j notation**; $x\mathbf{i} + y\mathbf{j}$
- Alternatively the two dimensions can be separated creating two systems of particles in a straight line

- Use the equations $\sum m_i x_i = \bar{x} \sum m_i$ and $\sum m_i y_i = \bar{y} \sum m_i$ to find the coordinates of the centre of mass, separately
- Remember to give your final answer in the form $(\bar{x}, \bar{y})$

STEP 1 Draw a diagram indicating the particles and their positions on the plane.

This does not need to be a scale diagram but should indicate the axes and origin.

(Remember you may have to create your own axes and origin.)

If given a diagram, add anything necessary to it.

STEP 2 Set up an equation for the position vector of the centre of mass using

$$\sum m_i r_i = \bar{r} \sum m_i$$

(This is the same equation as above but you may sometimes see it with the n and th $i=1$ removed, as we have done here.)

Alternatively, you can set up separate equations for $\bar{x}$ and $\bar{y}$.

STEP 3 Solve the equation(s) and answer the question.

This may be giving the position of the centre of mass as coordinates, a position vector or describing its position relative to an object or body (including the plane).

You can check your answer is sensible by comparing it to the location of the particles.

Worked Example

A system of two particles lies in the x-y plane. The first particle has mass 2.4 kg and is located at the point (-3, -2). The second particle has mass 7.6 kg and is located at the point (p, q). Given that the coordinates of the centre of mass of the system is (0.04, 3.32) find the values of p and q.

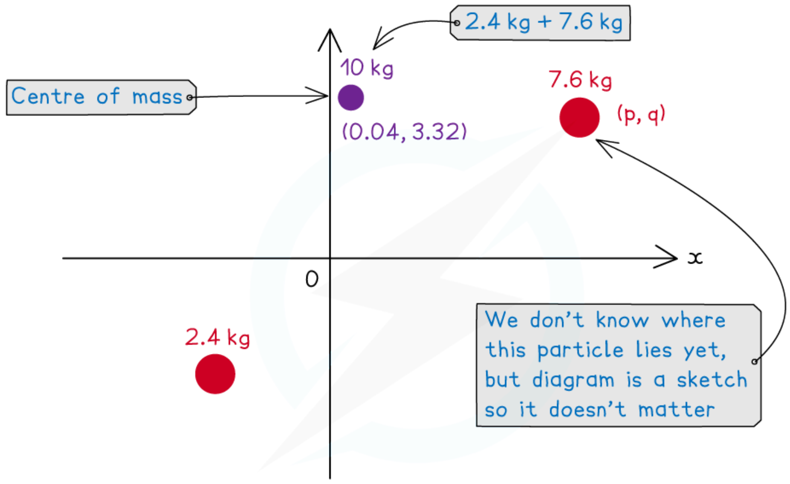
Answer:

Your notes

Notice in this question the coordinates of the centre of mass are given, and we asked to find the location of a particle

The method of solution is the same except the equation will need rearranging differently in order to solve it

Step 1: Draw a diagram, adding axes and the origin



Step 2: Set up an equation using $\Sigma m_i \mathbf{r}_i = \bar{\mathbf{r}} \Sigma m_i$

2.4 + 7.6

$2.4(-3\mathbf{i} - 2\mathbf{j}) + 7.6(p\mathbf{i} + q\mathbf{j}) = 10(0.04\mathbf{i} + 3.32\mathbf{j})$

Step 3: Solve the equation and answer in the format required

$7.6(p\mathbf{i} + q\mathbf{j}) = (0.4 + 7.2)\mathbf{i} + (33.2 + 4.8)\mathbf{j}$

$7.6(p\mathbf{i} + q\mathbf{j}) = 7.6\mathbf{i} + 38\mathbf{j}$

$\therefore p = 1, q = 5$

Examiner Tips and Tricks

- Sketch diagram(s) or add to any given in a question.
- If not referenced in a question create a coordinate system of your own, making it clear on your diagram(s) where the origin is.
- You do not have to use vector notation; you can treat each dimension separately. However, do ensure your final answer is in the required format.

Your notes

Your notes

Uniform Laminas

A brief reminder of centre of mass ...

- The **centre of mass** is the **point** at which the **total mass** of a body (or a system of bodies) can be considered to **act as one**

What is a lamina?

- A **lamina** is a (3D) body that has **one very small dimension** compared to the other **two dimensions** which means the third dimension is negligible and the body can be modelled as a **(2D) plane**
  - Common examples include
    - a sheet of paper, card, wood, metal, plastic, cloth, etc
    - a wall

What modelling assumptions are used with laminas?

- (For this Revision Note) all laminas are **uniform**
  - **uniform** means the **mass** per **unit area (*m* kg)** is **equal** at **every point** on the lamina
    - In the real world a uniform lamina is very difficult to achieve – particularly with natural materials such as timber where knots and other imperfections occur
    - The cloth on a professional snooker table needs to be as close as possible to a uniform lamina – it should have no imperfections (such as a small tear) if the table and balls are to play as a professional would expect
- In Mechanics 2 both **uniform** (this Revision Note) and **non-uniform** laminas (Revision Note 2.1.7) are considered

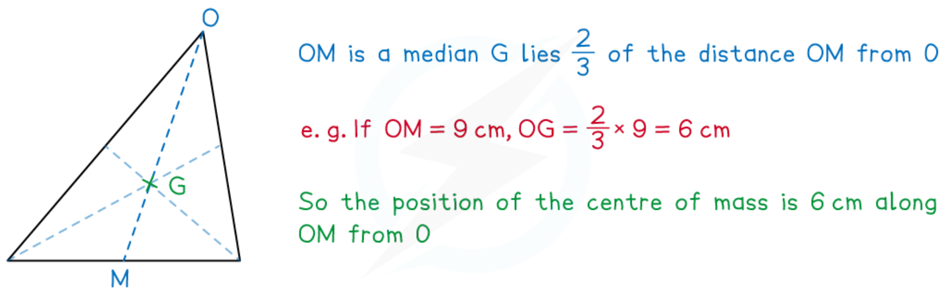
How do I find the centre of mass of a (standard) uniform lamina?

- If the **shape** of a **lamina** has an **axis** of **symmetry**, the position of the centre of mass will be somewhere on that **axis**
- If the **shape** of a **lamina** has **more than one axes** of **symmetry,** the position of the centre of mass will be the **intersection** of those **axes**
- The most **common** shapes – **rectangles** (including squares), **triangles**, **circles** and **sectors** (including **semi-circles**) are referred to as **standard** uniform laminas

How do I find the centre of mass of a triangular lamina?

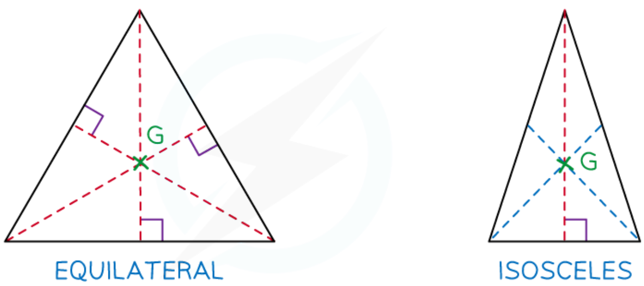
- Triangles are a little more complicated

- In **any** triangle a **median** is a line joining a **vertex** to the **midpoint** of the **opposite side**
- In **any uniform** triangle the position of the centre of mass
  - lies $\frac{2}{3}$ **of the way along a median** when measured from its **vertex**.
    This is stated in the formulae booklet
  - is the **point** of **intersection** of the **medians**



- The **coordinates** of the **centre** of **mass** of **any uniform** triangle can be found if the coordinates of its **three vertices** are known
  - The coordinates of the centre of mass of **any uniform triangular lamina** is the **mean** of the coordinates of its vertex
  - This only works for triangles and is **not true** for all other polygons

- In addition to the above, for **equilateral** and **isosceles** triangles only
  - In an **equilateral** triangle all three **medians** are also **axes** of **symmetry**
  - In an **isosceles** triangle one of the **medians** is an **axis** of **symmetry**
  - The position of the centre of mass will always lie on an **axis** of **symmetry** so for **both equilateral** and **isosceles** triangles only **one median** need be used
  - An **axis** of **symmetry** is **perpendicular** to the side they **bisect** so any missing lengths can be found using **Pythagoras' theorem** and/or basic **trigonometry**



Your notes

Your notes

Worked Example

A shop sign is made from a sheet of wood in the shape of an isosceles triangle with its two equal sides of length 60 cm meeting at the vertex O with an angle of $\frac{2\pi}{3}$ radians.

(a) State any modelling assumptions regarding the sheet of wood being used to make the shop sign.

(b) Describe the position of the centre of mass of the shop sign in relation to vertex O.

The shop sign is adapted slightly to make the shape of a sector of a circle with the two equal sides of the isosceles triangle forming two radii of the sector and vertex O becoming the centre of the circle

(c) Describe the new position of the centre of mass of the shop sign in relation to vertex O.

(a) State any modelling assumptions regarding the sheet of wood being used to make the shop sign.

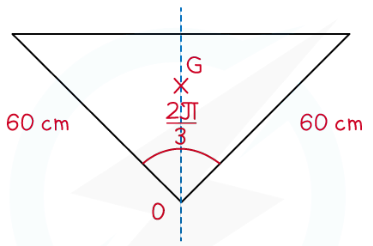
Answer:

a) Modelling assumptions would be that the sheet of wood is a LAMINA and is UNIFORM

(b) Describe the position of the centre of mass of the shop sign in relation to vertex O.

Answer:

b) Sketch a diagram...



Length of median through O $\quad 60\cos\left(\frac{\pi}{3}\right) = 30$

$\frac{2}{3}$ of distance of median $\quad \frac{2}{3} \times 30 = 20$

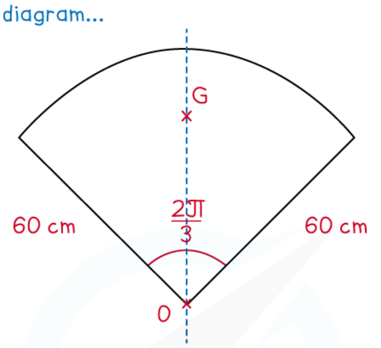
The position of the centre of mass is 20 cm from vertex O along the median through O

The shop sign is adapted slightly to make the shape of a sector of a circle with the two equal sides of the isosceles triangle forming two radii of the sector and vertex O becoming the centre of the circle

(c) Describe the new position of the centre of mass of the shop sign in relation to vertex O.

Answer:

c) Sketch a diagram...



"Sector of a circle, $\frac{2r\sin\alpha}{3\alpha}$ from centre"

$r = 60$

$2\alpha = \frac{2\pi}{3}, \alpha = \frac{\pi}{3}$

$\therefore \frac{2r\sin\alpha}{3\alpha} = \frac{120\sin\left(\frac{\pi}{3}\right)}{\frac{\cancel{3}\pi}{\cancel{3}}} = \frac{60\sqrt{3}}{\pi}$

The position of the centre of mass is $\frac{60\sqrt{3}}{\pi}$ cm

(33.1 cm to 3 s.f.) from vertex O along the median through O

Examiner Tips and Tricks

- Sketch diagram(s) or add to any given in a question – this may help you to see an axis of symmetry.
- If a triangle is involved, be clear about the type of triangle (and the information about it) you have been given.
- Remember that a semi-circle is a sector of a circle with angle π radians at the centre.
- Beware! The formula booklet lists the formulae for laminas and frameworks together.

Your notes

Composite Laminas

A brief reminder of centre of mass ...

- The **centre of mass** is the **point** at which the **total mass** of a body (or a system of bodies) can be considered to **act as one**

What is a composite lamina?

- A **composite lamina** is a non-standard shaped lamina that can be constructed from **two or more** of the **standard uniform laminas** – **rectangles**, **circles**, **sectors** and **triangles**
- Examples include
  - two rectangular laminas made into a L-shaped shelf
  - a semi-circular lamina cut from a larger semi-circular lamina to create a rainbow shaped sign
  - two overlapping circular laminas
- In Mechanics 2 both **uniform** (this Revision Note) and **non-uniform** laminas (Revision Note 2.1.7) are considered

What modelling assumptions are used with composite laminas?

- (For this Revision Note) all laminas making a composite lamina are made from the **same uniform material**
  - **uniform** means the **mass** per **unit area** (m kg) is **equal** at **every point** on a lamina
    - the **mass** of a lamina is **proportional** to its **area** (A m^2)
    - i.e. Mass of a lamina = Am kg
  - **every** lamina being **made** from the **same material** means that the mass per unit length will be the **same** for every lamina
    - i.e. m will be the **same** for every lamina
    - m's cancel in the equation for finding the position of the centre of mass
    - only the value of A(i.e. the **area** of each lamina) is needed in calculations
- A composite lamina is flat so is modelled as existing in a (2D) plane
  - the third dimension will be small compared to the other two so is negligible
- Any material/mass used in the process to join laminas is negligible

How do I find the centre of mass of a composite lamina?

- In short, find the **area** and **position** of the **centre** of **mass** for the **individual** parts of the composite lamina, then treating these as **particles** with **mass equal** to the **area**, use the equation below (Revision Note 2.1.1) to find the position vector of the centre of mass of the composite lamina

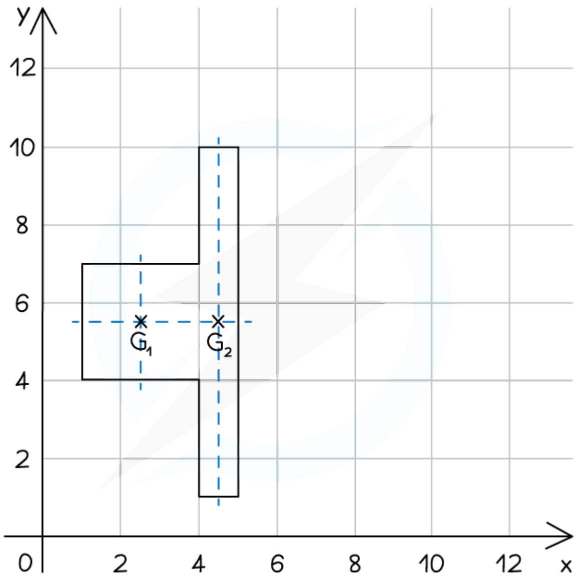
$$\sum m_i \mathbf{r}_i = \bar{\mathbf{r}} \sum m_i$$

STEP 1 Sketch a diagram, or add to a diagram if one has been given

Create your own axes if necessary

Split the composite lamina into two or more standard laminas

e.g.



STEP 2 Calculate the area and the position of the centre of mass for each standard lamina

List the results in a table to make the equation easier in the next step

	Square	Rectangle	Whole
Mass/Area	9	9	18
$\bar{x}$	2.5	5.5	$\bar{x}$
$\bar{y}$	4.5	5.5	$\bar{y}$

(For both squares and rectangles, the position of the centre of mass will be their centres; $G_1(2.5, 5.5)$ and $G_2(4.5, 5.5)$ which are easy to 'see', similarly the areas of these shapes are simple to calculate so the table can be completed directly.)

Your notes

STEP 3 Treat G_1 and G_2 as the positions of **particles** with **masses** the area of the standard laminas.

Use $\sum m_i r_i = \bar{r} \sum m_i$ to find the **position vector** of the centre of mass (G) of the composite lamina.

Remember to give the final answer in the required format.

e.g. $9(2.5\mathbf{i} + 5.5\mathbf{j}) + 9(4.5\mathbf{i} + 5.5\mathbf{j}) = 18(\bar{x}\mathbf{i} + \bar{y}\mathbf{j})$

$$18(\bar{x}\mathbf{i} + \bar{y}\mathbf{j}) = 63\mathbf{i} + 99\mathbf{j}$$

$$(\bar{x}\mathbf{i} + \bar{y}\mathbf{j}) = 3.5\mathbf{i} + 5.5\mathbf{j}$$

So the coordinates of the centre of mass (G) of the composite lamina are(3.5, 5.5)

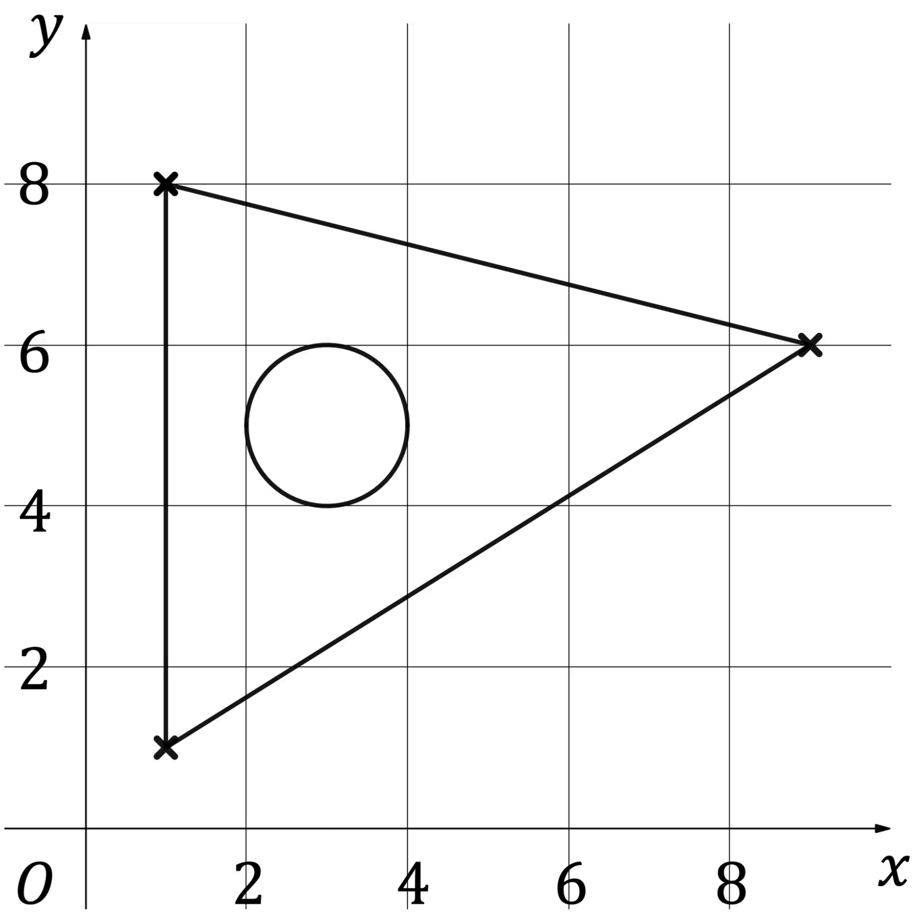
Looking back at the diagram this seems a sensible answer.

- In the example above
  - x and y could've been treated separately rather than use vector notation
  - the line $y = 5.5$ is an axis of symmetry of the **composite** lamina so the y-coordinate of the centre of mass could've been written down without any calculations being made
- **Composite laminas** also include cases where a **standard** lamina has been **removed** from another standard lamina – the **worked example** below demonstrates this

Worked Example

A uniform lamina in the shape of a triangle has a circle of radius 1 cm and centre (3, 5) removed from it as shown in the diagram below.

Your notes

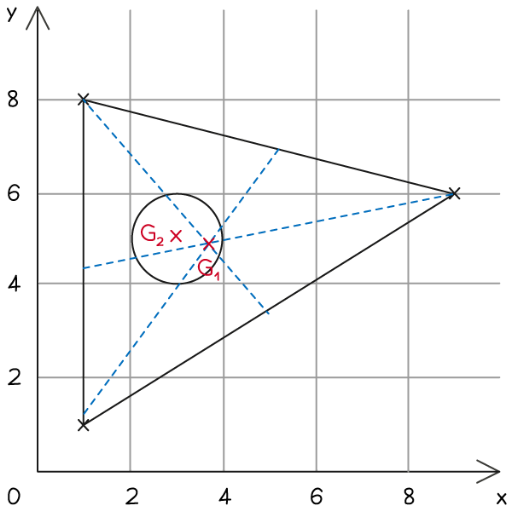


Find the coordinates of the centre of mass of the resulting lamina, giving your coordinates to three significant figures.

Answer:

Your notes

Step 1: Add useful information to the given diagram



Step 2: Calculate the area and position of the centre of mass for each standard lamina
Table the results

Triangle area: $\frac{1}{2} \times 7 \times 8 = 28$

Triangle centre of mass (G_1) $\left(\frac{1+1+9}{3}, \frac{1+8+6}{3}\right)$

$= \left(\frac{11}{3}, 5\right)$

(Circle is easy to calculate mentally, so results put directly into the table)

	Triangle	Circle	Whole
Area	28	$-\pi$	$28 - \pi$
$\bar{x}$	$\frac{11}{3}$	3	$\bar{x}$
$\bar{y}$	5	5	$\bar{y}$

$-\pi$ as the circle has been removed

Step 3: Treat as particles and use $\sum m_i \mathbf{r}_i = \bar{\mathbf{r}} \sum m_i$ to find the position vector of the centre of mass

$28\left(\frac{11}{3}\mathbf{i} + 5\mathbf{j}\right) - \pi(3\mathbf{i} + 5\mathbf{j}) = (28 - \pi)(\bar{x}\mathbf{i} + \bar{y}\mathbf{j})$

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = (3.750919...)\mathbf{i} + 5\mathbf{j}$

$\therefore$ The coordinates of the centre of mass of the resulting lamina is (3.75, 5) to 3 s.f.

Examiner Tips and Tricks

- Sketch diagram(s) or add to any given in a question.
- If not referenced in a question create a coordinate system of your own, making it clear on your diagrams where your origin is.

- For composite laminas that have parts removed, treat the parts removed as having negative area.
- You do not have to use vector notation; you can treat each dimension separately. However, do ensure your final answer is in the required format.
- Beware! The formula booklet lists the formulae for laminas and frameworks together.

Your notes

Frameworks

A brief reminder of centre of mass …

- The **centre of mass** is the **point** at which the **total mass** of a body (or a system of bodies) can be considered to **act as one**

What is a framework?

- A **framework** is created by joining thin-linear-shaped bodies (e.g. a pole or post) together to create a **flat** design or shape
- Such bodies can be **modelled mathematically** as **rods** or **wires**
- Examples include
  - the supporting structure for the roof of a house
  - four straight rope lights joined together to make a rectangular rope light
- In Mechanics 2 both **uniform** (this Revision Note) and **non-uniform** rods (Revision Note 2.1.7) are considered
- A framework would not be split into standard shapes as **perimeter** rather than the **area** is being worked with

What modelling assumptions are used with frameworks?

- (For this Revision Note) all rods (or wires) making a framework are made from the **same uniform material**
  - **uniform** means the **mass** per **unit length** (m kg) is **equal** at **every point** along a rod
    - the **mass** of a rod is **proportional** to its **length** (l m)
    - i.e. Mass of a rod = lm kg
  - **every** rod/wire being **made** from the **same material** means that the mass per unit length will be the **same** for every rod/wire
    - i.e. m will be the **same** for every rod
    - m's cancel in the equation for finding the position of the centre of mass
    - only the values of l (i.e. the **lengths** of each rod) are needed in calculations
- A framework is flat so is modelled as existing in a **(2D) plane**
  - the third dimension will be small compared to the other two so is negligible
- Any material/mass used in the process to join rods (or wires) is negligible

How do I find the centre of mass of a framework?

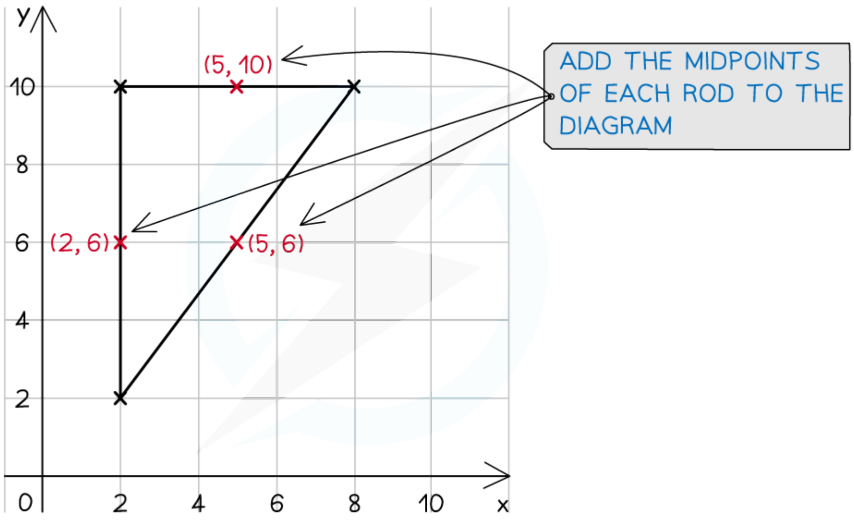
- In short, find the **length** and **position** of the **centre** of **mass** for each **individual** rod, then treating these each as a **particle** with **mass equal** to the **length** of the rod, use the equation $\sum m_i r_i = \bar{r} \sum m_i$ to find the position vector of the centre of mass of the framework (Revision Note 2.1.1)

STEP 1 Sketch a diagram, or add to a diagram if one has been given

Create your own axes if necessary

Identify the different rods making up the framework

e.g.



STEP 2 Find the length and the position of the centre of mass of each rod using midpoints

List the results in a table to make the equation easier in the next step

	Rod 1	Rod 2	Rod 3	Whole
Mass (Length)	8	6	10	24
$\bar{x}$	2	5	5	$\bar{x}$
$\bar{y}$	6	10	6	$\bar{y}$

The length of the diagonal rod (Rod 3) is found by Pythagoras' Theorem, $\sqrt{8^2 + 6^2} = 10$

Your notes

STEP 3 Treat the **midpoints** as the positions of **particles** with **masses** the length of the rods.

Use $\sum m_i r_i = \bar{r} \sum m_i$ to find the **position vector** of the centre of mass (G) of the framework.

Remember to give the final answer in the required format.

e.g. $8(2\mathbf{i} + 6\mathbf{j}) + 6(5\mathbf{i} + 10\mathbf{j}) + 10(5\mathbf{i} + 6\mathbf{j}) = 24(\bar{x}\mathbf{i} + \bar{y}\mathbf{j})$

$$24(\bar{x}\mathbf{i} + \bar{y}\mathbf{j}) = 96\mathbf{i} + 168\mathbf{j}$$

$$\bar{\mathbf{r}} = (\bar{x}\mathbf{i} + \bar{y}\mathbf{j}) = 4\mathbf{i} + 7\mathbf{j}$$

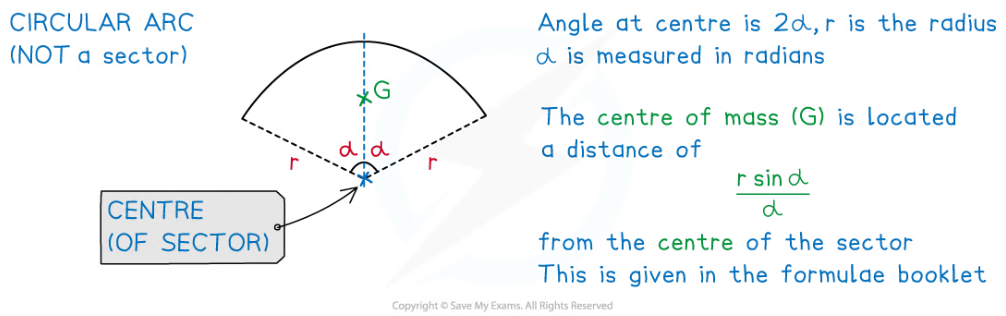
So the coordinates of the centre of mass (G) of the framework are (4, 7)

Looking back at the diagram this seems a sensible answer.

- In the example above x and y could've been treated separately rather than use vector notation

What if a rod or wire is curved (circular)?

- In Mechanics 2 if a **curved rod** or **wire** is involved in a **framework** it will be in the shape of a **circular arc**
- Recall that the formulae for the **length** of an **arc** is "$l = r\theta$" where *l* is the length of the arc, *r* is the radius and θ is the angle at the centre measured in radians
- The formula for finding the **position** of the **centre** of **mass** of a **circular arc** is **similar** but **different** to that for a **sector** of a **circle lamina**



- This is given in the formulae booklet
- Some questions may work with a **circular arc only**, some may work a **sector** of a **circle** that would include the two radii

Worked Example

Your notes

A light-up shop sign is a framework made from two straight wires and one circular arc wire creating the shape of a sector of a circle. The angle at the centre of the sector is $\frac{\pi}{3}$ radians and the two straight wires are both 60 cm in length. All three wires are made from the same uniform material.

(a) State the significance of all three wires being made from the **same uniform material**.

(b) Describe the position of the centre of mass of the shop sign in relation to the centre of the sector.

(a) State the significance of all three wires being made from the **same uniform material**.

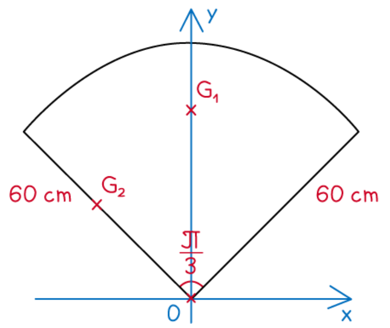
a) The three rods being made from the same uniform material is a modelling assumption that means the mass per unit length for each rod is the same at every point along its length and the mass per unit length is the same for every rod.
This means the length of each rod can be used as its mass in calculations to find the centre of mass of the framework

(b) Describe the position of the centre of mass of the shop sign in relation to the centre of the sector.

Answer:

Your notes

b) Step 1: Sketch a diagram, add axes



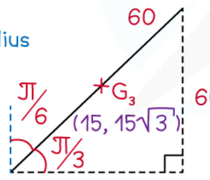
Step 2: Find the length and centre of mass for each rod
List results in a table

Circular arc $\alpha = \frac{\pi}{6}, r = 60$

"$l = r\theta$" $\text{length} = 60 \times \frac{\pi}{6} = 10\pi$

$\frac{\text{"}r\sin\alpha\text{"}}{\alpha}$ $\frac{r\sin\alpha}{\alpha} = \frac{60\sin\frac{\pi}{6}}{\frac{\pi}{6}} = \frac{180}{\pi}$

Radius



$60\sin\frac{\pi}{3} = 30\sqrt{3}$

The other radius should be obvious!

$60\cos\frac{\pi}{3} = 30$

	Arc	Radius 1	Radius 2	Whole
Mass / Length	10π	60	60	$120 + 10\pi$
$\bar{x}$	0	15	-15	$\bar{x}$
$\bar{y}$	$\frac{180}{\pi}$	$15\sqrt{3}$	$15\sqrt{3}$	$\bar{y}$

Your notes

Your notes

Step 3: Use $\Sigma m_i \mathbf{r}_i = \bar{\mathbf{r}}\Sigma m_i$ to find the centre of mass of the framework

$$10\pi\left(\frac{180}{\pi}\mathbf{j}\right) + 60\,(15\mathbf{i} + 15\sqrt{3}\,\mathbf{j}) + 60\,(-15\mathbf{i} + 15\sqrt{3}\,\mathbf{j})$$
$$= (120 + 10\pi)(\bar{x}\mathbf{i} + \bar{y}\mathbf{j})$$

This could've been deduced using symmetry

$$10(\pi + 12)(\bar{x}\mathbf{i} + \bar{y}\mathbf{j}) = 0\mathbf{i} + (1800 + 1800\sqrt{3})\mathbf{j}$$

$$\therefore \bar{\mathbf{r}} = (\bar{x}\mathbf{i} + \bar{y}\mathbf{j}) = \frac{180(1 + \sqrt{3})\mathbf{j}}{\pi + 12} \quad (32.478032...)$$

$\therefore$ The centre of mass of the shop sign lies 32.5 cm (3 s.f.) vertically (above) from the centre of the sector

We don't know which way up the shop sign is but in our diagram it is above O

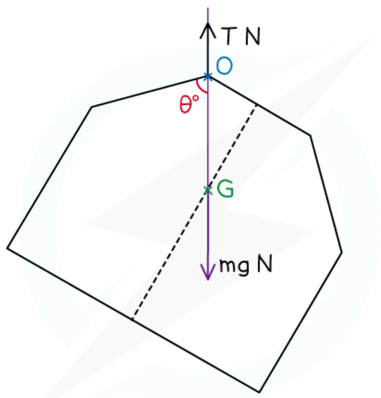
Examiner Tips and Tricks

- The methods for finding the position of the centre of mass for a sector and a circular arc are similar but different. Both are given in the formulae booklet. Ensure you are clear about which you are working with.
- Beware! The formula booklet lists the formulae for laminas and frameworks together.

Objects Suspended in Equilibrium

What is meant by objects suspended in equilibrium?

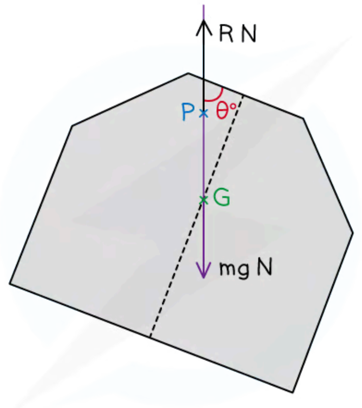
- An **object** can be **suspended** at a fixed point by a **string** such that it is allowed to **hang freely** and be in **equilibrium** (at rest)
  - e.g. A sign hung from the ceiling of a shop by single rope (string)
  - An **object** will either be a **uniform lamina** or a **uniform framework**
- **Equilibrium** means
  - there is **zero resultant** (net) **force** acting on the object
  - there is **zero resultant** (net) **moments** about the point of suspension
- Objects suspended in **equilibrium** have two forces acting on them
  - **tension** in the string (acting at the **fixed point**)
  - **weight** (acting at the position of the **centre** of **mass**)
- These forces act through a **vertical** line which passes through the **point** of **suspension** and the **position** of the **centre** of **mass**
- Problems often require the **angle** that is created between this **vertical** line and an edge of the **object** to be found



An **object, suspended** at a **fixed point** (O), at rest in equilibrium

- A **lamina** could be suspended at a **fixed point** from a **vertex** (as above) or along an **edge**, or, suspended by a **point** on its **surface** that it would then be able to **pivot** (rotate) around
- A **lamina** suspended at a **pivot** will have two forces acting on its
  - **reaction** force (acting at the **pivot**)

- **weight** (acting at the position of the **centre** of **mass**)
- As with the fixed-point scenario above, these forces will act through a vertical line which passes through the point of suspension and the position of the centre of mass



A **lamina, suspended** at a **pivot** (P), at rest in equilibrium

- The mathematics involved in finding the angle is the same for both the fixed-point and pivot scenarios but you should be aware of the different phrasing in questions

How do I solve problems involving freely suspended laminas and frameworks?

STEP 1 Sketch a diagram or use a given one and add to it as you make progress

STEP 2 Find the position of the centre of mass of the object

Ensure you know whether you are working with a

lamina or a **framework** and the **differences** in finding the

position of the **centre** of **mass**

STEP 3 On your diagram, add the **position** of the **centre** of **mass**

Draw a **line** to represent the ***downward*** **vertical**

This will be a **straight** line connecting the **point** at which the **body** is **suspended** to the point of the **centre** of **mass**

Identify the **angle** required

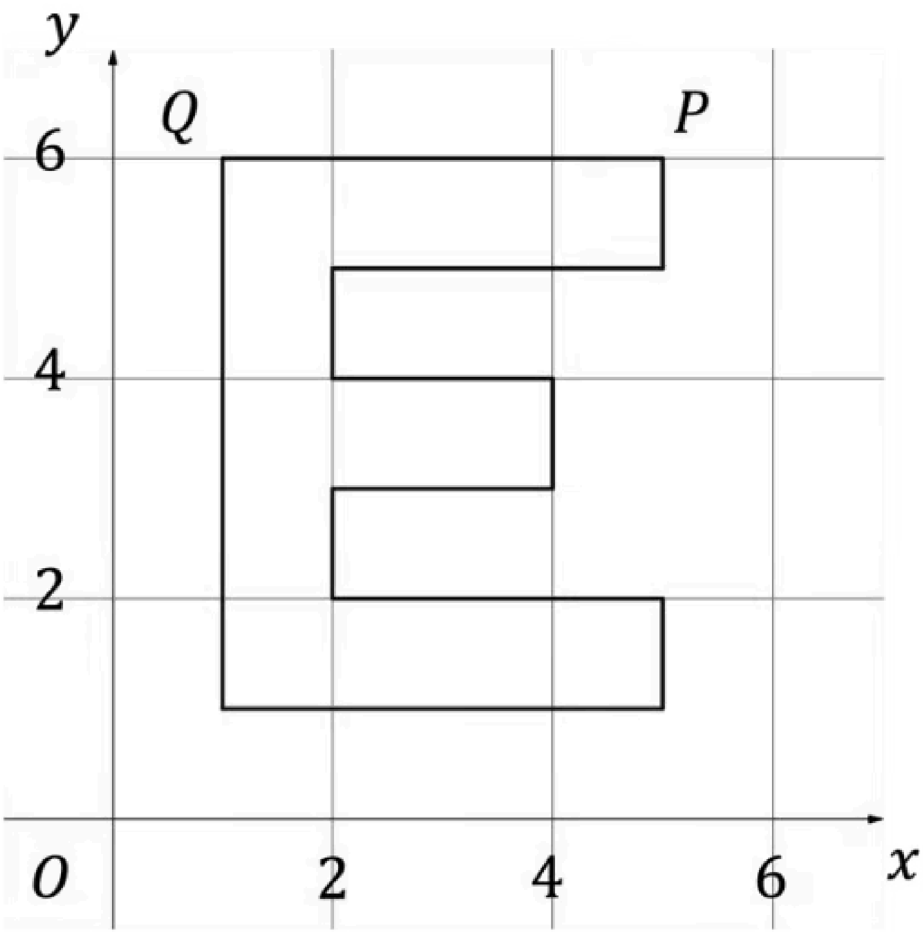
STEP 4 Use trigonometry to find the required angle

Interpret the result in the context of the problem and answer the question accordingly

Worked Example

Your notes

As part of the set in a theatre production, a large polystyrene letter E is to be produced and will hang above the stage by a single rope attached to the ceiling.



The polystyrene letter is modelled as a uniform lamina and the rope is modelled as a light inextensible string.

Find the angle the rope makes with the side PQ when the sign is at rest and suspended from the point P.

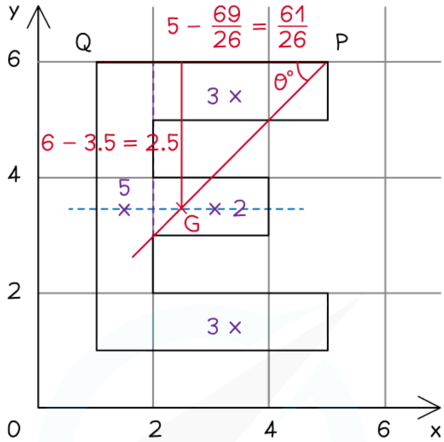
Answer:

Your notes

Your notes

Step 1: Add to the given diagram and do this throughout your solution

This would be the diagram at the end



Step 2: Find the position of the centre of mass

The centre of mass lies on an axis of symmetry, so $\bar{y} = 3.5$

Splitting the shape into four rectangles, using their areas and symmetry, only cosidering x−coordinates

$5 \times 1.5 + 3 \times 3.5 + 2 \times 3 + 3 \times 3.5 = \bar{x}(5 + 3 + 2 + 3)$

$\bar{x} = \frac{69}{26}$ (2.653846...)

Step 3: Add the point of the centre of mass the vertical, and the angle, $\theta°$, to the diagram

Step 4: Use trigonometry to find the required angle

$\tan \theta° = \frac{2.5}{\frac{61}{26}}$

$\theta° = 46.818302...$

$\therefore \theta° = 46.8°$ (1 d.p.)

Examiner Tips and Tricks

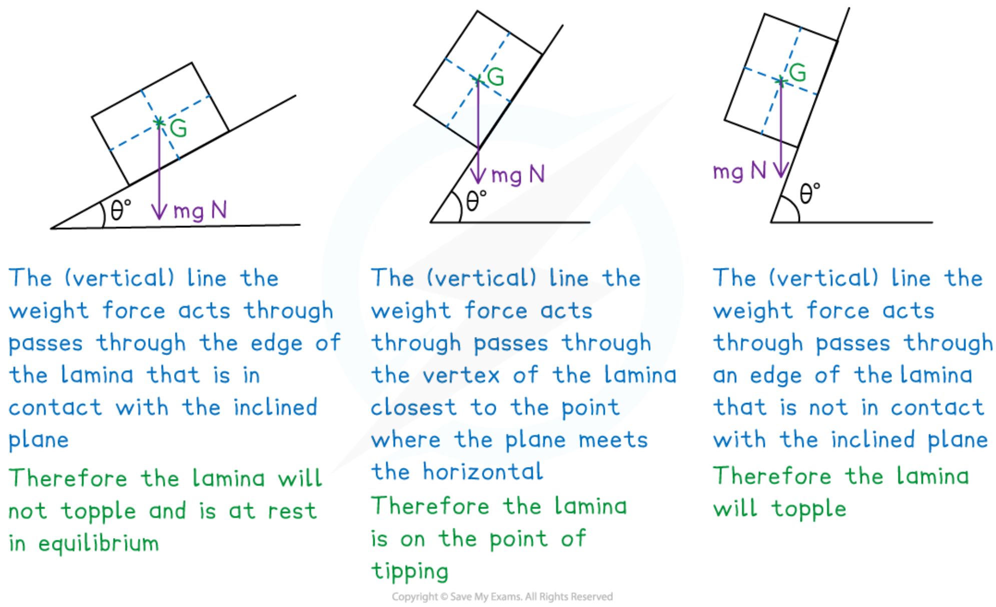
- Ensure you know whether you are working with a lamina or a framework.
- Once you have drawn the downward vertical line on your diagram turn your page (rather than trying to sketch it) to get an idea of how the lamina or framework would look like when suspended from the given point.

Your notes

Point of Tipping

What is meant by point of tipping/toppling?

- These types of problems involve a **lamina** being placed along one of its edges on a **rough inclined plane**
- The (main) **modelling assumption** here is that the **frictional force** between the **plane** and the **lamina** is **sufficient** such that the lamina will **not slide** (or **slip**) down the plane (regardless of the angle of incline)



How do I solve tipping/toppling problems?

STEP 1 Sketch or add anything useful to a given diagram

Define axes if necessary and add these to your diagram

- make the x-axis **parallel** to the plane
- make the y-axis **perpendicular** to the plane
- have the **origin** (O) at the **vertex** of the **lamina** that is in **contact** with the **plane** and **closest** to the **point** where the **plane** meets the **horizontal** ("bottom left")
- **Add** to the **diagram** as you **progress** through the question

STEP 2 Find the **position** of the **centre** of **mass** (G) of the lamina; use your diagram,

axes and symmetry (where possible) to help

The **lamina** may be a **standard shape** or a **composite lamina**

Add the **position** of the **centre** of **mass** to your diagram; also **add** a downward

vertical line (that weight acts through) but you **cannot** use your diagram to

accurately determine whether the lamina will topple

STEP 3 Use **trigonometry** (A new diagram of a right-angled triangle with OG as the hypotenuse can help) to determine the angle at which the **lamina** would be on the **point** of **tipping**

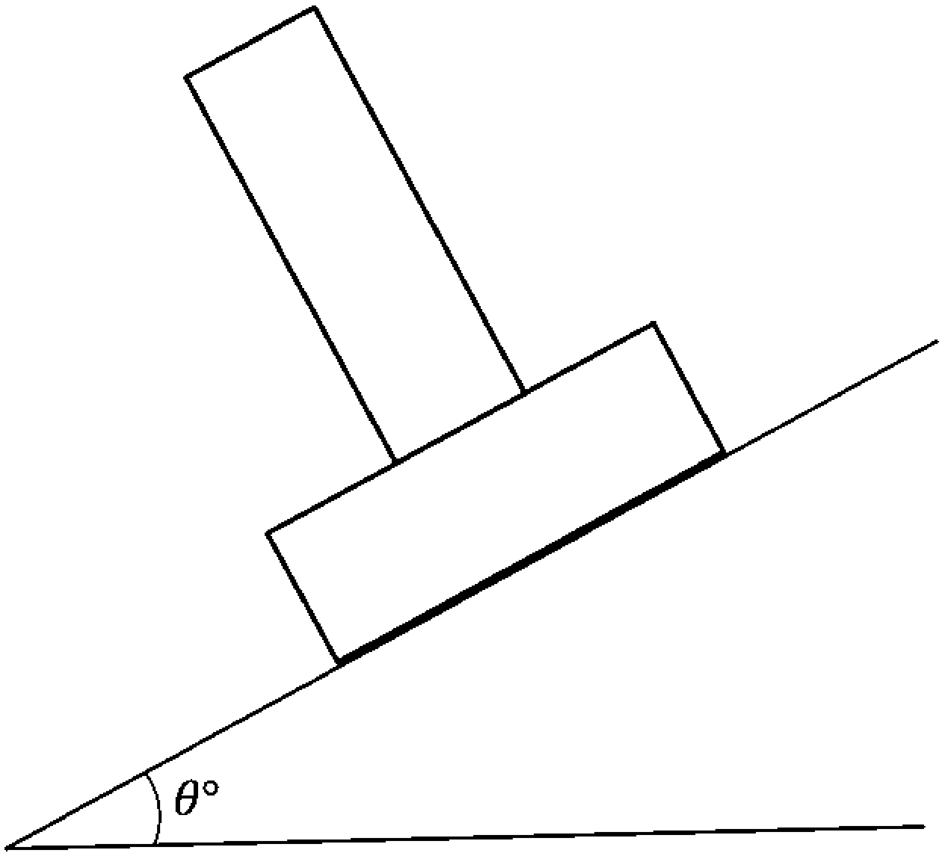
STEP 4 **Interpret** the result in the context of the problem and answer the question accordingly

Your notes

Worked Example

A symmetrical T-shaped lamina is made from two congruent 12 cm by 4 cm rectangles manufactured from the same uniform material.

The lamina is placed on a plane inclined at an angle of $\theta°$, as shown below.

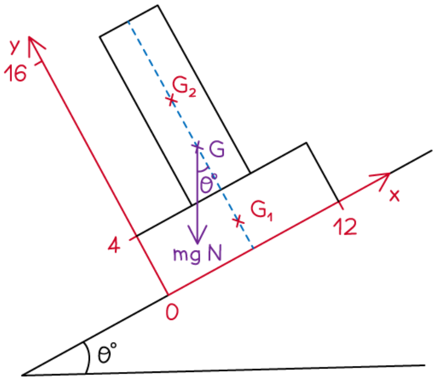


Determine the maximum angle that the plane can be raised to before the lamina will topple. You may assume that the frictional force between the lamina and the plane is sufficient to prevent the lamina sliding down the plane.

Answer:

Step 1: Add to the given diagram and define axes

The positive x−axis runs parallel to and up the inclined plane
The positive y−axis runs perpendicular to and above the inclined plane
The origin, O, is the "bottom left" vertex of the lamina



Step 2: Find the position of the centre of moss of the lamina

Use your diagram, and symmetry to help

By symmetry. $G_1(6, 2)$

$G_2(6, 10)$

$\bar{x} = 6$

Using $\Sigma m_i y_i = \bar{y}\Sigma m_i$

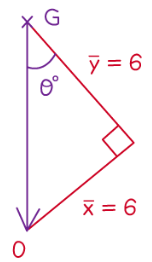
$(4 \times 12) \times 2 + (4 \times 12) \times 10 = \bar{y}(4 \times 12 + 4 \times 12)$

$96\bar{y} = 576$

$\bar{y} = 6$

Position of the centre of mass is **G(6,6)**
Add this to the diagram with the downward vertical line weight acts through

Step 3: Use trigometry to determine the angle at which the lamina is on the point of tipping
Sketch a right angled triangle with hypotenuse OG



$\tan\theta° = \frac{6}{6}$

$\theta° = 45°$

Step 4: Interpret the result and answer the question

The maximum angle that the plane can be raised before the lamina will topple is 45°

Your notes

Examiner Tips and Tricks

Your notes

- Whilst a diagram (sketched or given) will help you work through the stages of your solution, you cannot rely on it to accurately determine whether the (downward vertical) line that weight acts through passes through the edge of the lamina in contact with the plane.
- If defining your own axes, have the x-axis and the y-axis parallel and perpendicular to the plane, with the origin on the “bottom left” vertex of the lamina.

Non-unform Objects & Other Problems

What is meant by a non-uniform lamina/framework?

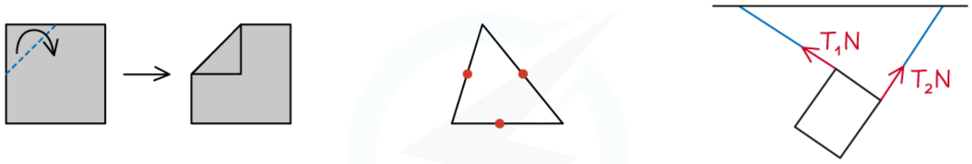
- Strictly speaking a **non-uniform lamina** would be one that has **different densities** at **different points** on its **surface** and a **non-uniform framework** would be one that is made from rods/wires that have **different densities along** their **lengths**.
    - e.g. a plank of wood (timber) may have knots in places on its surface
- However, for the problems you encounter in IAL Mechanics 2 **non-uniform means** each **individual lamina** making a **composite lamina** may be of a **different density** but each **individual lamina** is **uniform**
- Similarly, for a **framework**, each **individual rod/wire** will be **uniform**, but the **framework** may be **made** from rods/wires of **different densities**.

What other problems may I encounter in IAL Mechanics 2 (M2)?

- **Non-uniform problems**
    - Non-uniform (composite) **laminas** may be created by having uniform laminas **overlapping** each other or by taking **part** of a **lamina** and **folding** it back upon itself
    - Non-uniform **frameworks** would be made from **rods/wires** that have **different** (but related) **densities** or **thickness**
    - The **relationship** between the **different densities** will be **explained** in a **question**
        - e.g. part of a (uniform) lamina folded back on itself creates an area of double density
        - e.g. one cable in a framework is twice the thickness of the others
    - In solving problems remember, due to **uniformity**, we use can use **area** (**laminas**) and length (frameworks) rather than **mass** – so if we have **double density**, we **double** the **area/length** where necessary
- Another type of problem is where a **lamina** or a **framework** have an **object** (or **masses**) **attached** to them.
    - This will **affect** the **position** of the **centre** of **mass** of the lamina or framework
        - e.g. a framework made from metal rods could be holding up spotlights above a stage and so have lights (masses) attached to some (or all) rods
- Problems may involve laminas or frameworks being **suspended** by **more than one string** to keep them in a particular position
    - e.g. A sign hanging from a ceiling that needs to be pointing directly left

- Such problems normally involved finding the position of the centre of mass and/or the tension in the strings, using skills from various parts of the mechanics course

Your notes



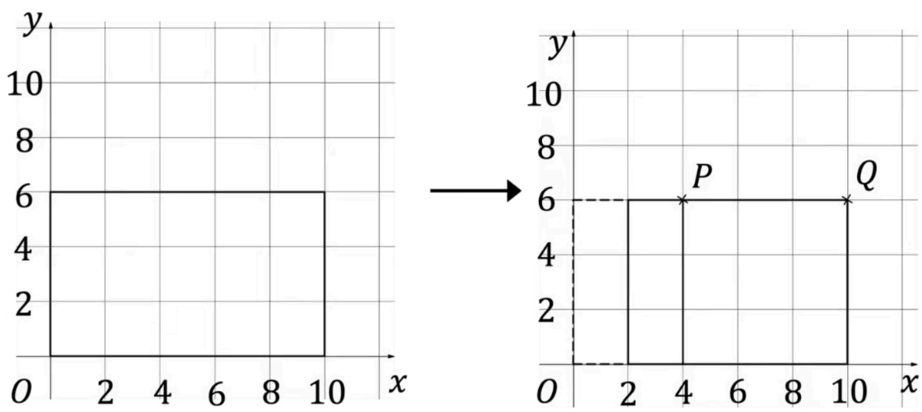
A square lamina has a triangular corner folded back on itself

A triangular framework has three masses located at the midpoints of each rod/wire

An object held in equilibrium by two strings

Worked Example

A rectangular uniform lamina has one edge folded back upon itself as shown in the diagrams below.



(a) Find the coordinates of the centre of mass of the folded lamina

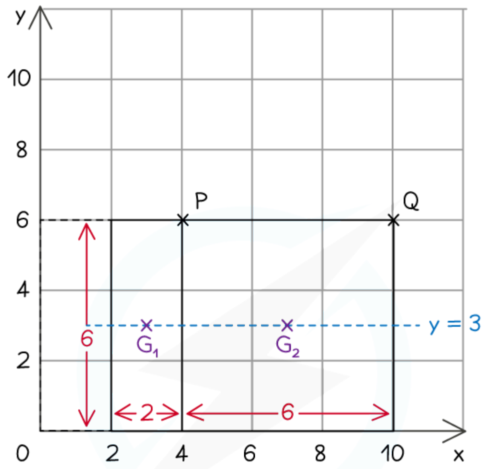
(b) The folded lamina is suspended from a ceiling by two strings attached at the points *P* and Q. The folded lamina remains horizontal in equilibrium. Find the tension in the two strings.

(a) Find the coordinates of the centre of mass of the folded lamina

Answer:

a) Add anything useful to the diagram

When folded, there will still be a horizontal line of symmetry, so the centre of mass will lie on the line $y = 3$ i. e. $\bar{y} = 3$



For the x−coordinate, use $\Sigma m_i x_i = \bar{x}\Sigma m_i$ remembering to use area for mass (for laminas) and that the area will be double for the folded part

Folder part: $A = 2 \times 2 \times 6 = 24$

CoM, G_1, by symmetry is (3,3)

Other part: $A = 6 \times 6 = 36$

CoM, G_2, by symmetry is (7,3)

$\therefore\ 24 \times 3 + 36 \times 7 = \bar{x}(24 + 36)$

$\bar{x} = 5.4$

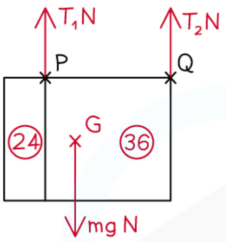
The coordinates of the centre of mass of the folded lamina are (5.4, 3)

(b) The folded lamina is suspended from a ceiling by two strings attached at the points *P* and Q. The folded lamina remains horizontal in equilibrium. Find the tension in the two strings.

Answer:

Your notes

b)



G (5.4, 3)

$m = 24 + 36 = 60$

Resolving forces, vertically (↑): $T_1 + T_2 = 60g$

Moments about P(↺): $(5.4 - 4) \times 60g - (10 - 4) \times T_2 = 0$

$\therefore T_2 = 14g$

$\therefore T_1 = 60g - 14g = 46g$

$\therefore$ Then tension in the string at P is $46g$ N and the tension in the string at Q is $14g$ N

Examiner Tips and Tricks

- Ensure you know whether you are working with a lamina or a framework.
- Folded and overlapping laminas may form shapes that have axes of symmetry which can cut down the work in finding the position of the centre of mass.

Your notes