Edexcel International A Level (IAL) Maths: Mechanics 2

Your notes

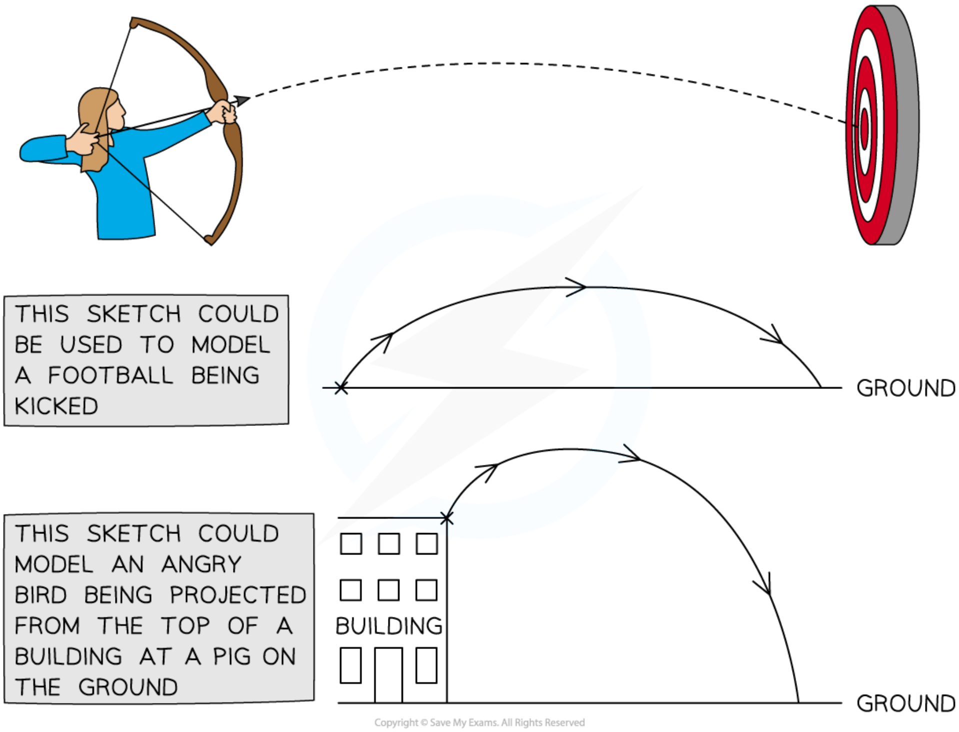
Projectiles

Contents

Horizontal & Vertical Components

What is a projectile?

- A projectile is a particle moving freely, under gravity, in a two-dimensional plane
- Examples of projectile motion include
  - Sports such as basketball, javelin, archery, etc
  - Computer games such as Angry Birds



Modelling assumptions with projectiles?

- The modelling assumptions with projectiles are
  - **No air resistance**, no horizontal forces
  - The projectile **moves freely** under gravity (no other vertical forces)
  - **Ignore spin** and the **rotation** of the projectile
  - Motion is **symmetrical** along the path of a **parabola**

What is the acceleration of a projectile?

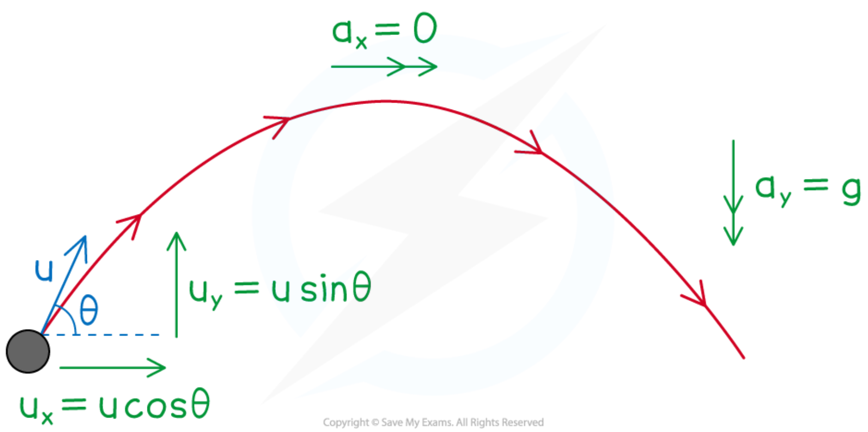
- As it is 2D there will be a horizontal component to acceleration ($\mathbf{a}_x$) and a vertical component ($\mathbf{a}_y$)

- There will be **no** horizontal acceleration (**a_x = 0**)
  - No forces are acting horizontally
  - **Horizontal speed** is **constant**
- There will be constant vertical acceleration (due to gravity, **$a_y = \pm g$**)
- **a_y** may be positive or negative depending on which direction is taken as positive
- The acceleration can be written as 2D vector

$$a = (0\mathbf{i} - g\mathbf{j})\ \mathrm{m\,s^{-2}}$$

What is the initial velocity of a projectile?

- As it is 2D there will be a horizontal component to the **initial velocity(u_x)** and a vertical component (**u_y**)
- A projectile is launched with **initial speed** U m s^{-1} at an angle θ to the horizontal
  - If it is projected **below** the horizontal then θ would be **negative**
- Its initial velocity, **u** m s^{-1}, is a vector with:
  - horizontal component, **u_x** = $U\cos\theta$
  - vertical component, **u_y** = $U\sin\theta$
- The initial velocity can have a **positive** or **negative vertical component** as an object can be projected **upwards** or **downwards**



Worked Example

A projectile is launched at an angle of 25° to the horizontal with speed 5 m s^{-1}.

Find the initial velocity, **u** m s^{-1}, of the projectile.

Answer:

Your notes

Draw a sketch

$u_x = 5\cos 25 = 4.531$

$u_y = 5\sin 25 = 2.113$

$u = (5\cos 25\,\mathbf{i} + 5\sin 25\,\mathbf{j})\ \text{m s}^{-1} = (4.53\,\mathbf{i} + 2.11\,\mathbf{j})\ \text{m s}^{-1}$ (3 sf)

Your notes

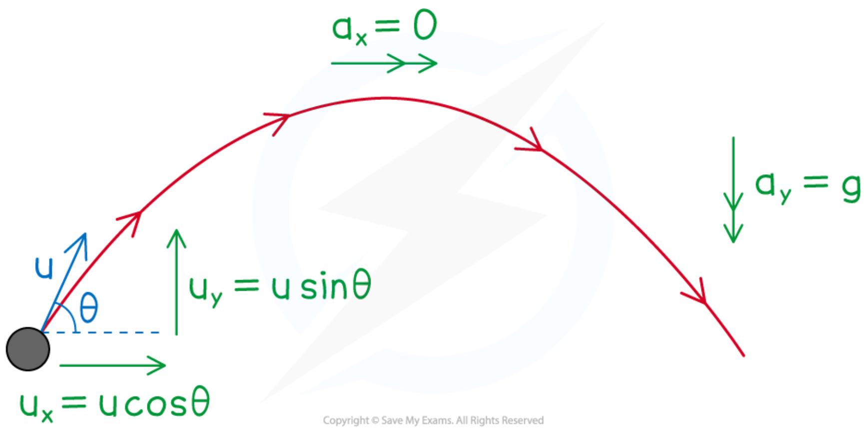
Examiner Tips and Tricks

- Sometimes they may give you the initial velocity as a vector already. To find the initial speed you would find the magnitude of the initial velocity and to find the angle of projection you could sketch a diagram and use trigonometry.
- The horizontal speed is always constant so there will never be a point during the motion when the speed of the object is zero. Its minimum speed will be at its maximum height when its vertical velocity is instantaneously zero.

Your notes

Using suvat with projectiles

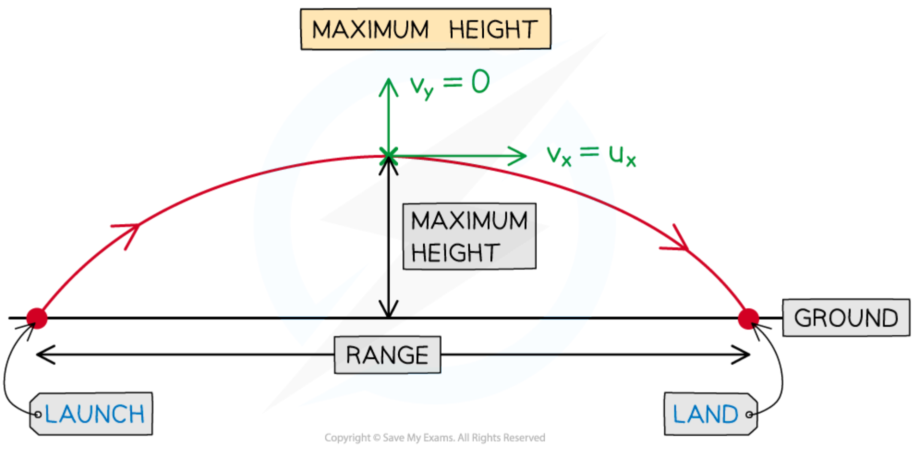
How do I use the suvat equations in projectile problems?



- For projectile motion with acceleration **a** m s^{-2} and initial velocity **u** m s^{-1}
  - $a_x = 0$, $a_y = \pm g$ (depending on which direction is positive)
  - $u_x = U\cos\theta$, $u_y = U\sin\theta$
- Acceleration is constant so the suvat equations apply
- Projectile motion is **2D** so **s**, **u**, **v** and **a** are **vectors**
  - **1D** suvat formulae can be applied to the directions **separately**

    (Note the formula $v^2 = u^2 + 2as$ still applies in 1D)
- Remember that acceleration is **different** for each direction
  - $a_x = 0$ so the suvat formulae reduce to horizontally

How do I solve problems involving the maximum height?



Your notes

- Projectile motion follows a **symmetrical** curve (a **parabola**)
- Therefore there will be a **maximum height** that the projectile reaches
- At the maximum height
  - **Horizontal** velocity, $\mathbf{v_x}$ m s^{-1}, will be the same as the initial horizontal velocity $\mathbf{u_x}$
  - **Vertical** velocity, $\mathbf{v_y}$ m s^{-1} will be instantaneously zero
  - If the projectile **launches and lands** at the same height then the **horizontal distance** to the **maximum height** is **half** of the **range of the projectile**
- Time **t** is the parameter that will be the same for both the horizontal and vertical components

How can projectiles questions be made harder?

- You will need all your skills from your previous work with **suvat**
  - Speed is the **magnitude** of velocity and can be found by using Pythagoras' Theorem
  - A good understanding of the relationship between **distance** and **displacement**
  - Simultaneous equations
  - Finding the point of intersection of two projectiles
- Sometimes the launch and landing points will be at different heights such as if a stone is thrown from the top of a cliff into the sea
  - Think carefully about the vertical displacement from the start to the end
- You might get asked to use the model of a projectile to decide whether an object passes over a certain height
  - Find the time taken to travel the horizontal distance and use this to find the maximum vertical height of the projectile

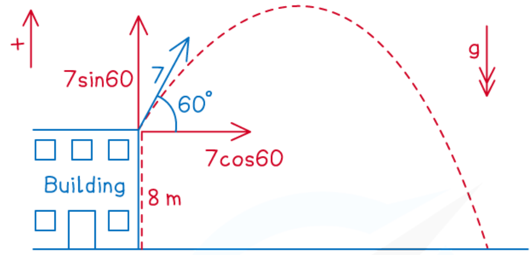
Worked Example

A stone is thrown from the top of a building which is 8 m tall. The stone is projected with speed 7 m s^{-1} at an angle of 60° to the horizontal. The stone is modelled as a particle and air resistance is ignored.

(a) Find the maximum height above the ground that the stone reaches.

Answer:

Draw a sketch



a) At maximum $v_y = 0$

Use suvat vertically (↑)

$s = s$ $\quad u = 7\sin 60$ $\quad v = 0$ $\quad a = -g$

$v^2 = u^2 + 2as$

$0 = (7\sin 60)^2 - 2gs$

$$s = \frac{(7\sin 60)^2}{2g} = 1.875$$

Add displacement to the height of the building

$8 + 1.875 = 9.875$

Maximum height above ground = 9.88 m (3 sf)

(b) Find the time taken for the stone to hit the ground.

Answer:

Your notes

Your notes

b) Vertical displacement from start to end is -8 m

Use suvat vertically ($\uparrow$)

$s = -8 \quad u = 7\sin 60 \quad a = -g \quad \underline{t = t}$

$$s = ut + \frac{1}{2}at^2$$

$$-8 = (7\sin 60)t - \frac{1}{2}gt^2$$

$$\frac{1}{2}gt^2 - (7\sin 60)t - 8 = 0$$

Solve the quadratic

$t = 2.0382...$ or $t = -0.8010...$ x Reject as time can't be negative

Time = 2.04 s (3 sf)

(c) Find the horizontal distance from the building to the point where the stone lands.

Answer:

c) Horizontally speed is constant so suvat becomes sut

$\underline{s = s} \quad u = 7\cos 60 \quad t = 2.0382...$ ← Use full answer from (b)

$$s = ut$$

$$s = (7\cos 60) \times 2.0382... = 7.1337...$$

Horizontal distance = 7.13 m (3 sf)

Examiner Tips and Tricks

- **Always** draw a diagram and add to it as you work through the question. Make it clear which direction you are using as positive.
- Questions could ask you to find the speed at an instant, to do this remember to first find both components of the velocity and then find the magnitude of the velocity.
- Not all projectiles are projected upwards, for example a cannonball fired from the turret of a castle aimed at an enemy on the ground!

Your notes

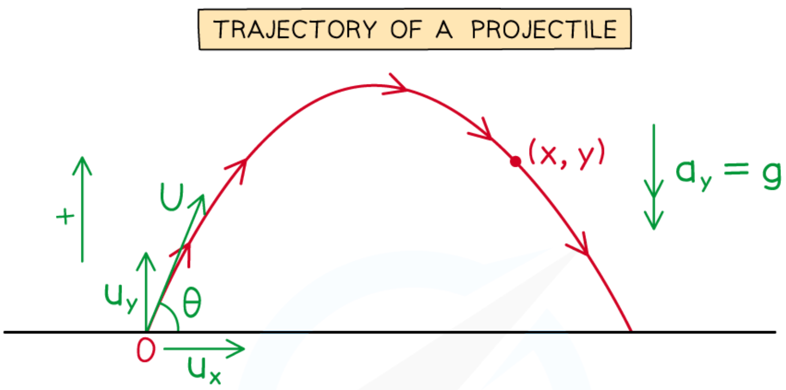
Equation of a trajectory

What is the trajectory of a projectile?

- The **trajectory** of a projectile is the **path** it follows during its motion
- The modelling **assumptions** mean it is **symmetrical**
- It follows a path called a **parabola**
- So far, motion has been described using the suvat formulae and we have produced parametric equations with time as the parameter
- A **Cartesian** equation links the horizontal (*x*) and vertical (*y*) components of the displacement y = f(x)

How do I find the equation for the trajectory of a projectile?

- *U* is the initial speed of the projectile at an angle of θ° to the horizontal



$$s = \begin{pmatrix} x \\ y \end{pmatrix} \qquad u = \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} U\cos\theta \\ U\sin\theta \end{pmatrix} \qquad a = \begin{pmatrix} 0 \\ -g \end{pmatrix} \qquad t = t$$

HORIZONTALLY: $s = ut$

$$x = (U\cos\theta)t \quad (1)$$

VERTICALLY: $s = ut + \frac{1}{2}at^2$

$$y = (U\sin\theta)t - \frac{1}{2}gt^2 \quad (2)$$

Your notes

REARRANGE ①: $t = \dfrac{x}{U\cos\theta}$

SUBSTITUTE INTO ②: $y = \dfrac{(U\sin\theta)x}{U\cos\theta} - \dfrac{1}{2}g\left(\dfrac{x}{U\cos\theta}\right)^2$

SIMPLIFY: $y = x\tan\theta - \dfrac{gx^2}{2U^2\cos^2\theta}$

THIS IS THE GENERAL EQUATION FOR A PROJECTILE'S TRAJECTORY

Worked Example

A particle is projected from a point on horizontal ground at a speed of V m s^{-1} at an angle α° to the horizontal. The trajectory of the particle is given by the equation

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$$

where y is the vertical height of the particle and x is the horizontal distance travelled by the particle.

(a) State two assumptions that have been made.

(b) In the case when $V = 16$ m s^{-1} and α = 70° find, to three significant figures, the height of the particle after it has travelled 12 m horizontally.

Answer:

a)
- Apart from gravitational forces, there are no vertical forces such as air resistance
- There is no spin/rotation on the particle

These are not the only two that you can mention

b) Substitute in all the values

$V = 16 \quad \alpha = 70 \quad x = 12$

$y = 12\tan 70 - \dfrac{g(12)^2}{2(16)^2\cos^2 70}$

$y = 9.4075...$

Height = 9.41 m (3 sf)

Your notes

Examiner Tips and Tricks

The steps are always the same so the only way the questions can be made difficult is for them to use confusing letters. They could use similar letters to suvat including capital letters. Alternatively, they could use unusual letters like $\boldsymbol{a}$, instead of θ, for angles.

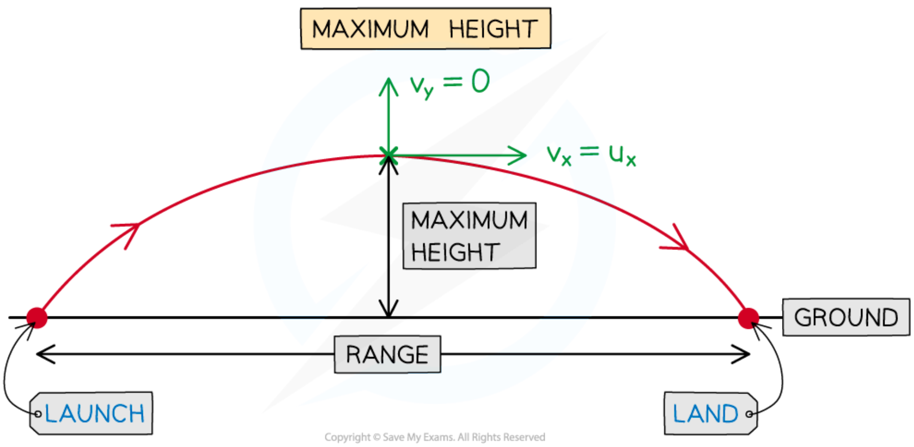
- Make sure you show enough working out to convince the examiner that you know the steps especially if it is a "show that" question.
- Sometimes you might have to use trigonometric identities from pure maths.

Your notes

Deriving projectile formulae

What projectile formulae do I need to be able to derive?

- In the previous notes the **equation** of the **trajectory** of a particle was **derived**
- Other equations you need to be able to derive are
  - $\text{time of flight (between launch and land)} = \frac{2U\sin\theta}{g}$
  - $\text{the (horizontal) range of the projectile} = \frac{U^2\sin 2\theta}{g}$
  - $\text{the time taken to reach the maximum height} = \frac{U\sin\theta}{g}$
  - $\text{the maximum height} = \frac{U^2\sin^2\theta}{2g}$
- These are derived using the suvat equations
- Whilst you do not need to memorise every step of the derivations, you should concentrate on the methods used and know how to approach each one
- There is no worked example with this revision note, instead see if you can use your knowledge of the suvat equations as well as geometrical and algebraic reasoning to derive the above equations before reading on



How do I derive the projectile formulae?

- A particle is projected with speed U m s^{-1} at an angle of $\theta°$ to the horizontal

Your notes

- All of the derivations assume that:
  - the launching and landing site are at the same vertical level
  - the projectile travels over horizontal ground
  - there are no forces acting on the particle other than the force due to gravity

$$s = \begin{pmatrix} x \\ y \end{pmatrix} \quad u = \begin{pmatrix} U\cos\theta \\ U\sin\theta \end{pmatrix} \quad v = \begin{pmatrix} U\cos\theta \\ v_y \end{pmatrix} \quad a = \begin{pmatrix} 0 \\ -g \end{pmatrix} \quad t = t$$

TIME OF FLIGHT

PROJECTILE RETURNS TO GROUND: $y = 0$

VERTICALLY: $s = ut + \frac{1}{2}at^2$

$$0 = (U\sin\theta)t - \frac{1}{2}gt^2$$

FACTORISE: $0 = t\left(U\sin\theta - \frac{1}{2}gt\right)$

SOLVE: $U\sin\theta - \frac{1}{2}gt = 0$ OR $t = 0$

THIS IS THE START

REARRANGE FOR t: $t = \dfrac{2U\sin\theta}{g}$

RANGE OF THE PROJECTILE

TIME OF FLIGHT: $t = \dfrac{2U\sin\theta}{g}$

HORIZONTALLY: $s = ut$

$$x = (U\cos\theta)\left(\frac{2U\sin\theta}{g}\right)$$

SIMPLIFY: $x = \dfrac{2U^2\sin\theta\cos\theta}{g}$

USE A TRIG IDENTITY: $2\sin\theta\cos\theta = \sin 2\theta$

$$x = \frac{U^2\sin 2\theta}{g}$$

TIME TO MAXIMUM HEIGHT

MAXIMUM HEIGHT: $v_y = 0$

VERTICALLY: $v = u + at$

$0 = U\sin\theta - gt$

REARRANGE FOR t: $t = \dfrac{U\sin\theta}{g}$

MAXIMUM HEIGHT

MAXIMUM HEIGHT: $v_y = 0$

VERTICALLY: $v^2 = u^2 + 2as$

$0 = U^2\sin^2\theta - 2gy$

REARRANGE FOR y: $y = \dfrac{U^2\sin^2\theta}{2g}$

Your notes

Examiner Tips and Tricks

- Always draw a sketch!
- The trickiest part about this is the algebra so do not rush these questions as that is when you are more likely to make mistakes.
- Make sure you can follow the derivations above as they could be asked. After reading through this revision note practice replicating the derivations from memory.