Questions

Q1.

The curve C has parametric equations

$$x = \theta - \tanh\theta, \quad y = \operatorname{sech}\theta \quad 0 \leq \theta \leq \ln 3$$

(a) Find

(i) $\dfrac{dx}{d\theta}$

(ii) $\dfrac{dy}{d\theta}$

(2)

The curve C is rotated through 2π radians about the x-axis.

(b) Find the exact area of the curved surface formed, giving your answer as a multiple of π.

(5)

(Total for question = 7 marks)

Q2.

The curve C has parametric equations

$$x = \cosh t + t, \quad y = \cosh t - t \quad 0 \leq t \leq \ln 3$$

(a) Show that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\cosh^2 t$$

(3)

The curve C is rotated through 2π radians about the x-axis. The area of the curved surface generated is given by S.

(b) Show that

$$S = 2\pi\sqrt{2}\int_0^{\ln 3}\left(\cosh^2 t - t\cosh t\right)dt$$

(2)

(c) Hence find the value of S, giving your answer in the form

$$\frac{\pi\sqrt{2}}{9}(a + b\ln 3)$$

where a and b are constants to be determined.

(7)

(Total for question = 12 marks)

Q3.

Without using a calculator, find

(a) $\int_{-2}^{1} \frac{1}{x^2 + 4x + 13} \, dx$, giving your answer as a multiple of π,

(5)

(b) $\int_{-1}^{4} \frac{1}{\sqrt{4x^2 - 12x + 34}} \, dx$, giving your answer in the form $p \ln (q + r\sqrt{2})$,

where p, q and r are rational numbers to be found.

(7)

(Total for question = 12 marks)

Q4.

Given that $y = \text{artanh}(\cos x)$

(a) show that

$$\frac{dy}{dx} = -\text{cosec}\, x$$

(2)

(b) Hence find the exact value of

$$\int_{0}^{\frac{\pi}{6}} \cos x \, \text{artanh}(\cos x) \, dx$$

giving your answer in the form $a \ln(b + c\sqrt{3}) + d\pi$, where a, b, c and d are rational numbers to be found.

(5)

(Total for question = 7 marks)

Q5.

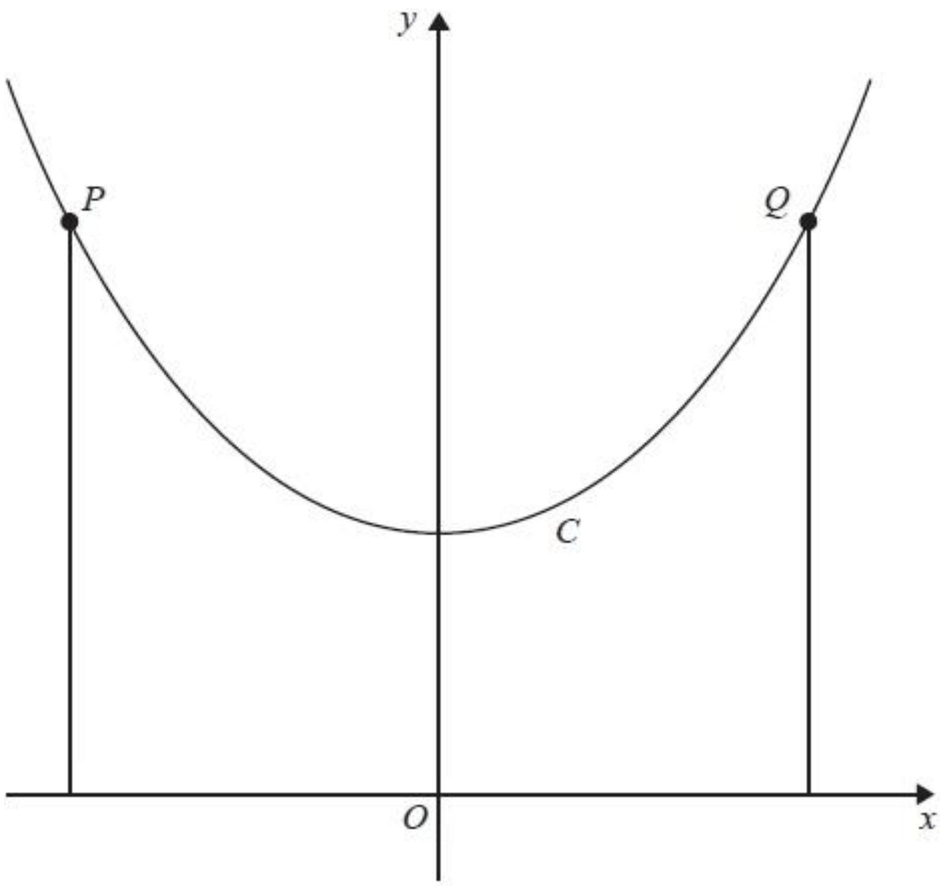

Figure 1

Figure 1 shows part of the curve *C* with equation

$$y = 3\cosh\left(\frac{x}{3}\right)$$

The point *P* and the point *Q* lie on the curve. The point *P* has *x* coordinate $-3a$ and the point *Q* has *x* coordinate $3a$.

(a) Find the length of the arc *PQ*, giving your answer as a multiple of $\sinh a$.

(5)

Given that the length of the arc *PQ* is 12

(b) show that the *x* coordinate of *Q* is $3\ln\left(p + \sqrt{q}\right)$, where *p* and *q* are integers to be found,

(2)

(c) show that the *y* coordinate of *Q* is $r\sqrt{s}$ where *r* and *s* are integers to be found.

(2)

(Total for question = 9 marks)

Q6.

$$I_n = \int \cosh^n x \, dx \qquad n \geqslant 0$$

(a) Show that, for $n \geq 2$

$$nI_n = \sinh x \cosh^{n-1} x + (n-1) I_{n-2}$$

(6)

(b) Hence find

$$\int \cosh^4 x \, dx$$

(4)

(Total for question = 10 marks)

Q7.

$$I_n = \int \cosh^n x \, dx, \qquad n \geq 0$$

(a) Show that, for $n \geq 2$,

$$nI_n = \sinh x \cosh^{n-1} x + (n-1) I_{n-2}$$

(6)

(b) Hence find the exact value of

$$\int_0^{\ln 2} \cosh^5 x \, dx$$

(4)

(Total for question = 10 marks)

Q8.

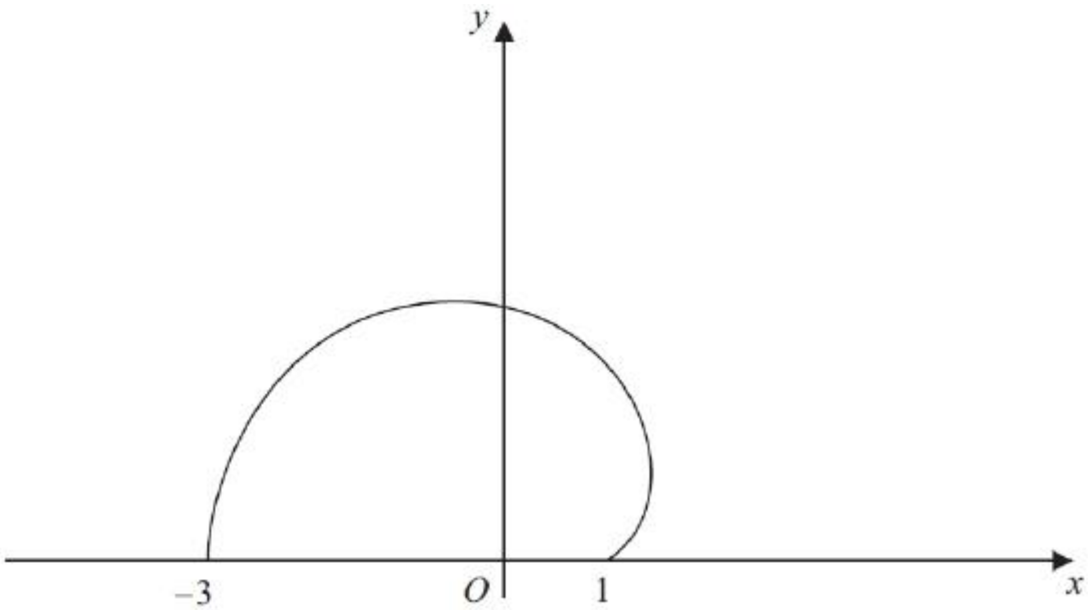

Figure 1

Figure 1 shows the curve C with parametric equations

$$x = 2\cos\theta - \cos 2\theta,\ y = 2\sin\theta - \sin 2\theta, \qquad 0 \le \theta \le \pi$$

(a) Show that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 8(1 - \cos\theta)$$

(5)

The curve C is rotated through 2π radians about the x-axis.

(b) Find the area of the surface generated, giving your answer in the form $k\pi$, where k is a rational number.

(5)

(Total for question = 10 marks)

Q9.

(a) Show that, under the substitution $x = \frac{3}{4} \sinh u$,

$$\int \frac{x^2}{\sqrt{16x^2+9}}\,\mathrm{d}x = k\int (\cosh 2u - 1)\,\mathrm{d}u$$

where k is a constant to be determined.

(6)

(b) Hence show that

$$\int_0^1 \frac{64x^2}{\sqrt{16x^2+9}}\,\mathrm{d}x = p + q\ln 3$$

where p and q are rational number to be found.

(5)

(Total for question = 11 marks)

Q10.

The curve C has parametric equations

$$x = 3t^4,\ y = 4t^3, \qquad 0 \leq t \leq 1$$

The curve C is rotated through 2π radians about the x-axis. The area of the curved surface generated is S.

(a) Show that

$$S = k\pi \int_0^1 t^5 (t^2 + 1)^{\frac{1}{2}}\,\mathrm{d}t$$

where k is a constant to be found.

(4)

(b) Use the substitution $u^2 = t^2 + 1$ to find the value of S, giving your answer in the form $p\pi\,(11\sqrt{2} - 4)$ where p is a rational number to be found.

(7)

(Total for question = 11 marks)

Q11.

$$I_n = \int_0^{\ln 2} \tanh^{2n} x \, dx, \quad n \geqslant 0$$

(a) Show that, for $n \geq 1$

$$I_n = I_{n-1} - \frac{1}{2n-1}\left(\frac{3}{5}\right)^{2n-1}$$

(5)

(b) Hence show that

$$\int_0^{\ln 2} \tanh^4 x \, dx = p + \ln 2$$

where p is a rational number to be found.

(5)

(Total for question = 10 marks)

Q12.

(a) Find

$$\int \frac{5 + x}{\sqrt{4 - 3x^2}} \, dx$$

(5)

(b) Hence find the exact value of

$$\int_0^1 \frac{5 + x}{\sqrt{4 - 3x^2}} \, dx$$

giving your answer in the form $p\pi\sqrt{3} + q$, where p and q are rational numbers to be found.

(3)

(Total for question = 8 marks)

Q13.

The curve C has parametric equations

$$x = \theta - \sin\theta, \qquad y = 1 - \cos\theta, \qquad 0 \leq \theta \leq 2\pi$$

The curve C is rotated through 2π radians about the x-axis. The area of the curved surface generated is given by S.

(a) Show that

$$S = 2\pi\sqrt{2}\int_0^{2\pi} (1 - \cos\theta)^{\frac{3}{2}}\,d\theta$$

(4)

(b) Hence find the exact value of S.

(6)

(Total for question = 10 marks)

Q14.

Find

(a) $\displaystyle\int \frac{1}{8 + 4x + x^2}\,dx$

(3)

(b) $\displaystyle\int \frac{1}{\sqrt{(8 - 4x - x^2)}}\,dx$

(4)

(Total for question = 7 marks)

Q15.

$$I_n = \int \frac{x^n}{\sqrt{(x^2 + k^2)}}\,\mathrm{d}x$$ where k is a constant and $n \in \mathbb{Z}^+$

(a) Show that, for $n \geq 2$

$$I_n = \frac{x^{n-1}}{n}(x^2 + k^2)^{\frac{1}{2}} - \frac{(n-1)}{n}k^2 I_{n-2}$$

(7)

(b) Hence find the exact value of

$$\int_0^1 \frac{x^5}{\sqrt{(x^2 + 1)}}\,\mathrm{d}x$$

(5)

(Total for question = 12 marks)

Q16.

$$I_n = \int x^n \cos x\,\mathrm{d}x$$

(a) Show that, for $n \geq 2$

$$I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2}$$

(4)

(b) Hence find the functions f(x) and g(x) such that

$$\int x^4 \cos x\,\mathrm{d}x = \mathrm{f}(x)\sin x + \mathrm{g}(x)\cos x + c$$

where c is an arbitrary constant.

(5)

(Total for question = 9 marks)

Q17.

$$I_n = \int \operatorname{cosec}^n x \, dx, \quad 0 < x < \frac{\pi}{2}, \quad n \geqslant 0$$

(a) Show that, for n ≥ 2

$$I_n = \frac{n-2}{n-1} I_{n-2} - \frac{1}{n-1} \cot x \operatorname{cosec}^{n-2} x$$

(5)

(b) Hence, or otherwise, find

$$\int \operatorname{cosec}^4 x \, dx$$

giving your answer in terms of cot *x*.

(4)

(Total for question = 9 marks)

Examiner's Report

Q1.

Differentiation of $x = \theta - \tanh\theta$ and $y = \text{sech}\theta$ with respect to θ was generally successful though $-\text{sech}^2\theta$ and $+ \text{sech}\theta \tanh\theta$ were seen. A few students used a variable x for θ in their answers. The correct surface area formula was generally used and most students substituted their derivatives correctly. A number of students found the simplification to $\text{sech}\theta \tanh\theta$ quite challenging.

A few solutions used the $2\pi\int y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ approach but then failed to replace dx with $\left(\frac{dx}{d\theta}\right)d\theta$.
Quite a few solutions wrote $\int \text{sech}\theta \tanh\theta \, d\theta = \text{sech}\theta$ and were unable to explain why the final answer was -0.8π.
A few examples were seen where 2π was used as one of the limits. Some students need to show more working when evaluating their limits so that, in particular, the correct use of the limit $x = 0$ is evident.

Q2.

In part (a) many good solutions to verifying $(dx/dt)^2 + (dy/dt)^2 = 2\cosh^2 t$ were seen. dx/dt and dy/dt were correct in most cases and the expressions squared accurately. There were some careless slips in reaching $2\sinh^2 t + 2$. The final accuracy mark was often lost as an intermediate step $2\sinh^2 t + 2 = 2(\sinh^2 t + 1)$ or $2\sinh^2 t + 2 = 2(\cosh^2 t - 1) + 2$ was required since this was a "show that" question.
The derivation of the formula for surface area was well done in part (b) though the occasional bracketing error lost the accuracy mark.
Evaluation of the integral in part (c) caused some difficulties. Candidates who attempted the derivation of a four term expression for an indefinite integral generally avoided sign errors. Evaluation of $\cosh^2 t$ was generally done using the double angle formula though a failure to write down $\cosh 2t = 2\cosh^2 t - 1$ led to sign errors or a missing ½. Attempts using exponentials made progress though did cause difficulties later. Integrating $t\cosh t$ by parts was well done though errors arose due to a failure to show full working. Solutions which evaluated the definite integrals separately often lost accuracy marks when combining as they overlooked the fact that the second part was $-\int t\cosh t \, dt$.
The last two marks for evaluation of the integral were also lost by the assumption that the lower limit value was zero.

Q3.

(a) was usually correct, even if some students made life difficult for themselves by using the substitution $x + 2 = 3\tan t$ rather than quoting the standard integral result. The $4x^2$ in (b) proved to be much more of a problem. Many simply divided the whole expression by 4 and integrated $\frac{1}{\left(x-\frac{3}{2}\right)^2+\frac{25}{4}}$. Those who completed the square correctly to get $(2x - 3)^2 + 25$ then often lost the necessary half on integration.

Use of limits and logarithmic form were usually correct but those who tried to rationalise the denominator within their logarithm quite often failed to do so correctly.

Q4.

(a) was generally well done by the majority of students. Most students used the chain rule directly, some used the substitution $u = \cos x$ and correctly obtained the given result. Other students rearranged to get $\tanh y = \cos x$ and differentiated implicitly though, and while a few did not substitute for $\text{sech}^2 y$ to obtain an expression in x, most were able to rearrange to obtain the given result, showing all the necessary steps in their working.

Many excellent solutions were seen to this part of the problem. Some students tried to use integration by parts in the wrong direction and made no progress. They seldom realised the need to restart with "u" = artanh(cos x) and $"\frac{dv}{dx}" = \cos x$. The most common error here was to lose the factor of ½ when using the ln form of artanh x. This happened most frequently when students tried to miss lines of working and jump directly from their integral $\left[\sin x \operatorname{artanh}(\cos x) + x\right]_0^{\frac{\pi}{6}}$ to substituting the limits and writing in the ln form in a single step.

Q5.

This arc length question produced a slightly more mixed response. In part (a) most obtained the correct $\frac{dy}{dx}$ and used the correct formula to reach the correct integrand. A common error was to integrate cosh $\frac{x}{3}$ into $\frac{1}{3}$ sinh $\frac{x}{3}$ rather than 3sinh $\frac{x}{3}$. Some who used 3a and –3a as limits did not give their answer as a multiple of sinh a, leaving it as 3sinh a – 3sinh (–a). Most went on in part (b) to use the given arc length to obtain a value for the required x-coordinate of 3a. In (c), the majority successfully obtained a value of the y-coordinate of 3cosh a using the exponential definition of cosh with only a small number unable to obtain it in the right form. The more direct route of using 3cosh $a = 3\sqrt{1 + \sinh^2 a}$ was not widely seen.

Q6.

This reduction formula question saw good scoring for the most part. In part (a), most sensibly opted for Way 1, performing the "split" correctly and applying parts in the correct direction. The most common error occurred when differentiating $\cosh^{n-1}x$, obtaining $(n - 1)\cosh^{n-2}x$ rather than $(n - 1)\cosh^{n-2}x \sinh x$. Those who obtained the required $\sinh^2 x$ invariably replaced it with the correct $\cosh^2 x - 1$. A few sign errors were seen but most who scored the first three marks had little difficulty reaching the given answer correctly. Those who used the Way 2 "split" were far more likely to encounter difficulties such as knowing how to apply parts correctly to integrate $\sinh x \sinh x \cosh^{n-2}x$.

It was common to be awarding marks in part (b) even for those who made little progress in (a). Most used the reduction formula twice and obtained the correct I_0. Only a small number found I_2 directly. Errors usually revolved around slips with bracketing leading to incorrect coefficients.

Q7.

Many correct proofs were seen in part (a). The most common approach was to write $\cosh^n x$ as $\cosh^{n-1}x \cosh x$ and then attempt parts although a significant number of students differentiated $\cosh^{n-1}x$ as $(n-1)\cosh^{n-2}x$ and potentially lost a significant number of subsequent marks. Some students wrote $\cosh^n x$ as $\cosh^{n-2}x \cosh^2 x$ and then used $\cosh^2 x = 1 + \sinh^2 x$ and proceeded correctly. A significant minority of students incorrectly attempted parts using $\frac{dv}{dx} = 1$ and some students failed to give a convincing proof because one or more vital steps were missing.

Students often applied the reduction formula well in part (b) although there were some arithmetic slips and very occasionally the reduction formula was applied as $I_n = \sinh x \cosh^{n-1}x + (n-1)I_{n-2}$ rather than $nI_n = \sinh x \cosh^{n-1}x + (n-1)I_{n-2}$.

Q8.

Part (a) was potentially a source of 5 easy marks for many students. The majority found correct derivatives although there were some sign and coefficient errors. Those with correct derivatives could usually establish the printed result although quite a few got lost with the trigonometry or made minor slips. Success in part (b) was varied and it was quite often the case that once students had reached $S = 2\pi\int(2\sin\theta - \sin 2\theta)\sqrt{8(1-\cos\theta)}\,d\theta$ they could not see how to proceed.

Students often split the integral into two and made failed attempts to integrate by parts. Those who eliminated the double angle often made further progress and there were other successful methods employed such as the use of half angles and integration by substitution. This part discriminated well.

Q9.

Many students scored well on this question. In part (a), the substitution of $x = \frac{3}{4}\sinh u$ was usually sound and a fully correct substitution was frequently seen. Attempts to simplify were also usually accurate although some students struggled to simplify $\sqrt{9 + 16 \times \frac{9}{16}\sinh^2 u}$ correctly. Many students could obtain an integrand that was a multiple of $\sinh^2 u$ but sometimes lost the factor of ½ when using the double angle identity.

Many students could make progress in part (b) with or without a correct value for k and could obtain an expression of the required form. The correct u limits were frequently seen although some students used x limits throughout.

Q10.

The derivatives in (a) were almost always correct, though a few students got no further than this. The majority quoted/ used a correct formula, though it occasionally lacked a *y*. Those who used $\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$ managed to negotiate their way to the required form. Successful factorisation usually followed, so it was usual for those who progressed beyond the basic derivatives to score full marks.

Puzzlingly in (b) $u^2 = t^2 + 1$ sometimes led to $t^2 = u^2 + 1$. Other than this, differentiation was, again, usually correct though very complicated versions resulting from $u = (t^2+1)^{\frac{1}{2}}$ or $t = \left(u^2-1\right)^{\frac{1}{2}}$ were common. It was no surprise that substitution of "d*t*" caused problems either by being upside down or by being simply changed to "d*u*", but a significant number successfully completed to the correct integral in terms of *u*.

From here, some used integration by parts and were occasionally successful, while those who tried to expand and integrate were sometimes puzzled by the u^2 in $(u^2 - 1)^2$. New limits were usually found correctly but the bottom one seemed to revert back to 0 from time to time.

For those who had navigated everything else successfully, the final hurdle was to factorise their expression to the required form. As with (b), there was often a lack of appreciation that we can't simply multiply/divide at will when an expression does not equal 0, so it was quite common to simplify the fractions by multiplying by the common denominator.

Q11.

Many students found (a) of this question challenging. There were many cases where (a) or the whole question was not attempted. There were also many cases of students trying to integrate in (a) by first writing $\tanh^{2n}x$ as $1 \times \tanh^{2n}x$ or $\tanh^{2n-1}x \tanh x$. The cases of it being written $\tanh^{2n}x = \tanh^{2(n-1)}x \tanh^2 x$ were in the minority. In the main students who started off by using one of the incorrect expressions did not go on to try an alternative. Of those students who did write $\tanh^{2n}x$ as $\tanh^{2n}x \tanh^{2(n-1)}x \tanh^2 x$ some then made no further progress. Many did substitute $\tanh^2 x = 1 - \text{sech}^2 x$ and split their expression into 2 integrals, however they then did not recognise this as the derivative of $\tanh^{2n-1}x$ and tried to integrate $\tanh^{2n-2}x\,\text{sech}^2 x$ by parts. While, of course, this is possible many mistakes occurred on the way.

Attempts at (b) were much better and there were many students who were unsuccessful in (a) who went on to complete (b) with full marks. The most common errors here were using $n = 4$ rather than $n = 2$ and sign errors if students calculated I_1 rather than I_0.

Q12.

This proved a challenging question for many students. In part (a) the idea of splitting the integrand into two separate terms was not well-known and so students achieved very few marks for the whole question. Of those who did split the integrand there were many perfect solutions seen with only a minority having problems with the various powers and square root terms. The alternative substitution method was rarely seen but of those who chose this route there were many completely correct solutions seen. Some tried to integrate by parts and were unable to progress very far. Arcsin was usually obtained by students with the odd slip up with either p or q. The other part of the integral was nearly always reached but a minority of students gave a function of ln rather than a square root.

Part (b) was dependent on part (a). Most students used a sound method but errors in part (a) led to a failure to gain the A marks here.

Q13.

In part (a), students could apply the formula for the required surface area producing a majority of perfectly correct solutions.

However part (b) proved very difficult with very few correct solutions seen. Indeed there were a lot of students who left this part blank which may have been due to the lack of time. Those who chose to express the integrand in terms of a sine function of the half angle could then usually proceed to a correct answer.
There were attempts at solutions seen in which a calculator had clearly been used to get the area as no working was shown and these gained no credit. Many started part (b) using a substitution but mainly substituting incorrectly. Very few substituted $u = 1 - \cos\theta$ and those who did generally failed to achieve an expression in terms of u only as they did not replace $d\theta$ correctly. A novel method used once that substitution had been done was to rewrite the numerator as $-(2 - u - 2)$ and then proceed to split the integrand. This avoided the use of integration by parts. Very few students brought part (b) to a correct conclusion.

Q14.

The two integrations required here were completed successfully by the vast majority. It was rare to see any candidate attempt to progress without first completing the square for the quadratics. Full substitutions were not common with most using the forms from the formula book. The multiplier of $\frac{1}{2}$ was occasionally missing in final answers to part (a). Part (b) saw slightly more errors – often from the erroneous extraction of a minus sign beyond the square root and arithmetic errors when attempting to complete the square.

Q15.

A large number of students made no progress as a result of splitting the integrand as x^n and $\frac{1}{\sqrt{x^2+k^2}}$.

Students splitting as x^{n-1} and $\frac{x}{\sqrt{x^2+k^2}}$ generally applied the integration by parts method correctly. The next hurdle was to write $\sqrt{x^2+k^2}$ as $\frac{x^2+k^2}{\sqrt{x^2+k^2}}$. Solutions which managed this generally continued to produce a fully correct solution. A few students copied the answer from the question when it bore no resemblance to preceding work.

Evaluation of I_5 was attempted by most students in part (b) but sometimes careless arithmetic led to the wrong fractions in the formula linking it to I_3. The problem was overcomplicated by some students not writing $k = 1$ at the beginning.

Students who approached in the order I_5, I_3 I_1 generally proceeded well until they evaluated I_1; $\sqrt{x^2+1}$ was often not achieved. Evaluation of I_1 was frequently seen as 2 with the use of lower limit $x = 0$ omitted. Combination of the various results frequently had bracketing errors and so only a few solutions reached a final answer of $\frac{7\sqrt{2}}{15} - \frac{8}{15}$.

Students working in the order I_1, I_3 then I_5 often ended up substituting limits twice into part of the expression.

Q16.

There were many excellent solutions to this question. Some candidates made little progress in part (a) but achieved marks in (b).

The final accuracy mark was occasionally lost in (a) through poor bracketing or missing a function in the final line. A few solutions split the parts as x^{n-1} and $x\cos x$. The second integration by parts caused a few of these attempts to falter.

Part (b) was well done by most candidates. The general method was to express I_4 in terms of I_2 and then I_2 in terms of I_0. I_0 was occasionally evaluated as 0, 1 or x. The final answer occasionally had one of the x^n terms with a wrong power or $\sin x/\cos x$ absent. Sometimes a value of I_0 was embedded in the work and it was difficult to establish its value when the various multiples were mixed up.

A few candidates decided to ignore the result of part (a) and used repeated integration by parts. They were generally successful.

Q17.

Proof of the reduction formula in part (a) was challenging for a very high proportion of the students and many failed to score any marks here.

Quite a few attempted to split the expression $\text{cosec}^{n-2}x$ as $\text{cosec}^{n-1}x\,\text{cosec}\,x$ and integrate by parts. They soon realised that further progress was impossible.

Many students identified a suitable split as $\text{cosec}^{n-2}x\ \text{cosec}^2x$ and proceeded to integrate by parts. Lack of a starting formula or specific reference to the four parts $u, v, \frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$ (or f, g, f ′ and g′) meant it was difficult to be sure a correct method was employed. Three negative quantities were involved and many solutions tried to simplify before writing down the complete expression. Many students realised the need to replace $\cot^2x$ with $\text{cosec}^2x - 1$ though sign errors in the formula were not unusual. Students reaching this point generally recognised how I_n and I_{n-2} appeared and further progress was made.

An alternative strategy favoured by a number of students was to replace $\text{cosec}^{n-2}x\ \text{cosec}^2x$ with $\text{cosec}^{n-2}x(1 + \cot^2x)$. Few following this method were then able to split up and integrate by parts.

In part (b) most student managed to score marks for the evaluation of I_4 and there were a large number of correct solutions. Few errors were made in applying the reduction formula to express I_4 in terms of I_2. Evaluation of I_2 was well done when approached as $\int \text{cosec}^2 x\,\mathrm{d}x$ though a few solutions using the reduction formula for a second time thought that $\int \text{cosec}^0 x\,\mathrm{d}x$ was needed when it should have been multiplied by zero. The formula $\cot^2x = \text{cosec}^2x - 1$ was occasionally incorrect or not used. A few solutions left $-\frac{2}{3}\cot x - \frac{1}{3}\cot x$ in the final answer.

Mark Scheme

Q1.

Question Number	Scheme	Notes	Marks
	$x = \theta - \tanh\theta, \quad y = \operatorname{sech}\theta, \qquad 0 \le \theta \le \ln 3$		
(a)(i)	$\left(\frac{dx}{d\theta}\right) = 1 - \operatorname{sech}^2\theta$	Correct derivative	B1
(ii)	$\left(\frac{dy}{d\theta}\right) = -\operatorname{sech}\theta\tanh\theta$ oe	Correct derivative	B1
	If both derivatives are in terms of a different variable but otherwise correct, allow B1B0. If one (or both) incorrect award B0B0		
			(2)
(b)	$S = (2\pi)\int \operatorname{sech}\theta\sqrt{(1-\operatorname{sech}^2\theta)^2 + (-\operatorname{sech}\theta\tanh\theta)^2}\,(d\theta)$	Uses the correct formula with their derivatives 2π not needed	M1
	$S = 2\pi\int \operatorname{sech}\theta\sqrt{1-\operatorname{sech}^2\theta}\,d\theta$		
	$S = 2\pi\int \operatorname{sech}\theta\tanh\theta\,d\theta$	Correct integral after full simplification – 2π and limits not needed	A1
	$S = 2\pi[-\operatorname{sech}\theta]$	Correct integration – limits not needed	A1
	$S = -2\pi(\operatorname{sech}(\ln 3) - \operatorname{sech}(0)) = 0.8\pi$	dM1: Include 2π and use limits (0 to ln3) correctly in a multiple of $\operatorname{sech}\theta$ A1: cao and cso	dM1A1cao and cso
	Use of calculator: Correct integral, inc correct limits, shown followed by correct answer (multiple of π) scores full marks. No need to simplify the initial integral shown but if simplified incorrectly, only M mark can be awarded regardless of final answer. Incorrect answer given, mark as scheme.		
	Allow h (eg from tanh) to disappear as long as the functions are treated as hyperbolics.		
			(5)
			Total 7

Q2.

Question Number	Scheme	Notes	Marks
	$x = \cosh t + t, \quad y = \cosh t - t$		
(a)	$\frac{dx}{dt} = \sinh t + 1, \quad \frac{dy}{dt} = \sinh t - 1$	Correct derivatives	B1
	$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sinh^2 t + 2\sinh t + 1 + \sinh^2 t - 2\sinh t + 1$ $= 2\sinh^2 t + 2$ M1: Squares correctly, cancels and collects terms		M1
	$= 2(1 + \sinh^2 t) = 2\cosh^2 t$*	Uses $\cosh^2 t = 1 + \sinh^2 t$ to complete the proof with no errors	A1*
			(3)
(b)	$S = 2\pi \int y \, ds = 2\pi \int (\cosh t - t)\sqrt{2} \cosh t \, dt$	Uses $S = 2\pi \int y \, ds$ with the given y and the result from part (a)	M1
	$= 2\sqrt{2}\pi \int_0^{\ln 3} (\cosh^2 t - t\cosh t) \, dt$*	Correct proof with no errors	A1*
			(2)
(c)	$\int \cosh^2 t \, dt = \int \pm\frac{1}{2} \pm \frac{1}{2}\cosh 2t \, dt$	Uses $\cosh^2 t = \pm\frac{1}{2} \pm \frac{1}{2}\cosh 2t$	M1
	$\int t\cosh t \, dt = t\sinh t - \int \sinh t \, dt$	Attempts integration by parts the right way round on $t\cosh t$	M1
		Correct expression	A1
	$S = (2\sqrt{2}\pi)\int (\cosh^2 t - t\cosh t) \, dt = (2\sqrt{2}\pi)\left[\frac{1}{2}t + \frac{1}{4}\sinh 2t - t\sinh t + \cosh t\right]$ A1: 2 correct terms A1: All correct		A1A1
	$(S =) 2\sqrt{2}\pi \left\{\left(\frac{1}{2}\ln 3 + \frac{10}{9} - \frac{4}{3}\ln 3 + \frac{5}{3}\right) - (1)\right\}$ dM1: Correct use of limits 0 and ln3 depends on both preceding M marks		dM1
	$S = \frac{1}{9}\sqrt{2}\pi(32 - 15\ln 3)$	cao	A1 (7)
			Total 12

	Alternative for (c)		
	$\int \cosh^2 t\,\mathrm{d}t = \int \left(\frac{\mathrm{e}^t + \mathrm{e}^{-t}}{2}\right)^2 \mathrm{d}t$ $= \frac{1}{4}\int \left(\mathrm{e}^{2t} + 2 + \mathrm{e}^{-2t}\right)\mathrm{d}t$	Substitutes the exponential form and attempts to square	M1
	$\int t\cosh t\,\mathrm{d}t = \frac{1}{2}\int t\left(\mathrm{e}^t + \mathrm{e}^{-t}\right)\mathrm{d}t$ $= \frac{1}{2}t\mathrm{e}^t - \frac{1}{2}\int t\mathrm{e}^t\mathrm{d}t - \left\{\frac{1}{2}t\mathrm{e}^{-t} - \frac{1}{2}\int \mathrm{e}^{-t}\mathrm{d}t\right\}$	Substitutes the exponential form and attempts integration by parts the right way round Correct expression	M1 A1
	$(S=)\left(2\sqrt{2}\pi\right)\left\{\frac{1}{4}\left(\frac{1}{2}\mathrm{e}^{2t} + 2t - \frac{1}{2}\mathrm{e}^{-2t}\right) - \frac{1}{2}t\mathrm{e}^t + \frac{1}{2}\mathrm{e}^t + \frac{1}{2}t\mathrm{e}^{-t} - \frac{1}{2}\mathrm{e}^{-t}\right\}$ A1: either integral correct A1: other integral correct but both must be in a complete expression for S		A1A1
	Depends on both M marks above	Correct use of limits 0 and ln3	dM1
	$S = \frac{1}{9}\sqrt{2}\pi(32 - 15\ln 3)$	cao	A1

	Alternative for the first 3 marks of (c)		
	$= 2\sqrt{2}\pi\int \left(\cosh^2 t - t\cosh t\right)\mathrm{d}t$ $= 2\sqrt{2}\pi\int \cosh t\left(\cosh t - t\right)\mathrm{d}t$		
	$2\sqrt{2}\pi\left(\left[\sinh t\left(\cosh t - t\right)\right] - \int \sinh t\left(\sinh t - 1\right)\mathrm{d}t\right.$		
	$2\sqrt{2}\pi\left(\left[\sinh t\left(\cosh t - t\right)\right] - \left[\cosh t\left(\sinh t - 1\right)\right] + \int \cosh^2 t\,\mathrm{d}t\right)$ M1 (2nd on e-PEN): Use parts twice A1 Correct expression		M1A1
	$\int \cosh^2 t\,\mathrm{d}t = \int \pm\frac{1}{2} \pm \frac{1}{2}\cosh 2t\,\mathrm{d}t$	Uses $\cosh^2 t = \pm\frac{1}{2} \pm \frac{1}{2}\cosh 2t$	M1 (1st on e-PEN)
	Rest as main scheme		

Q3.

Question Number	Scheme	Notes	Marks
(a)	$x^2+4x+13 \equiv (x+2)^2+9$		B1
	$\int \frac{1}{(x+2)^2+9}\,dx = \frac{1}{3}\arctan\left(\frac{x+2}{3}\right)$	M1: $k\arctan f(x)$. A1: Correct expression	M1A1
	$\left[\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)\right]_{-2}^{1} = \frac{1}{3}(\arctan 1 - \arctan 0)$	Correct use of limits arctan0 need not be shown	M1
	$\frac{\pi}{12}$	cao	A1
			(5)

ALT:	**Sub** $x+2=3\tan t$		
	$x^2+4x+13 \equiv (x+2)^2+9$		B1
	$\frac{dx}{dt}=3\sec^2 t \qquad x=-2,\ \tan t=0,\ t=0;\ x=1, \tan t=1,\ t=\frac{\pi}{4}$		
	$\int \frac{3\sec^2 t}{9\tan^2 t+9}\,dt = \frac{1}{3}\int dt = \frac{1}{3}t$	M1 sub and integrate inc use of $\tan^2+1=\sec^2$ A1 Correct expression Ignore limits	M1A1
	$..\left[\frac{\pi}{12}\right]_0^{\frac{\pi}{4}}.$	Either change limits and substitute Or reverse substitution and substitute original imits	M1
	$\frac{\pi}{12}$	cao	A1

(b)	$4x^2-12x+34 = 4\left(x-\frac{3}{2}\right)^2+25$ or $(2x-3)^2+25$	M1: $4(x\pm p)^2 \pm q,\ (p,q\neq 0)$ A1: $4\left(x-\frac{3}{2}\right)^2+25$	M1A1
	$\int \frac{1}{\sqrt{4(x-\frac{3}{2})^2+25}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{(x-\frac{3}{2})^2+\frac{25}{4}}}\,dx = \frac{1}{2}\operatorname{arsinh}\left(\frac{x-\frac{3}{2}}{\frac{5}{2}}\right)$ M1: $k\operatorname{arsinh} f(x)$. A1: Correct expression		M1A1
	$\left[\frac{1}{2}\operatorname{arsinh}\left(\frac{x-\frac{3}{2}}{\frac{5}{2}}\right)\right]_{-1}^{4} = \frac{1}{2}(\operatorname{arsinh}(1)-\operatorname{arsinh}(-1))$	Correct use of limits	M1
	$=\frac{1}{2}\left(\ln\left(1+\sqrt{2}\right)-\ln\left(-1+\sqrt{2}\right)\right)$	Uses the logarithmic form of arsinh	M1
	$=\frac{1}{2}\ln\left(3+2\sqrt{2}\right)$ or $\ln\left(1+\sqrt{2}\right)$	cao	A1
			(7)
			Total 12

ALT:	Second M1A1		
	Sub $2x-3=u$ or $2x-3=5\sinh u$		
	$\int_{\operatorname{arsinh}-1}^{\operatorname{arsinh}1} \frac{1}{\sqrt{25\sinh^2 u+25}}5\cosh u\,du = \left[\frac{1}{2}\operatorname{arsinh}\left(\frac{u}{5}\right)\right]_{-5}^{5}$	$\int_{-5}^{5}\frac{1}{2\sqrt{u^2+25}}\,du = \left[\frac{1}{2}\operatorname{arsinh}\left(\frac{u}{5}\right)\right]$	M1A1

Q4.

Question Number	Scheme	Notes	Marks
(a)	$y=\text{artanh}(\cos x)$		
	$\dfrac{dy}{dx}=\dfrac{1}{1-\cos^2 x}\times -\sin x$	Correct use of the chain rule	M1
	$=\dfrac{-\sin x}{\sin^2 x}=\dfrac{-1}{\sin x}=-\text{cosec}\,x$ *	A1: Correct completion with no errors	A1
			(2)
	Alternative 1		
	$\tanh y=\cos x\Rightarrow \text{sech}^2 y\dfrac{dy}{dx}=-\sin x$		
	$\dfrac{dy}{dx}=\dfrac{-\sin x}{\text{sech}^2 y}=\dfrac{-\sin x}{1-\cos^2 x}$	Correct differentiation to obtain a function of x	M1
	$=\dfrac{-\sin x}{\sin^2 x}=\dfrac{-1}{\sin x}=-\text{cosec}\,x$ *	A1: Correct completion with no errors	A1
	Alternative 2		
	$\text{artanh}(\cos x)=\dfrac{1}{2}\ln\left(\dfrac{1+\cos x}{1-\cos x}\right)$		
	$\dfrac{dy}{dx}=\dfrac{1}{2}\times\dfrac{1-\cos x}{1+\cos x}\times\dfrac{-\sin x(1-\cos x)-\sin x(1+\cos x)}{(1-\cos x)^2}$	Correct differentiation to obtain a function of x	M1
	$=\dfrac{-2\sin x}{2(1-\cos^2 x)}=-\text{cosec}\,x$ *	A1: Correct completion with no errors	A1
(b)	$\displaystyle\int \cos x\,\text{artanh}(\cos x)\,dx=\sin x\,\text{artanh}(\cos x)-\int \sin x\times -\text{cosec}x\,dx$ M1: Parts in the correct direction A1: Correct expression		M1A1
	$\left[\sin x\,\text{artanh}(\cos x)+x\right]_0^{\frac{\pi}{6}}=\dfrac{1}{2}\text{artanh}\left(\dfrac{\sqrt{3}}{2}\right)+\dfrac{\pi}{6}(-(0))$ M1: Correct use of limits on either part (provided both parts are integrated). Lower limit need not be shown		M1
	$=\dfrac{1}{4}\ln\left(\dfrac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right)+\dfrac{\pi}{6}$	Use of the logarithmic form of artanh	M1
	$=\dfrac{1}{4}\ln\left(7+4\sqrt{3}\right)+\dfrac{\pi}{6}$ or $\dfrac{1}{2}\ln\left(2+\sqrt{3}\right)+\dfrac{\pi}{6}$	Cao (oe)	A1
	The last 2 M marks may be gained in reverse order.		(5)
			Total 7

Q5.

Question Number	Scheme		Marks
(a)	$\frac{dy}{dx} = \sinh\frac{x}{3}$	Correct expression for dy/dx seen explicitly or used	B1
	$\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int \sqrt{1+\sinh^2\left(\frac{x}{3}\right)}\,dx = \int \cosh\left(\frac{x}{3}\right)dx$ Uses the correct formula for arc length and reaches: $k\int \pm\cosh\left(\frac{x}{3}\right)dx$		M1
	$= 3\sinh\left(\frac{x}{3}\right)$	Correct integration	A1
	$\text{length} = \left[3\sinh\left(\frac{x}{3}\right)\right]_{-3a}^{3a} = 3\left(\sinh a - \sinh(-a)\right) = ...$ **or** $\text{length} = 2\left[3\sinh\left(\frac{x}{3}\right)\right]_{0}^{3a} = 2\times 3\left(\sinh a - \sinh(0)\right) = ...$ Correct use of limits – in the second case, the lower (0) limit does not have to been seen used		M1
	$= 6\sinh a$	Correct expression	A1
	Do not be overly concerned if a "sinh" becomes "sin" or there is a missing "dx" along the way but the final answer must be correct.		
			(5)
(b)	$6\sinh a = 12 \Rightarrow \sinh a = 2 \Rightarrow x_Q = 3a = 3\text{arsinh}2$ Uses their arc length in terms of sinh a and the 12 to find a in terms of arsinh or ln and multiplies by 3		M1
	$= 3\ln\left(2+\sqrt{5}\right)$	Correct answer **including brackets** and no other answers.	A1
			(2)
(c)	$y_Q = 3\cosh a = 3\sqrt{1+\sinh^2 a} = 3\sqrt{1+"2"^2}$ or $y_Q = 3\cosh a = 3\cosh\left(\ln\left(2+\sqrt{5}\right)\right) = 3\left(\frac{e^{\ln(2+\sqrt{5})}+e^{-\ln(2+\sqrt{5})}}{2}\right) = ...$ Use the curve equation with $x = 3a$ or $x = -3a$ and their value for a or sinh a to obtain a numerical **value** for y_Q i.e. no e's or ln's or cosh's etc.		M1
	$= 3\sqrt{5}$	Cao	A1
			(2)
			[Total 9]

Q6.

Question Number	Scheme	Marks
(a)	$I_n = \int \cosh^n x\,dx = \int \cosh x \cosh^{n-1} x\,dx$ $(I_n =)\ \int \cosh x \cosh^{n-1} x\,dx$ seen explicitly or used	B1
	$I_n = \pm\cosh^{n-1} x \sinh x \pm k\int \cosh^{n-2} x \sinh^2 x\,dx$ Uses parts in the correct direction to obtain an expression of the above form	M1
	$I_n = \cosh^{n-1} x \sinh x - \int (n-1)\cosh^{n-2} x \sinh^2 x\,dx$ Correct expression	A1
	$I_n = \cosh^{n-1} x \sinh x - \int (n-1)\cosh^{n-2} x(\cosh^2 x - 1)\,dx$ Use of $\sinh^2 x = \pm\cosh^2 x \pm 1$ **Dependent on the previous method mark**	dM1
	$I_n = \cosh^{n-1} x \sinh x - (n-1)I_n + (n-1)I_{n-2}$ Sub for I_n and I_{n-2} and collect terms **Dependent on both previous method marks**	ddM1
	$nI_n = \sinh x \cosh^{n-1} x + (n-1)I_{n-2}$ *	A1cso
Do not be overly concerned with notational errors e.g. if a "cosh" becomes a "cos" or a "sinh" becomes a "sin" and is then recovered or if e.g. a $\cosh^2 x$ appears as $\cosh x^2$ but is then recovered or if the odd "x" or "dx" disappears etc. as long as the intention is clear. However, if there are any obvious errors such as sign errors then the final mark should be withheld.		
		(6)

(a) Way 2	$I_n = \int \cosh^n x\,dx = \int \cosh^2 x \cosh^{n-2} x\,dx$ $(I_n =)\ \int \cosh^2 x \cosh^{n-2} x\,dx$ seen explicitly or used	B1
	$= \int (1 + \sinh^2 x)\cosh^{n-2} x\,dx$ Use of $\cosh^2 x = \pm\sinh^2 x \pm 1$	M1
	$= \int \cosh^{n-2} x\,dx + \int \sinh x \sinh x \cosh^{n-2} x\,dx$	
	$= \left(\int \cosh^{n-2} x\,dx + \right)\sinh x \dfrac{\cosh^{n-1} x}{n-1} - \int \cosh x \dfrac{\cosh^{n-1} x}{n-1}\,dx$ M1: Integrates $\sinh x \sinh x \cosh^{n-2} x$ to obtain an expression of the form $\pm P\sinh x \cosh^{n-1} x \pm Q\int \cosh x \cosh^{n-1} x\,dx$ **Dependent on the previous method mark** A1: For $\ldots + \sinh x \dfrac{\cosh^{n-1} x}{n-1} - \int \cosh x \dfrac{\cosh^{n-1} x}{n-1}\,dx$ Mark in the order on ePEN so that the dM1 is the A1 on ePEN and the A1 is the M2 on ePEN	dM1A1 **Note that these 2 marks are reversed for this way.**
	$\Rightarrow (n-1)I_n = (n-1)I_{n-2} + \sinh x \cosh^{n-1} x - I_n$ Sub for I_n and I_{n-2} and collect term **Dependent on both previous method marks**	ddM1
	$nI_n = \sinh x \cosh^{n-1} x + (n-1)I_{n-2}$ *	A1cso

(b)	$\int \cosh^4 x\,dx = \frac{1}{4}\left[\cosh^3 x \sinh x + 3I_2\right]$ or $4\int \cosh^4 x\,dx = \cosh^3 x \sinh x + 3I_2$ Applies the reduction formula to I_4	M1
	$\int \cosh^4 x\,dx = \frac{1}{4}\left[\cosh^3 x \sinh x + 3\left(\frac{1}{2}\left[\cosh x \sinh x + I_0\right]\right)\right]$ or $4\int \cosh^4 x\,dx = \cosh^3 x \sinh x + 3\times\frac{1}{2}\left[\cosh x \sinh x + I_0\right]$ Attempts to use the reduction formula again to obtain I_2 in terms of I_0 May be seen embedded in their I_4 **This is a method mark so allow confusion with the constants.** This mark can also be scored by an attempt to integrate $\cosh^2 x$: Either $\int \cosh^2 x\,dx = \frac{1}{2}\int (\pm\cosh 2x \pm 1)\,dx = \alpha \sinh 2x + \beta x$ or $\int \cosh^2 x\,dx = \int \left(\frac{e^x + e^{-x}}{2}\right)^2 dx = \frac{1}{4}\int \left(e^{2x} + 2 + e^{-2x}\right)dx = \frac{1}{8}e^{2x} + \frac{1}{2}x - \frac{1}{8}e^{-2x}$	M1
	Note that the final 2 A marks are only to be awarded once I_0 has been evaluated and substituted and they depend on having scored at least one method mark. **Examples:** • $\int \cosh^4 x\,dx = \frac{1}{4}\cosh^3 x \sinh x + \frac{3}{8}\cosh x \sinh x + \frac{3}{8}x(+c)$ • $\int \cosh^4 x\,dx = \frac{1}{4}\cosh^3 x \sinh x + \frac{3}{16}\sinh 2x + \frac{3}{8}x(+c)$ • $\int \cosh^4 x\,dx = \frac{1}{4}\cosh^3 x \sinh x + \frac{3}{32}e^{2x} - \frac{3}{32}e^{-2x} + \frac{3}{8}x(+c)$ A1: $\frac{1}{4}\cosh^3 x \sinh x + 1$ other correct term But note that $\frac{3}{32}e^{2x} - \frac{3}{32}e^{-2x}$ counts as 1 term A1: Fully correct Correct answer – **constant of integration not required.** **NOTE:** $\frac{1}{4}\left[\cosh^3 x \sinh x + \frac{3}{2}(\sinh x \cosh x + I_0)\right]$ scores M1M1A0A0 $\frac{1}{4}\left[\cosh^3 x \sinh x + \frac{3}{2}(\sinh x \cosh x + x)\right]$ scores M1M1A1A1	A1A1
		(4)
		[Total 10]

Note that part (b) can be done in reverse order, in which case the method marks are awarded the other way round e.g.

$$\left(I_0 = \int dx = x\right)$$

Second M: $I_2 = \frac{1}{2}\left[\sinh x \cosh x + x\right]$ **or as defined in the main scheme**

First M: $I_4 = \frac{1}{4}\left[\sinh x \cosh^3 x + 3\left(\frac{1}{2}\left[\cosh x \sinh x + x\right]\right)\right]$

$$\int \cosh^4 x\,dx = \frac{1}{4}\cosh^3 x \sinh x + \frac{3}{8}\cosh x \sinh x + \frac{3}{8}x(+c)$$

A1A1: As defined in the main scheme

Q7.

Question	Scheme	Notes	Marks
(a)	$\int \cosh^n x\,dx = \int \cosh x \cosh^{n-1} x\,dx$	Writes $\cosh^n x$ as $\cosh x \cosh^{n-1} x$	B1
	$\int \cosh x \cosh^{n-1} x\,dx = \sinh x \cosh^{n-1} x - \int (n-1)\cosh^{n-2} x \sinh^2 x\,dx$ M1: Parts in the correct direction (if the method is unclear or formula not quoted only allow sign errors) A1: Correct expression		M1A1
	$= \sinh x \cosh^{n-1} x - (n-1)\int \cosh^{n-2} x(\cosh^2 x - 1)\,dx$ Writes $\sinh^2 x$ as $\cosh^2 x - 1$		dM1
	$= \sinh x \cosh^{n-1} x - (n-1)\int \cosh^n x\,dx + (n-1)\int \cosh^{n-2} x\,dx$		
	$= \sinh x \cosh^{n-1} x - (n-1)\mathrm{I}_n + (n-1)\mathrm{I}_{n-2}$	Substitutes for I_n and I_{n-2}	ddM1
	$(1+n-1)\mathrm{I}_n = \sinh x \cosh^{n-1} x + (n-1)\mathrm{I}_{n-2}$		
	$n\mathrm{I}_n = \sinh x \cosh^{n-1} x + (n-1)\mathrm{I}_{n-2}$ *	Correct answer with at least one intermediate line of working and no errors seen	A1*
	Condone omission of "dx" and the occasional invisible x throughout but the final answer must be as printed.		**(6)**
(b)	$I_5 = \frac{1}{5}\left[\sinh x \cosh^4 x\right]_0^{\ln 2} + \frac{4}{5}\mathrm{I}_3$ or $I_3 = \frac{1}{3}\left[\sinh x \cosh^2 x\right]_0^{\ln 2} + \frac{2}{3}I_1$ One application of reduction formula (I_5 in terms of I_3 **or** I_3 in terms of I_1)		M1
	$I_5 = \frac{1}{5}\left[\sinh x \cosh^4 x\right]_0^{\ln 2} + \frac{4}{5}\mathrm{I}_3$ **and** $I_3 = \frac{1}{3}\left[\sinh x \cosh^2 x\right]_0^{\ln 2} + \frac{2}{3}I_1$ Second application of reduction formula (I_5 in terms of I_3 **and** I_3 in terms of I_1)		M1
	$I_1 = \int_0^{\ln 2} \cosh x\,dx = \left[\sinh x\right]_0^{\ln 2} = \frac{3}{4}$	$I_1 = \frac{3}{4}$	B1
	$= \frac{1}{5}.\frac{3}{4}.\left(\frac{5}{4}\right)^4 + \frac{4}{5}\left(\frac{1}{3}.\frac{3}{4}.\left(\frac{5}{4}\right)^2 + \frac{2}{3}.\frac{3}{4}\right)$		
	$\int_0^{\ln 2} \cosh^5 x\,dx = \frac{5523}{5120}$	Must be exact	A1
	NB $I_5 = \frac{1}{5}\sinh x \cosh^4 x + \frac{4}{15}\sinh x \cosh^2 x + \frac{8}{15}I_1$ could score M1M1B0A0		
			(4)
			Total 10

(a) Way 2	$\int \cosh^n x\,dx = \int \cosh^2 x \cosh^{n-2} x\,dx = \int (1+\sinh^2 x)\cosh^{n-2} x\,dx$ Writes $\cosh^n x = (1+\sinh^2 x)\cosh^{n-2} x$		B1
	$\int \sinh x \sinh x \cosh^{n-2} x\,dx = \frac{1}{n-1}\sinh x \cosh^{n-1} x - \frac{1}{n-1}\int \cosh^n x\,dx$ M1: Parts in the correct direction (if the method is unclear or formula not quoted only allow sign errors) A1: Correct expression		M1A1
	$\int \cosh^n x\,dx = \int \cosh^{n-2} x\,dx + \frac{1}{n-1}\sinh x \cosh^{n-1} x - \frac{1}{n-1}\int \cosh^n x\,dx$ Adds I_{n-2} to their integration by parts		dM1
	$I_n = I_{n-2} + \frac{1}{n-1}\sinh x \cosh^{n-1} x - \frac{1}{n-1}I_n$	Substitutes for I_n and I_{n-2}	ddM1
	$(n-1)I_n = (n-1)I_{n-2} + \sinh x \cosh^{n-1} x - I_n$		
	$n\mathrm{I}_n = \sinh x \cosh^{n-1} x + (n-1)\mathrm{I}_{n-2}$ *	Correct answer with at least one intermediate line of working and no errors seen	A1*

Q8.

Question Number	Scheme	Notes	Marks
	$x = 2\cos\theta - \cos 2\theta,\ \ y = 2\sin\theta - \sin 2\theta$		
(a)	$\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -2\sin\theta + 2\sin 2\theta,\ \ \dfrac{\mathrm{d}y}{\mathrm{d}\theta} = 2\cos\theta - 2\cos 2\theta$	Correct derivatives	B1, B1
	$\left(\dfrac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 =$ $4\sin^2\theta - 8\sin\theta\sin 2\theta + 4\sin^2 2\theta + 4\cos^2\theta - 8\cos\theta\cos 2\theta + 4\cos^2 2\theta$ Squares **and adds** their derivatives		M1
	$= 8 - 8(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta)$		
	$= 8 - 8\cos(2\theta - \theta) = 8(1 - \cos\theta)$ *	M1: Use of at least one correct trig identity A1*: Correct proof with **no errors**	M1A1*
			(5)
(b)	$S = 2\pi\int y\sqrt{\left(\left(\dfrac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2\right)}\,\mathrm{d}\theta$		
	$S = 2\pi\int(2\sin\theta - \sin 2\theta)\sqrt{8(1-\cos\theta)}\,\mathrm{d}\theta$	Substitutes $y = 2\sin\theta - \sin 2\theta$ and $8(1-\cos\theta)$ into a correct formula	M1
	$= 2\pi\int 2\sin\theta(1-\cos\theta)\sqrt{8(1-\cos\theta)}\,\mathrm{d}\theta$		
	$= 8\pi\sqrt{2}\int \sin\theta(1-\cos\theta)^{\frac{3}{2}}\,\mathrm{d}\theta$	Processes to reach an integrand of the form $k\sin\theta(1-\cos\theta)^{\frac{3}{2}}$	M1
	$= 8\sqrt{2}\,\pi\left[\dfrac{2}{5}(1-\cos\theta)^{\frac{5}{2}}\right]_0^{\pi}$	Integrates to obtain an expression of the form $\alpha(1-\cos\theta)^{\frac{5}{2}}$. Dependent on the previous method mark. (May be done by substitution)	dM1
	$= 8\pi\sqrt{2}\left(\dfrac{2}{5}(2)^{\frac{5}{2}} - 0\right)$	Use of limits 0 and π and subtracts Dependent on all previous method marks.	dddM1
	$= \dfrac{128}{5}\pi$	Allow equivalents e.g. 25.6π	A1
			(5)
			Total 10

Q9.

Question Number	Scheme	Notes	Marks
(a)	$\dfrac{\mathrm{d}x}{\mathrm{d}u}=\dfrac{3}{4}\cosh u$	Correct expression	B1
	$\displaystyle\int\frac{x^2}{\sqrt{16x^2+9}}\,\mathrm{d}x=\int\frac{\frac{9}{16}\sinh^2 u}{\sqrt{16.\frac{9}{16}\sinh^2 u+9}}.\frac{3}{4}\cosh u\,\mathrm{d}u$ M1: A **complete** substitution attempt. A1: Correct expression		M1A1
	$=k\displaystyle\int\sinh^2 u\,\mathrm{d}u$		A1
	$\sinh^2 u=\dfrac{\cosh 2u-1}{2}$	Use of $\sinh^2 u=\pm\frac{1}{2}\cosh 2u\pm\frac{1}{2}$ May be implied by their integration	M1
	$=\dfrac{9}{128}\displaystyle\int(\cosh 2u-1)\,\mathrm{d}u$		A1
			(6)
(b)	$x=0\Rightarrow u=0,\ x=1\Rightarrow u=\operatorname{arsinh}\left(\frac{4}{3}\right)$	Correct limits	B1
	$\displaystyle\int(\cosh 2u-1)\,\mathrm{d}u=\left[\frac{1}{2}\sinh 2u-u\right]$	Attempt integration of the form $\alpha\sinh 2u+\beta u$	M1
	$\displaystyle\int_0^1\frac{64x^2}{\sqrt{16x^2+9}}\,\mathrm{d}x=\frac{9}{2}\left[\frac{1}{2}\sinh 2u-u\right]_0^{\operatorname{arsinh}\left(\frac{4}{3}\right)}$		
	$=\dfrac{9}{2}\left\{\left(\dfrac{4}{3}\sqrt{1+\dfrac{16}{9}}\right)-\ln\left(\dfrac{4}{3}+\sqrt{1+\dfrac{16}{9}}\right)(-0)\right\}$	Substitute u limits and subtract the right way round. **Dependent on the previous M** (Condone omission of "-0")	dM1
	$=10,-\dfrac{9}{2}\ln 3$	cao	A1, A1
			(5)
			Total 11
	Alternative – changes back to x		
	$I=\dfrac{9}{2}\left[\sinh u\cosh u-u\right]_0^{\operatorname{arsinh}\left(\frac{4}{3}\right)}$		
	$I=\dfrac{9}{2}\left[\dfrac{4x}{3}\sqrt{\dfrac{16x^2}{9}+1}-\operatorname{arsinh}\left(\dfrac{4x}{3}\right)\right]_0^1$	Use of $\dfrac{4x}{3}$	B1
	$=\dfrac{9}{2}\left\{\left(\dfrac{4}{3}\sqrt{1+\dfrac{16}{9}}\right)-\ln\left(\dfrac{4}{3}+\sqrt{1+\dfrac{16}{9}}\right)(-0)\right\}$	Substitute x limits and subtract the right way round. **Dependent on the previous M** (Condone omission of "-0")	dM1
	$=10,-\dfrac{9}{2}\ln 3$		A1, A1

Q10.

Question Number	Scheme	Notes	Marks
	$x = 3t^4,\quad y = 4t^3$		
(a)	$\frac{dx}{dt} = 12t^3,\ \frac{dy}{dt} = 12t^2$	Correct derivatives	B1
	$S = (2\pi)\int y\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{\frac{1}{2}} dt = (2\pi)\int 4t^3\sqrt{(12t^3)^2 + (12t^2)^2}\, dt$ $\left(= (2\pi)\int 4t^3\left(144t^6 + 144t^4\right)^{\frac{1}{2}} dt\right)$ M1: Substitutes their derivatives into a correct formula (2π not required)		M1
	$S = (2\pi)\int 4t^3\left(144t^4\right)^{\frac{1}{2}}\left(t^2+1\right)^{\frac{1}{2}} dt$	Attempt to factor out at least t^4 - numerical factor may be left	M1
	$S = 96\pi\int_0^1 t^5\left(t^2+1\right)^{\frac{1}{2}} dt$	Correct completion	A1
			(4)

Question Number	Scheme	Notes	Marks
(b)	$u^2 = t^2 + 1 \Rightarrow 2u\frac{du}{dt} = 2t$ or $2u = 2t\frac{dt}{du}$	Correct differentiation	B1
	$t = 0 \Rightarrow u = 1,\ t = 1 \Rightarrow u = \sqrt{2}$	Correct limits **ALT:** reverse the substitution later. (Treat as M1 in this case and award later when work seen)	B1
	$S = (96\pi)\int t^5 \times u \times \frac{u}{t} du$		
	$S = (96\pi)\int \left(u^2 - 1\right)^2 \times u^2\, du$	M1: Complete substitution A1: Correct integral in terms of u. Ignore limits, need not be simplified	M1A1
	$S = (96\pi)\int \left(u^6 - 2u^4 + u^2\right) du = (96\pi)\left[\frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3}\right]$ M1: Expands and attempts to integrate		dM1
	$S = 96\pi\left[\frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3}\right]_1^{\sqrt{2}} = 96\pi\left\{\left(\frac{\sqrt{2}^7}{7} - \frac{2\sqrt{2}^5}{5} + \frac{\sqrt{2}^3}{3}\right) - \left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right)\right\}$ M1: Correct use of their changed limits (both to be changed) **ALT:** If sub reversed, substitute the original limits		ddM1
	$S = \frac{192\pi}{105}\left(11\sqrt{2} - 4\right)$	Cao eg $\frac{64\pi}{35}$	A1
			(7)
			Total 11

Q11.

Question Number	Scheme	Notes	Marks
	$I_n = \int_0^{\ln 2} \tanh^{2n} x \, dx, \quad n \geq 0$		
(a)	$\tanh^{2n} x = \tanh^{2(n-1)} x \tanh^2 x$		B1
	$\tanh^{2n} x = \pm\tanh^{2(n-1)} x\left(1-\operatorname{sech}^2 x\right)$		M1
	$I_n = \int_0^{\ln 2} \tanh^{2(n-1)} x \, dx - \int_0^{\ln 2} \tanh^{2(n-1)} x \operatorname{sech}^2 x \, dx$		
	$I_n = I_{n-1} - \left[\frac{1}{2n-1}\tanh^{2n-1} x\right]_0^{\ln 2}$	M1: Correctly substitutes for I_{n-1} and obtains $\int \tanh^{2(n-1)} x \operatorname{sech}^2 x \, dx = k\tanh^{2n-1} x$ A1: Correct expression	M1A1
	$= I_{n-1} - \frac{1}{2n-1}\left(\frac{3}{5}\right)^{2n-1}$ *	Correct completion with no errors	A1*
			(5)

ALT:	$I_n - I_{n-1} = \int_0^{\ln 2} \left(\tanh^{2n} x - \tanh^{2(n-1)} x\right) dx$		
	$= \int_0^{\ln 2} \tanh^{2(n-1)} x\left(\tanh^2 x - 1\right) dx$		B1
	$= \int_0^{\ln 2} \tanh^{2(n-1)} x\left(-\operatorname{sech}^2 x\right) dx$	$= \int_0^{\ln 2} \tanh^{2(n-1)} x\left(\pm\operatorname{sech}^2 x\right) dx$	M1
	$I_n - I_{n-1} = -\left[\frac{1}{2n-1}\tanh^{2n-1} x\right]_0^{\ln 2}$	M1: Obtains $\int \tanh^{2(n-1)} x \operatorname{sech}^2 x \, dx = k\tanh^{2n-1} x$ A1: Correct expression	M1A1
	$= I_{n-1} - \frac{1}{2n-1}\left(\frac{3}{5}\right)^{2n-1}$ *	Correct completion with no errors	A1*

(b)	$I_0 = \ln 2$	The integration must be seen.	B1
	$I_2 = I_1 - \frac{1}{3}\left(\frac{3}{5}\right)^3$	Applies the reduction formula once	M1
	$I_2 = I_0 - \frac{1}{1}\left(\frac{3}{5}\right)^1 - \frac{1}{3}\left(\frac{3}{5}\right)^3$	M1: Second application of the reduction formula A1: Correct expression	M1A1
	$I_2 = \ln 2 - \frac{84}{125}$	cao	A1
	Special Case:		
	If I_4 is found award B1 for I_0 or I_1 and M1M0A0A0		

	(b) Alternative		
	$I_1 = \int_0^{\ln 2} \tanh^2 x \, dx = \int_0^{\ln 2} \left(1 - \operatorname{sech}^2 x\right) dx$		
	$I_1 = \left[x - \tanh x\right]_0^{\ln 2}$	Correct integration	B1
	$I_2 = I_1 - \frac{1}{3}\left(\frac{3}{5}\right)^3$	Applies the reduction formula once	M1
	$I_1 = \ln 2 - \tanh\left(\ln 2\right) = \ln 2 - \frac{3}{5}$	M1: Uses limits A1: Correct expression	M1A1
	$I_2 = \ln 2 - \frac{3}{5} - \frac{1}{3}\left(\frac{3}{5}\right)^3$		
	$= \ln 2 - \frac{84}{125}$		A1
			(5)
			Total 10

Q12.

Question Number	Scheme	Notes	Marks
	$\int \frac{5+x}{\sqrt{4-3x^2}}\,dx$		
(a)	$\int \frac{5+x}{\sqrt{4-3x^2}}\,dx = \int \frac{5}{\sqrt{4-3x^2}}\,dx + \int \frac{x}{\sqrt{4-3x^2}}\,dx$	Splits (Denominators to be correct) Can be evidenced by the two separate integrals below.	M1
	$\int \frac{5}{\sqrt{4-3x^2}}\,dx = \frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}x$	M1: $p\arcsin qx$ Depends on 1st M mark (p can = 1) A1: $\frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}x$	dM1A1
	$\int \frac{x}{\sqrt{4-3x^2}}\,dx = -\frac{1}{3}\left(4-3x^2\right)^{\frac{1}{2}}$	M1: $k\left(4-3x^2\right)^{\frac{1}{2}}$ Depends on 1st M mark A1: $-\frac{1}{3}\left(4-3x^2\right)^{\frac{1}{2}}$	dM1A1
	$\int \frac{5+x}{\sqrt{4-3x^2}}\,dx = \frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}x - \frac{1}{3}\left(4-3x^2\right)^{\frac{1}{2}}(+c)$		
			(5)

	Alternative (a)		
	$x = \frac{2}{\sqrt{3}}\sin u \Rightarrow \int \frac{5+x}{\sqrt{4-3x^2}}\,dx = \int \frac{5+\frac{2}{\sqrt{3}}\sin u}{\sqrt{4-4\sin^2 u}}\frac{2}{\sqrt{3}}\cos u\,du$ M1: Attempt $x = k\sin u$ including replacing dx		M1
	$= \frac{5}{\sqrt{3}}u - \frac{2}{3}\cos u\,(+c)$	M1: ku or $k\cos u$ Depends on 1st M A1: Both correct	dM1A1
	$= \frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}x - \frac{1}{3}\left(4-3x^2\right)^{\frac{1}{2}}(+c)$ or $\frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}x - \frac{2}{3}\cos\left[\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right]$	M1: Changes back to x Depends on both preceeding M marks A1: Fully correct (Allow equivalent correct forms)	ddM1A1

	Can be done by sub $x = \frac{2}{\sqrt{3}}\tanh\theta$.		
(b)	$\left[\frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}x - \frac{1}{3}\left(4-3x^2\right)^{\frac{1}{2}}\right]_0^1$		
	$\left[\frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2} - \frac{1}{3}\right] - \left[\frac{5}{\sqrt{3}}\arcsin 0 - \frac{1}{3}\times 2\right]$	Substitute the limits 0 and 1 (or 0 and $\frac{\pi}{3}$ if in terms of u) in **both** parts of their integral from (a) and subtract the right way round.	M1
	$= \frac{5}{9}\pi\sqrt{3}, +\frac{1}{3}$	Any exact equivalent	A1, A1
			(3)
			Total 8

Q13.

<table>
<tr><th>Question Number</th><th colspan="2">Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td></td><td colspan="3">$x=\theta-\sin\theta,\quad y=1-\cos\theta,\qquad 0\le\theta\le 2\pi$</td><td></td></tr>
<tr><td>(a)</td><td colspan="2">$\frac{dx}{d\theta}=1-\cos\theta,\qquad \frac{dy}{d\theta}=\sin\theta$</td><td>Both</td><td>B1</td></tr>
<tr><td></td><td colspan="2">$(S=2\pi)\int(1-\cos\theta)\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}\ d\theta$
or $\int(1-\cos\theta)\sqrt{(1-\cos\theta)^2+\sin^2\theta}\ d\theta$</td><td>M1: Uses the correct formula with their derivatives.
Integrand must be a function of θ
A1: Correct integrand</td><td>M1A1</td></tr>
<tr><td></td><td colspan="2">$(S=2\pi)\int(1-\cos\theta)\sqrt{2(1-\cos\theta)}\ d\theta$</td><td></td><td></td></tr>
<tr><td></td><td colspan="2">$S=2\pi\sqrt{2}\int_0^{2\pi}(1-\cos\theta)^{\frac{3}{2}}d\theta$*</td><td>cso with at least one intermediate step shown</td><td>A1*cso</td></tr>
<tr><td></td><td colspan="2"></td><td></td><td>(4)</td></tr>
<tr><td>(b)</td><td colspan="2">$S=2\pi\sqrt{2}\int\left(1-\left(1-2\sin^2\frac{\theta}{2}\right)\right)^{\frac{3}{2}}d\theta$</td><td>Uses $\cos\theta=\pm1\pm2\sin^2\frac{\theta}{2}$</td><td>M1</td></tr>
<tr><td></td><td colspan="2">$=8\pi\int\sin^3\frac{\theta}{2}d\theta$</td><td>Correct expression</td><td>A1</td></tr>
<tr><td></td><td colspan="2">$=8\pi\int\sin\frac{\theta}{2}\left(1-\cos^2\frac{\theta}{2}\right)d\theta$</td><td>Uses Pythagoras</td><td>dM1</td></tr>
<tr><td></td><td colspan="2">$=8\pi\left[-2\cos\frac{\theta}{2}+\frac{2}{3}\cos^3\frac{\theta}{2}\right]_0^{2\pi}$</td><td>Integrates to obtain $p\cos\frac{\theta}{2}+q\cos^3\frac{\theta}{2},\ p,q\ne0$</td><td>ddM1</td></tr>
<tr><td></td><td colspan="2">$=16\pi\left[\left(-\frac{1}{3}+1\right)-\left(\frac{1}{3}-1\right)\right]$</td><td>Include 16π and use limits correctly</td><td>dddM1</td></tr>
<tr><td></td><td colspan="2">$=\frac{64\pi}{3}$</td><td></td><td>A1 (6)</td></tr>
<tr><td></td><td colspan="2"></td><td></td><td>Total 10</td></tr>
</table>

	Alternatives for (b)		
1	$S=2\pi\sqrt{2}\int\left(1-\left(1-2\sin^2\frac{\theta}{2}\right)\right)^{\frac{3}{2}}\mathrm{d}\theta$	Uses $\cos\theta=\pm1\pm2\sin^2\frac{\theta}{2}$	M1
	$=8\pi\int\sin^3\frac{\theta}{2}\mathrm{d}\theta$	Correct expression	A1
	$\sin^3\frac{\theta}{2}=\frac{3}{4}\sin\frac{\theta}{2}-\frac{1}{4}\sin\frac{3\theta}{2}$		
	$8\pi\int\left(\frac{3}{4}\sin\frac{\theta}{2}-\frac{1}{4}\sin\frac{3\theta}{2}\right)\mathrm{d}\theta$	Uses the above identity (sign errors allowed)	dM1
	$=8\pi\left(-\frac{3}{4}\times2\cos\frac{\theta}{2}+\frac{1}{4}\times\frac{2}{3}\cos\frac{3\theta}{2}\right)$	Integrates to obtain $p\cos\frac{\theta}{2}+q\cos\frac{3\theta}{2}$	ddM1
	$8\pi\left[\left(\frac{3}{2}-\frac{1}{6}\right)-\left(\frac{3}{2}+\frac{1}{6}\right)\right]=\frac{64\pi}{3}$	Correct use of limits	dddM1 A1

2.	$u=1-\cos\theta;\ \mathrm{d}u=\sin\theta\mathrm{d}\theta$		
	$\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-(1-u)^2}$ $=u^{\frac{1}{2}}(2-u)^{\frac{1}{2}}$		
	$\int(1-\cos\theta)^{\frac{3}{2}}\mathrm{d}\theta=\int u(2-u)^{-\frac{1}{2}}\mathrm{d}u$	M1 Attempt the substitution. Integral to be in terms of u only. A1 Correct integral in terms of u	M1A1
	$=-2u(2-u)^{\frac{1}{2}}-\int-2(2-u)^{\frac{1}{2}}\times1\mathrm{d}u$		
	$=-2u(2-u)^{\frac{1}{2}}-2\times\frac{2}{3}(2-u)^{\frac{3}{2}}$	Integrate by parts	dM1
	$2\pi\sqrt{2}\int_0^{2\pi}(1-\cos\theta)^{\frac{3}{2}}\mathrm{d}\theta$ $=2\times2\pi\sqrt{2}\int_0^{\pi}(1-\cos\theta)^{\frac{3}{2}}\mathrm{d}\theta$		
	$2\times2\pi\sqrt{2}\int_0^2u(2-u)^{-\frac{1}{2}}\mathrm{d}u$ $=2\times2\pi\sqrt{2}\times2\times\frac{2}{3}\times2^{\frac{3}{2}}$	Include the constant multiplier of the integral and EITHER: M1 Change the limits M1 Substitute limits for u OR: M1 Reverse the substitution M1 Substitute limits for θ	ddM1 ddM1
	$=\frac{64\pi}{3}$		A1

3	**Using Integration by parts:**		
	$S = 8\pi \int \sin^3 \frac{\theta}{2} \mathrm{d}\theta$	See main scheme	M1A1
	$u = \sin^2 \frac{\theta}{2}$ $\quad \frac{\mathrm{d}v}{\mathrm{d}\theta} = \sin \frac{\theta}{2}$ $\frac{\mathrm{d}u}{\mathrm{d}\theta} = \sin \frac{\theta}{2} \cos \frac{\theta}{2}$ $\quad v = -2\cos \frac{\theta}{2}$		
	$= -2\sin^2 \frac{\theta}{2} \cos \frac{\theta}{2} + \int 2\sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} \mathrm{d}\theta$	Integrate by parts	dM1
	$\int 2\sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} \mathrm{d}\theta \to k\cos^3 \frac{\theta}{2}$		ddM1
	$= \frac{64\pi}{3}$	M1: Include the constant multiplier and use limits (0 and 2pi) correctly A1: Correct answer	dddM1A1

Q14.

Question Number	Scheme		Marks
(a)	$8+4x+x^2=(x+2)^2+4$	Correct completion of the square	B1
	$\int \frac{1}{(x+2)^2+4}\,dx=\frac{1}{2}\arctan\frac{x+2}{2}\quad(+c)$ M1: For obtaining $k\arctan f(x)$ Accept other notation for arctan e.g. artan, tan^{-1} etc. A1: Correct result oe e.g. $\frac{1}{2}\arctan\left(\frac{x}{2}+1\right)(+c)$ **(must see brackets in this case)** The constant of integration is **not** required		M1A1
	May see substitution e.g. $x+2=2\tan u\Rightarrow\int\frac{1}{(x+2)^2+4}\,dx=\int\frac{1}{4\tan^2u+4}2\sec^2u\,du$ $=\frac{1}{2}\int du=\frac{1}{2}u(+c)=\frac{1}{2}\arctan\left(\frac{x+2}{2}\right)(+c)$ For M1 this requires a complete method using a **correct substitution** and including the reversal of the substitution and A1 as already defined Accept other notation for arctan e.g. artan, tan^{-1} etc.		
			(3)

(b)	$8-4x-x^2=12-(x+2)^2$	For an attempt to complete the square. Allow $8-4x-x^2=\alpha-(\pm x\pm 2)^2$ **Where $\alpha>0$**	M1
		$12-(x+2)^2$	A1
	$\int\frac{1}{\sqrt{\{12-(x+2)^2\}}}\,dx=\arcsin\frac{x+2}{\sqrt{12}}\quad(+c)$ M1: For obtaining $\alpha\arcsin g(x)$ Accept other notation for arcsin e.g. arsin, sin^{-1} etc. A1: Correct result oe e.g. $\arcsin\frac{x+2}{2\sqrt{3}}\quad(+c)$ The constant of integration is **not** required Accept other notation for arcsin e.g. arsin, sin^{-1} etc.		M1A1
	May see substitution e.g. $x+2=\sqrt{12}\sin u\Rightarrow\int\frac{1}{\sqrt{12-(x+2)^2}}\,dx=\int\frac{1}{\sqrt{12-12\sin^2u}}\sqrt{12}\cos u\,du$ $=\int du=u(+c)=\arcsin\left(\frac{x+2}{\sqrt{12}}\right)(+c)$ For M1 this requires a complete method using a **correct substitution** and including the reversal of the substitution and A1 as already defined		
	May see substitution e.g. $x+2=\sqrt{12}\cos u\Rightarrow\int\frac{1}{\sqrt{12-(x+2)^2}}\,dx=\int\frac{-1}{\sqrt{12-12\cos^2u}}\sqrt{12}\sin u\,du$ $=-\int du=u(+c)=-\arccos\left(\frac{x+2}{\sqrt{12}}\right)(+c)$ For M1 this requires a complete method using a correct substitution and including the reversal of the substitution and A1 as already defined Accept other notation for arccos e.g. arcos, cos^{-1} etc.		
			(4)
			[Total 7]

Q15.

Question Number	Scheme	Notes	Marks
	$I_n = \int \frac{x^n}{\sqrt{(x^2+k^2)}}\,dx$		
(a)	$I_n = \int x^{n-1}x(x^2+k^2)^{-\frac{1}{2}}\,dx$	Separates correctly (Without this there will be no progress.)	B1
	$I_n = x^{n-1}(x^2+k^2)^{\frac{1}{2}} - \int (n-1)x^{n-2}(x^2+k^2)^{\frac{1}{2}}\,dx$	M1: Parts in the correct direction A1: Correct expression	M1A1
	$= ... - (n-1)\int \frac{x^{n-2}(x^2+k^2)}{\sqrt{(x^2+k^2)}}\,dx$	Writes $(x^2+k^2)^{\frac{1}{2}}$ as $\frac{(x^2+k^2)}{\sqrt{(x^2+k^2)}}$	dM1
	$= ... - (n-1)\int \frac{x^n}{\sqrt{(x^2+k^2)}}\,dx - (n-1)\int \frac{k^2x^{n-2}}{\sqrt{(x^2+k^2)}}\,dx$	Correct separation	A1
	$I_n = x^{n-1}(x^2+k^2)^{\frac{1}{2}} - (n-1)I_n - (n-1)k^2I_{n-2}$	Introduces I_n and I_{n-2} on rhs depends on both M marks above	ddM1
	$I_n = \frac{x^{n-1}}{n}(x^2+k^2)^{\frac{1}{2}} - \frac{(n-1)}{n}k^2I_{n-2}$ *	Cso (Given answer!)	A1*
			(7)

(b)	$I_5 = \int \frac{x^5}{\sqrt{(x^2+1)}}\,dx = \frac{x^4}{5}(x^2+1)^{\frac{1}{2}} - \frac{4}{5}I_3$	Correct first application of the reduction formula Can have k^2 instead of 1	M1
	$I_3 = \frac{x^2}{3}(x^2+1)^{\frac{1}{2}} - \frac{2}{3}I_1$	Correct second application of the reduction formula Can have k^2 instead of 1	M1
	$I_1 = \int \frac{x}{\sqrt{(x^2+1)}}\,dx = \left[\sqrt{x^2+1}\right] \Rightarrow I_5 = ...$	$\int \frac{x}{\sqrt{(x^2+1)}}\,dx = a\sqrt{x^2+1}$ And attempt I_5 using correct limits (k^2 or 1)	ddM1
	$\int_0^1 \frac{x^5}{\sqrt{(x^2+1)}}\,dx = \frac{7}{15}\sqrt{2} - \frac{8}{15}$	A1: Either term correct A1: Both terms correct	A1A1 (5) **Total 12**
(b)			
Way 2	$I_1 = \int \frac{x}{\sqrt{(x^2+1)}}\,dx = \sqrt{x^2+1}$	$\int \frac{x}{\sqrt{(x^2+1)}}\,dx = a\sqrt{x^2+1}$ (k^2 or 1)	M1
	$I_3 = \frac{x^2}{3}(x^2+1)^{\frac{1}{2}} - \frac{2}{3}I_1$	Attempt I_3 by using the reduction formula (k^2 or 1)	M1
	$I_5 = \int \frac{x^5}{\sqrt{(x^2+1)}}\,dx = \frac{x^4}{5}(x^2+1)^{\frac{1}{2}} - \frac{4}{5}I_3$ $= \frac{x^4}{5}(x^2+1)^{\frac{1}{2}} - \frac{4}{5}\left(\frac{x^2}{3}(x^2+1)^{\frac{1}{2}} - \frac{2}{3}(x^2+1)^{\frac{1}{2}}\right)$	Form a complete statement for I_5 and use the correct limits (k^2 or 1)	ddM1
	$\int_0^1 \frac{x^5}{\sqrt{(x^2+1)}}\,dx = \frac{7}{15}\sqrt{2} - \frac{8}{15}$	A1: Either term correct A1: Both terms correct	A1A1

Q16.

Question Number	Scheme	Notes	Marks
	$I_n = \int x^n \cos x \, dx$		
(a)	$\int x^n \cos x \, dx = x^n \sin x - \int nx^{n-1} \sin x \, dx$ M1: Parts in the correct direction A1: Correct expression		M1A1
	$= x^n \sin x - \left\{-nx^{n-1} \cos x + \int n(n-1)x^{n-2} \cos x \, dx\right\}$ Uses integration by parts again (**dependent on the first M**)		dM1
	$= x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2}$ * Fully correct proof with no errors		A1*
			(4)
ALT			
	$I_n = \int x^n \cos x \, dx = \int x^{n-1}(x \cos x) \, dx$		
	$= x^n \sin x + x^{n-1} \cos x - (n-1)\int x^{n-2}(x \sin x + \cos x) \, dx$ M1: Parts in the correct direction A1: Correct expression		M1A1
	$= x^n \sin x + x^{n-1} \cos x - (n-1)\int x^{n-1} \sin x \, dx - (n-1)I_{n-2}$		
	$= x^n \sin x + x^{n-1} \cos x - (n-1)\left\{-x^{n-1} \cos x + (n-1)I_{n-2}\right\} - (n-1)I_{n-2}$ Uses integration by parts again (**dependent on the first M**)		dM1
	$= x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2}$ * Fully correct proof with no errors		A1*
(b)	$I_0 = \sin x \ \ (+k)$		B1
	$I_4 = x^4 \sin x + 4x^3 \cos x - 12I_2$	Applies the reduction formula once for I_4 or I_2	M1
	$= x^4 \sin x + 4x^3 \cos x - 12\left(x^2 \sin x + 2x \cos x - 2I_0\right)$ Applies the reduction formula again and obtains an expression for I_4 which can include I_0 but not I_2		M1
	$= \left(x^4 - 12x^2 + 24\right)\sin x + \left(4x^3 - 24x\right)\cos x + c$ Award A1 for either bracket and A1 for the other If the answer is not factorised but is otherwise correct, award A1A0		A1A1
			(5)
			Total 9

Q17.

Question Number	Scheme	Notes	Marks
	$I_n = \int \text{cosec}^n x \, dx = \int \text{cosec}^{n-2} x \, \text{cosec}^2 x \, dx$		
(a)	$= -\cot x \, \text{cosec}^{n-2} x - \int (n-2) \text{cosec}^{n-2} x \cot^2 x \, dx$	M1: Parts in the correct direction A1: Correct unsimplified expression	M1A1
	$= -\cot x \, \text{cosec}^{n-2} x - \int (n-2) \text{cosec}^{n-2} x (\text{cosec}^2 x - 1) dx$	Uses $\cot^2 x = \text{cosec}^2 x - 1$ (incorrect signs allowed)	dM1
	$= -\cot x \, \text{cosec}^{n-2} x - (n-2) \int \text{cosec}^n x \, dx + (n-2) \int \text{cosec}^{n-2} x \, dx$		
	$= -\cot x \, \text{cosec}^{n-2} x - (n-2) I_n + (n-2) I_{n-2}$	Introduces I_n and I_{n-2}	ddM1
	$I_n = \frac{n-2}{n-1} I_{n-2} - \frac{1}{n-1} \cot x \, \text{cosec}^{n-2} x$ *	Correct completion with no errors (apart from possible omission of dx)	A1* cso
			(5)

(a) Way 2	$I_n = \int \text{cosec}^n x \, dx = \int \text{cosec}^{n-2} x \, \text{cosec}^2 x \, dx$		
	$I_n = \int \text{cosec}^{n-2} x \, \text{cosec}^2 x \, dx = \int \text{cosec}^{n-2} x (1 + \cot^2 x) dx$ Uses $\cot^2 x = \text{cosec}^2 x - 1$ (incorrect signs allowed)		M1
	$= \int \text{cosec}^{n-2} x \, dx + \int \cot^2 x \, \text{cosec}^{n-2} x \, dx$		
	$\int \cot^2 x \, \text{cosec}^{n-2} x \, dx = -\frac{1}{(n-2)} \cot x \, \text{cosec}^{n-2} x - \frac{1}{(n-2)} \int \text{cosec}^n x \, dx$ M1: Parts in the correct direction A1: Correct expression		dM1 (2nd M on e-PEN) A1 (1st A mark on e-PEN)
	$= I_{n-2} - \frac{1}{(n-2)} \cot x \, \text{cosec}^{n-2} x - \frac{1}{(n-2)} I_n$	Introduces I_n and I_{n-2}	ddM1
	$I_n = \frac{n-2}{n-1} I_{n-2} - \frac{1}{n-1} \cot x \, \text{cosec}^{n-2} x$ *	Correct completion with no errors	A1*cso
			(5)

(b)	$I_4 = \frac{2}{3} I_2 - \frac{1}{3} \cot x \, \text{cosec}^2 x$	Correct application of the given reduction formula	M1
	$I_2 = \int \text{cosec}^2 x \, dx = -\cot x$ or $= -\cot x \, \text{cosec}^0 x$	By integration or use of reduction formula	B1
	$I_4 = \frac{2}{3}(-\cot x) - \frac{1}{3} \cot x (1 + \cot^2 x)$	Uses their I_2 and $\text{cosec}^2 x = 1 + \cot^2 x$ (incorrect signs allowed)	M1
	$I_4 = -\cot x - \frac{1}{3} \cot^3 x (+c)$	+ c not required	A1cso
			(4)
	(b) Alternative		
	$\int \text{cosec}^4 x \, dx = \int (1 + \cot^2 x) \text{cosec}^2 x \, dx$	$\text{cosec}^4 x = (1 + \cot^2 x) \text{cosec}^2 x$	M1
	$= \int (\text{cosec}^2 x + \text{cosec}^2 x \cot^2 x) dx$		
	$I_4 = -\cot x - \frac{1}{3} \cot^3 x (+c)$	B1: $\int \text{cosec}^2 x \, dx = -\cot x$ M1: $\int \text{cosec}^2 x \cot^2 x \, dx = k \cot^3 x$ A1: $\int \text{cosec}^2 x \cot^2 x \, dx = -\frac{1}{3} \cot^3 x$	B1M1A1cso
			Total 9