<u>Questions</u>

Q1.

Find the exact values of x for which

$$\cosh 2x - 7\sinh x = 5$$

giving your answers as natural logarithms.

(7)

(Total for question = 7 marks)

Q2.

Using the definitions of sinh x and cosh x in terms of exponentials,

(a) prove that

$$\cosh^2 x - \sinh^2 x \equiv 1$$

(2)

(b) find algebraically the exact solutions of the equation

$$2\sinh x + 7\cosh x = 9$$

giving your answers as natural logarithms.

(5)

(Total for question = 7 marks)

Q3.

The curve C has equation

$$y = 9\cosh x + 3\sinh x + 7x$$

Use differentiation to find the exact x coordinate of the stationary point of C, giving your answer as a natural logarithm.

(6)

(Total for question = 6 marks)

Q4.

Solve the equation

$$18\cosh x + 14\sinh x = 11 + e^x$$

Give your answers in the form ln a, where a is rational.

(5)

(Total for question = 5 marks)

Q5.

Solve the equation

$$15\,\text{sech}^2 x + 7\tanh x = 13$$

Give your answers in terms of simplified natural logarithms.

(Total for question = 6 marks)

Q6.

(a) Starting from the definitions of sinh x and cosh x in terms of exponentials, prove that

(i) $\cosh 2x \equiv 2\cosh^2 x - 1$
(ii) $\sinh 2x \equiv 2\sinh x \cosh x$

(4)

(b) Solve the equation

$$\cosh 2x - 7\cosh x = -7$$

giving your answers as exact logarithms.

(5)

(Total for question = 9 marks)

Q7.

(a) Use the definition of sinh x in terms of exponentials to show that

$$\sinh 3x \equiv 4\sinh^3 x + 3\sinh x$$

(2)

(b) Hence determine the exact coordinates of the points of intersection of the curve with equation $y = \sinh 3x$ and the curve with equation $y = 19\sinh x$, giving your answers as simplified logarithms where necessary.

(5)

(Total for question = 7 marks)

Q8.

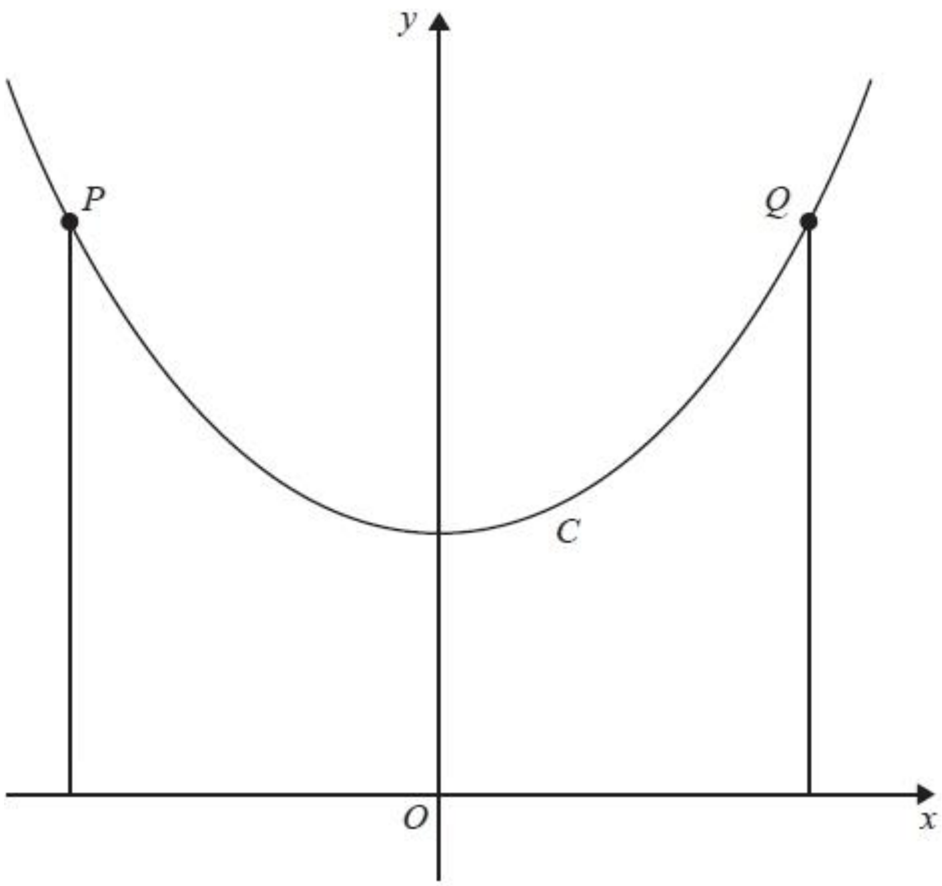

Figure 1

Figure 1 shows part of the curve C with equation

$$y = 3\cosh\left(\frac{x}{3}\right)$$

The point P and the point Q lie on the curve. The point P has x coordinate $-3a$ and the point Q has x coordinate $3a$.

(a) Find the length of the arc PQ, giving your answer as a multiple of $\sinh a$.

(5)

Given that the length of the arc PQ is 12

(b) show that the x coordinate of Q is $3\ln\left(p+\sqrt{q}\right)$, where p and q are integers to be found,

(2)

(c) show that the y coordinate of Q is $r\sqrt{s}$ where r and s are integers to be found.

(2)

(Total for question = 9 marks)

Q9.

Given that

$$y = x - \operatorname{artanh}\left(\frac{2x}{1+x^2}\right)$$

(a) show that

$$1 - \frac{dy}{dx} = \frac{k}{1-x^2}$$

where k is a constant to be found.

(4)

(b) Hence, or otherwise, show that

$$\frac{d^2y}{dx^2} + x\left(1 - \frac{dy}{dx}\right)^2 = 0$$

(4)

(Total for question = 8 marks)

Examiner's Report

Q1.

The vast majority of students correctly used the identity cosh $2x = 1 + 1 + 2\sinh^2 x$ to obtain a quadratic in sinh$x$. Most then used the logarithmic form of arsinh to obtain the final answers. Some students wrote sinhx in terms of exponentials and proceeded to solve the resulting quadratics in e^x and sometimes ended up with extra solutions that were not rejected. A significant number of students attempted to solve the given equation by expressing it in terms of exponentials. Such solutions usually stopped once a quartic in e^x was reached. Quite often, students who adopted this approach, realised that any progress would be difficult and so resorted to using the identity cosh $2x = 1 + 2\sinh^2 x$.

Q2.

Although most candidates followed the instructions given in the question to use the exponential forms of sinh x and cosh x to establish the identity in part (a) and to solve the equation in (b), many penalised themselves by not following the instructions carefully and did not use exponential functions in part (b).

In part (a), most succeeded in proving this standard equality although some candidates did not show sufficient working to justify the accuracy mark. Examiners expected to see at least one line of working in between the substitution of exponentials and the "= 1".

In (b), those who used the exponential form of the hyperbolic functions, as instructed, quickly managed to collect the terms and form a 3 term quadratic in e^x. The quadratic was solved by either using the formula or less frequently, by factorisation. Once the values for e were found very few made errors in finding the values of x. A significant number of candidates chose to make a rearrangement of the given equation and square both sides and solve their resulting equation without reference to exponential functions. Some, however, did revert to exponential functions but candidates need to be aware that the process of squaring can introduce extra spurious answers to the original equation.

Q3.

This question was found to be accessible. Most students could differentiate correctly although there was the occasional student who thought that cosh x differentiated to − sinh x. There was also the occasional error in signs for the expression of sinh x and cosh x in terms of e^x and e^{-x}. Most then realised they needed to rearrange their expressions to obtain a quadratic in e^x and were able to solve their quadratic correctly although the occasional one had incorrect signs. Most realised that they should only chose the positive value for e^x and then got the correct answer although some still wrote $x = lm\left(\frac{-3}{2}\right)$ which does not exist as well as the correct answer. Very few used the alternate method of squaring and attempting the quadratic in cosh x but most who used this method got to the correct answer although again some had cosh x as negative which is not possible.

Q4.

This was a good straightforward question to start the examination. There were very few errors and most students gained full marks.

All but a very small minority used the correct exponential forms for cosh and sinh. A small number of students made algebraic slips when substituting into the given equation and collecting terms. Of those who simplified to a quadratic equation the vast majority solved by using factorisation with only a small minority attempting the quadratic formula. A small number of students gave one of their answers as − ln3 which was not in the required form and lost the final mark.

Q5.

Most students could make progress on this question and a high proportion of fully correct solutions were seen. The preferred approach was using the identity $\text{sech}^2 x = 1 - \tanh^2 x$ to form and solve a quadratic in tanh*x*. A number of students wrote down the formula incorrectly thus losing all the accuracy marks.

Solving the equations $\tanh x = -\frac{1}{5}$ or $\frac{2}{3}$ was frequently done from first principles using exponential formulae for sinhx and coshx rather than using $\text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ as given in the formula book.

Students who preferred to use exponentials from the beginning found it more challenging and errors such as $\text{sech}^2 x = \frac{2}{e^x + e^{-x}}$ were seen on a number of occasions. Solutions which initially multiplied by $\cosh^2 x$ before using exponentials found the algebra easier. Algebraic errors left a number of students unable to obtain a three term quadratic equation. Solving for e^{2x} made this way easier to find values for x.

Q6.

In part (a), both (i) and (ii) were well answered with the majority sensibly working from right to left. A small number offered proofs that did not involve the exponential definitions of sinh and cosh and so could not score the marks available.

Part (b) was also a good source of marks for almost all. The last mark was less accessible since candidates occasionally missed the negative solutions if they used the logarithmic form of arcosh rather than exponentials. Those who attempted to use exponentials before solving for cosh x often reached a quartic in e^x but generally made no further progress.

Q7.

Most candidates made a good start to this question.

(a) Instructions to start with the definition $\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$ were overlooked by some who used sinh3x = sinh(2x + x) = sinh2$x$cosh$x$ + cosh2$x$sinh$x$. Most students tackling the problem correctly scored full marks. Sign errors in the expansion of the cubic $(e^{x} - e^{-x})^3$ or missing terms were seen. It is important that candidates who work on the left and right sides of the identity separately reach a conclusion "Shown", "√ ", "qed" or "lhs = rhs" are all acceptable.

(b) The result of part (a) was generally used to reach the cubic equation $4\sinh^3 x - 16\sinh x = 0$. Most students misread the question and only attempted the x values of the intersection of the two curves. The solution of sinh$x$ = 0 was frequently lost but solution of $\sinh^2 x = 4$ generally reached two correct values. $\sinh^2 x = 16$ was also seen. The easiest way of calculating y was 19(+/-sinh2) though the use of the exponential form for sinh 3x or $4\sinh^3 x + 3\sinh x$ was seen; hard work, though a few succeeded. Some candidates rejected sinh$x$ = -2. Values ln 2 + √5 (missing brackets) and ln (+/-2+√3) were seen. Students who used exponential functions rather than the result of part (a) were generally unsuccessful. Solution of sinh$x$ = 2 would have been quicker using the formula in the formula book rather than working through the exponentials.

Q8.

This arc length question produced a slightly more mixed response. In part (a) most obtained the correct $\frac{dy}{dx}$ and used the correct formula to reach the correct integrand. A common error was to integrate cosh $\frac{x}{3}$ into $\frac{1}{3}$ sinh $\frac{x}{3}$ rather than 3sinh $\frac{x}{3}$. Some who used 3*a* and –3*a* as limits did not give their answer as a multiple of sinh *a*, leaving it as 3sinh *a* – 3sinh (–*a*). Most went on in part (b) to use the given arc length to obtain a value for the required *x*-coordinate of 3*a*. In (c), the majority successfully obtained a value of the *y*-coordinate of 3cosh *a* using the exponential definition of cosh with only a small number unable to obtain it in the right form. The more direct route of using 3cosh $a = 3\sqrt{1 + \sinh^2 a}$ was not widely seen.

Q9.

This question proved to be a challenging one for a lot of students. The need to use the chain rule and either the quotient or product rule in part (a) was usually understood though some students did forget to include one of the two elements. Manipulating the details, including keeping track of the signs, was often done badly. Very few students attempted the alternative methods.

Those who scored all the marks in part (a) usually went on to answer part (b) correctly. A small number who had made no progress in part (a) still attempted part (b) using *k* with some success. However, only 2 of the 4 marks for part (b) were available to students who did not use a numerical value of *k* which they had obtained in part (a).

Mark Scheme

Q1.

Question Number	Scheme	Notes	Marks
	$1+2\sinh^2 x-7\sinh x=5$	Replaces $\cosh 2x$ by $1+2\sinh^2 x$ or replaces $\cosh 2x$ with $\cosh^2 x+\sinh^2 x$ and then $\cosh^2 x$ with $1+\sinh^2 x$. There must be no incorrect identities used.	M1
	$2\sinh^2 x-7\sinh x-4=0$	Correct quadratic	A1
	$(2\sinh x+1)(\sinh x-4)=0\Rightarrow\sinh x=$	Attempt to solve 3TQ in $\sinh x$ (usual rules)	M1
	$\sinh x=-\frac{1}{2}, 4$	Both (allow un-simplified e.g. $\frac{7\pm9}{4}$)	A1
	$\text{arsinh}x=\ln\left(x+\sqrt{x^2+1}\right)$	Use of the **correct** log form of arsinh	M1
	This mark may also be gained by using the exponential form of $\sinh x$ and attempting to solve to give x in terms of ln		
	$x=\ln\left(-\frac{1}{2}+\sqrt{\frac{5}{4}}\right),\ \ln\left(4+\sqrt{17}\right)$	A1: One correct exact value of x. Allow equivalent exact answers which may be un-simplified. A1: Both values correct and exact and no incorrect values. Allow equivalent exact answers which may be un-simplified. **Condone missing brackets.**	A1, A1
	Correct work giving $x=\ln\left(-\frac{1}{2}\pm\sqrt{\frac{5}{4}}\right),\ \ln\left(4\pm\sqrt{17}\right)$ would generally lose the final mark		
			(7)
			Total 7

	Alternative:		
	$\left(\frac{e^{2x}+e^{-2x}}{2}\right)-7\left(\frac{e^{x}-e^{-x}}{2}\right)=5$	M1: Substitutes the correct exponential definitions for $\cosh 2x$ and $\sinh x$ A1: Correct expression	M1A1
	$e^{4x}-7e^{3x}-10e^{2x}-7e^{x}+1=0$	M1: Multiplies by e^{2x} A1: Correct quartic in e^x	M1A1
	$\left(e^{2x}+e^{x}-1\right)\left(e^{2x}-8e^{x}-1\right)=0\Rightarrow e^{x}=...$ $\Rightarrow x=...$	Solves their quartic as far as $e^x=...$ and then converts to give x in terms of ln. There must be a recognisable attempt to solve a quartic with at least 4 terms as e.g. the product of two 3TQ's in e^x.	M1
	$x=\ln\left(-\frac{1}{2}+\sqrt{\frac{5}{4}}\right),\ \ln\left(4+\sqrt{17}\right)$	A1: One correct exact value of x. Allow equivalent exact answers which may be un-simplified. A1: Both values correct and exact and no incorrect values. Allow equivalent exact answers which may be un-simplified. **Condone missing brackets**	A1, A1

Q2.

Question Number	Scheme		Marks
(a)	$\{\frac{1}{2}(e^x+e^{-x})\}^2-\{\frac{1}{2}(e^x-e^{-x})\}^2=\{\frac{1}{4}(e^{2x}+2+e^{-2x})\}-\{\frac{1}{4}(e^{2x}-2+e^{-2x})\}$		M1
	M1: Uses the correct exponential forms for cosh and sinh and squares both brackets obtaining 3 terms each time		
	$\frac{1}{2}+\frac{1}{2}=1$	**At least one line of intermediate working** (e.g. combines fractions with a common denominator) with no errors seen and concludes = 1	A1
			(2)
(b)	$(e^x-e^{-x})+7\times\frac{1}{2}(e^x+e^{-x})=9$ $\Rightarrow\frac{9}{2}e^x+\frac{5}{2}e^{-x}-9=0$	M1: Uses exponential forms **and** collects terms A1: Any correct form with terms collected	M1A1
	$\Rightarrow 9e^{2x}-18e^x+5=0$ so $e^x=\ldots$	Solves their three term quadratic in e^x as far as $e^x=$	M1
	$e^x=\frac{1}{3}$ or $\frac{5}{3}$	Both values correct	A1
	$x=\ln\frac{1}{3}$ and $\ln\frac{5}{3}$	Both values correct (accept equivalents)	A1
			(5)
			Total 7
	Alternatives for (b) – Special Cases		
Way 2	$2\sinh x=9-7\cosh x\Rightarrow 45\cosh^2 x-126\cosh x+85=0$		M1A1
	M1: Attempt to square both sides A1: Correct quadratic in cosh*x*		
	$(15\cosh x-17)(3\cosh x-5)=0\Rightarrow\cosh x=\frac{17}{15}$ or $\cosh x=\frac{5}{3}$		
	$\frac{e^x+e^{-x}}{2}=\frac{17}{15}\Rightarrow 15e^{2x}-34e^x+15=0,\ \frac{e^x+e^{-x}}{2}=\frac{5}{3}\Rightarrow 3e^{2x}-10e^x+3=0$		
	$(5e^x-3)(3e^x-5)=0\Rightarrow e^x=\frac{3}{5},\ e^x=\frac{5}{3}$ $(3e^x-1)(e^x-3)=0\Rightarrow e^x=\frac{1}{3},\ e^x=3$ $e^x=\frac{5}{3}$ and $e^x=\frac{1}{3}$	M1: Solves at least one of their three term quadratics in e^x as far as $e^x=\ldots$, having used the correct exponential form for cosh*x* A1: $e^x=\frac{5}{3}$ and $e^x=\frac{1}{3}$ seen	M1A1
	$x=\ln\frac{1}{3}$ and $\ln\frac{5}{3}$	These values only with $\ln 3$ and $\ln\frac{3}{5}$ rejected	A1
Way 3	$7\cosh x=9-2\sinh x\Rightarrow 45\sinh^2 x+36\sinh x-32=0$		
	M1: Attempt to square both sides A1: Correct quadratic in sinh*x*		M1A1
	$(15\sinh x-8)(3\sinh x+4)=0\Rightarrow\sinh x=\frac{8}{15}$ or $\sinh x=-\frac{4}{3}$		
	$\frac{e^x-e^{-x}}{2}=\frac{8}{15}\Rightarrow 15e^{2x}-16e^x-15=0,\ \frac{e^x-e^{-x}}{2}=-\frac{4}{3}\Rightarrow 3e^{2x}+8e^x-3=0$		
	$(3e^x-5)(5e^x+3)=0\Rightarrow e^x=\frac{5}{3},\ e^x=-\frac{3}{5}$ $(3e^x-1)(e^x+3)=0\Rightarrow e^x=\frac{1}{3},\ e^x=-3$ $e^x=\frac{5}{3}$ and $e^x=\frac{1}{3}$	M1: Solves at least one of their three term quadratics in e^x as far as $e^x=\ldots$, having used the correct exponential form for sinh*x* A1: $e^x=\frac{5}{3}$ and $e^x=\frac{1}{3}$ seen	M1A1
	$x=\ln\frac{1}{3}$ and $\ln\frac{5}{3}$	These values only	A1
	Note: For these special cases, if they use the ln form of arcosh or arsinh from their cosh = … or sinh = … then only the first 2 marks are available as they are not using exponentials.		

Q3.

Question Number	Scheme	Notes	Marks
	$y = 9\cosh x + 3\sinh x + 7x$		
	$\frac{dy}{dx} = 9\sinh x + 3\cosh x + 7$	Correct derivative	B1
	$9\frac{(e^x - e^{-x})}{2} + 3\frac{(e^x + e^{-x})}{2} + 7 = 0$	Replaces sinh$x$ and cosh$x$ by the correct exponential forms	M1
	Note that the first 2 marks can score the other way round: M1: $y = 9\frac{(e^x+e^{-x})}{2} + 3\frac{(e^x - e^{-x})}{2} + 7x$ B1: $\frac{dy}{dx} = 9\frac{(e^x - e^{-x})}{2} + 3\frac{(e^x + e^{-x})}{2} + 7$		
	$12e^{2x} + 14e^x - 6 = 0$ oe	M1: Obtains a quadratic in e^x A1: Correct quadratic	M1A1
	$(3e^x - 1)(2e^x + 3) = 0 \Rightarrow e^x = ...$	Solves their quadratic as far as $e^x = ...$	M1
	$x = \ln\left(\frac{1}{3}\right)$	cso (Allow $-\ln 3$) $e^x = -\frac{3}{2}$ need not be seen. Extra answers, award A0	A1
			(6)
	Alternative		
	$\frac{dy}{dx} = 9\sinh x + 3\cosh x + 7$	Correct derivative	B1
	$9\sinh x = -3\cosh x - 7 \Rightarrow 81\sinh^2 x = 9\cosh^2 x + 42\cosh x + 49$		
	$72\cosh^2 x - 42\cosh x - 130 = 0$	Squares and attempts quadratic in cosh$x$	M1
	$(3\cosh x - 5)(12\cosh x + 13) = 0 \Rightarrow \cosh x = \frac{5}{3}$	M1: Solves quadratic A1: Correct value	M1A1
	$x = \ln\left(\frac{5}{3} \pm \sqrt{\left(\frac{5}{3}\right)^2 - 1}\right)$	Use of ln form of arcosh	M1
	$x = \ln\left(\frac{1}{3}\right)$	cso (Allow $-\ln 3$)	A1

NB: Ignore any attempts to find the y coordinate

Q4.

Question Number	Scheme	Notes	Marks
	$18\cosh x + 14\sinh x = 11 + e^x$		
	$18\left(\frac{e^x + e^{-x}}{2}\right) + 14\left(\frac{e^x - e^{-x}}{2}\right) = 11 + e^x$	Uses the correct exponential forms	M1
	$9e^{2x} + 9 + 7e^{2x} - 7 = 11e^x + e^{2x}$		
	$15e^{2x} - 11e^x + 2 \ (=0)$ or $15e^x - 11 + 2e^{-x} \ (=0)$	M1: Collects terms to obtain a 3 term equation A1:Correct equation in either of the forms shown	M1A1
	$(5e^x - 2)(3e^x - 1) = 0 \Rightarrow e^x = ...$ or $\left(5e^{\frac{x}{2}} - 2e^{\frac{-x}{2}}\right)\left(3e^{\frac{x}{2}} - e^{\frac{-x}{2}}\right)$ or $(5e^x - 2)(3 - e^{-x})$	Attempt to solve their 3TQ Depends on the second M mark	dM1
	$x = \ln\frac{2}{5},\ \ln\frac{1}{3}$	Both; $\ln\frac{2}{5}$ or $\ln 0.4$; $\ln\frac{1}{3}$ or $\ln 0.3$rec -ln3 scores A0	A1
			(5)
			Total 5

Q5.

Question Number	Scheme	Notes	Marks
	$15\text{sech}^2 x + 7\tanh x = 13$		
	$15(1-\tanh^2 x)+7\tanh x = 13$	Uses $\text{sech}^2 x = 1-\tanh^2 x$	M1
	$15\tanh^2 x - 7\tanh x - 2 = 0$	Correct 3 term quadratic, terms in any order	A1
	$(5\tanh x+1)(3\tanh x-2)=0$ $\Rightarrow \tanh x = -\frac{1}{5},\ \frac{2}{3}$	M1: Solves their 3 term quadratic to obtain at least one value for tanh*x* Correct answers implies method A1: Both correct values If solved by formula accept $\frac{7\pm13}{30}$	M1A1
	$x=\frac{1}{2}\ln\frac{2}{3},\ \frac{1}{2}\ln 5$	A1: One correct exact answer A1: Both exact answers correct Allow equivalent answers e.g. $x=\frac{1}{2}\ln 2-\frac{1}{2}\ln 3, \ln\frac{\sqrt{6}}{3}, \ln\sqrt{\frac{2}{3}}, \ln\sqrt{5}$ etc	A1, A1
			(6)
			Total 6
	Alternative Using Exponentials		
	$15\left(\frac{2}{e^x+e^{-x}}\right)^2+7\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)=13$	Substitutes the correct exponential forms The equation may have been re-arranged before substitution. ½s may have been cancelled.	M1
	$6e^{2x}-34+20e^{-2x}=0$	Correct 3 term quadratic in e^{2x}	A1
	$3e^{4x}-17e^{2x}+10=0$		
	$(3e^{2x}-2)(e^{2x}-5)=0$ or $(3e^x-2e^{-x})(e^x-5e^{-x})=0$ $\Rightarrow e^{2x}=\frac{2}{3}$ or 5	M1: Solves their 3 term quadratic to obtain at least one value for e^{2x} A1: Both correct values	M1A1
	$x=\frac{1}{2}\ln\frac{2}{3},\ \frac{1}{2}\ln 5$	A1: One correct answer A1: Both answers correct Allow equivalent answers e.g. $x=\frac{1}{2}\ln 2-\frac{1}{2}\ln 3$	A1, A1

Solving quadratics by calculator: check their solutions if the equation is incorrect. If the solution is correct for their equation, award M1

Q6.

Question Number	Scheme		Marks
(a)(i)	$2\cosh^2 x-1=2\frac{\left(e^x+e^{-x}\right)^2}{4}-1=\frac{\left(e^{2x}+2e^x\times e^{-x}+e^{-2x}\right)}{2}-1$ Substitutes the correct definition for cosh$x$ into the rhs and squares - full expansion must be seen but allow 2 for $2e^x\times e^{-x}$		M1
	$=\frac{\left(e^{2x}+e^{-2x}\right)}{2}+1-1=\cosh 2x*$	Correct completion with no errors seen.	A1
	Working from left to right:		
	$\cosh 2x=\frac{\left(e^{2x}+e^{-2x}\right)}{2}=\frac{\left(e^x+e^{-x}\right)^2-2}{2}$ Uses the correct definition for cosh$2x$ on lhs and expresses in terms of $\left(e^x+e^{-x}\right)^2$.		M1
	$2\cosh^2 x-1*$	Correct completion with no errors seen.	A1
(ii)	$2\sinh x\cosh x=2\frac{\left(e^x-e^{-x}\right)}{2}\times\frac{\left(e^x+e^{-x}\right)}{2}=...$ Use both correct definitions on rhs and attempts to multiply $2\sinh x\cosh x=\frac{1}{2}\left(e^x-e^{-x}\right)\left(e^x+e^{-x}\right)=...\text{scores M0}$ as the definitions for sinh$x$ and cosh$x$ have not been seen		M1
	$\frac{\left(e^{2x}-e^{-2x}\right)}{2}=\sinh 2x*$	Correct completion with no errors seen.	A1
	Working from left to right:		
	$\sinh 2x=\frac{\left(e^{2x}-e^{-2x}\right)}{2}=\frac{\left(e^x+e^{-x}\right)\left(e^x-e^{-x}\right)}{2}$ Uses the correct definition for sinh$2x$ on lhs and uses the difference of 2 squares.		M1
	$2\sinh x\cosh x*$	Correct completion with no errors seen.	A1

If they work from both ends then a clear link must be established as a conclusion e.g. lhs = rhs, tick QED etc.

(b)	$2\cosh^2 x - 1 - 7\cosh x + 7 = 0$	Use the identity for cosh2x	M1
	$2\cosh^2 x - 7\cosh x + 6 = 0 \Rightarrow (2\cosh x - 3)(\cosh x - 2) = 0 \Rightarrow \cosh x = ...$ Solve their 3TQ in $\cosh x$ (the usual rules for solving can be applied if necessary)		M1
	$\cosh x = \frac{3}{2}, 2$	Correct answers, both needed	A1
	$\cosh x = \alpha \Rightarrow x = \ln\left(\alpha + \sqrt{\alpha^2 - 1}\right)$ **or** $\frac{e^x + e^{-x}}{2} = 2 \Rightarrow e^{2x} - 4e^x + 1 = 0$ or $\frac{e^x + e^{-x}}{2} = \frac{3}{2} \Rightarrow e^{2x} - 3e^x + 1 = 0$ $\Rightarrow e^x = \frac{4 \pm \sqrt{12}}{2}$ or $e^x = \frac{3 \pm \sqrt{5}}{2}$ $\Rightarrow x = \ln...$ Changes at least one arcosh to ln form either using the correct ln form of arcosh or by returning to the correct exponential form of cosh and solving a quadratic in e^x. (Note that returning to exponentials is more likely to give all 4 answers below)		M1
	$x = \ln\left(\frac{3}{2} + \sqrt{\frac{5}{4}}\right), \ -\ln\left(\frac{3}{2} + \sqrt{\frac{5}{4}}\right)\left(\text{or } \ln\left(\frac{3}{2} - \sqrt{\frac{5}{4}}\right)\right),$ $\ln\left(2 + \sqrt{3}\right), -\ln\left(2 + \sqrt{3}\right)\left(\text{or } \ln\left(2 - \sqrt{3}\right)\right)$ All 4 correct, must be exact logarithms but can be any equivalent to those shown **with brackets**. Allow unsimplified if necessary and apply isw e.g. allow $\ln\left(\frac{3}{2} + \sqrt{\left(\frac{3}{2}\right)^2 - 1}\right)$ for $\ln\left(\frac{3}{2} + \sqrt{\frac{5}{4}}\right)$		A1
			(5)
			[Total 9]

	Alternative for (b) using exponentials:	
	$\cosh 2x - 7\cosh x = -7 \Rightarrow \frac{e^{2x}+e^{-2x}}{2} - 7\left(\frac{e^{x}+e^{-x}}{2}\right) = -7$ $\Rightarrow e^{4x} - 7e^{3x} + 14e^{2x} - 7e^{x} + 1 = 0$ Substitutes the correct exponential forms and forms quartic in e^x	M1
	$e^{4x} - 7e^{3x} + 14e^{2x} - 7e^{x} + 1 = 0 \Rightarrow \left(e^{2x} - 4e^{x} + 1\right)\left(e^{2x} - 3e^{x} + 1\right) = 0$ $\Rightarrow e^{x} = \frac{4 \pm \sqrt{12}}{2}$ or $e^{x} = \frac{3 \pm \sqrt{5}}{2}$ M1: Attempts to solve one of their quadratics in e^x which has come from their quartic in e^x to **obtain exact values** for e^x A1: For at least 2 **exact** values of e^x	M1A1
	$\Rightarrow e^{x} = \frac{4 \pm \sqrt{12}}{2}$ or $e^{x} = \frac{3 \pm \sqrt{5}}{2}$ $\Rightarrow x = \ln \ldots$ Change at least one exponential form to ln form	M1
	$\Rightarrow x = \ln\left(\frac{4 \pm \sqrt{12}}{2}\right)$ and $x = \ln\left(\frac{3 \pm \sqrt{5}}{2}\right)$ All 4 correct, must be exact logarithms but can be any equivalent to those shown with brackets if necessary but e.g. they would not be required in the above forms.	A1

Q7.

Question Number	Scheme	Notes	Marks
(a)	$4\sinh^3 x+3\sinh x=4\left(\frac{e^x-e^{-x}}{2}\right)^3+3\left(\frac{e^x-e^{-x}}{2}\right)$ $=4\left(\frac{e^{3x}-3e^x+3e^{-x}-e^{-3x}}{8}\right)+3\left(\frac{e^x-e^{-x}}{2}\right)$ Uses $\sinh x=\frac{e^x-e^{-x}}{2}$ on both sinh terms and attempts to cube the bracket (min accepted is a linear x a quadratic bracket)		M1
	$=\frac{1}{2}e^{3x}-\frac{3}{2}e^x+\frac{3}{2}e^{-x}-\frac{1}{2}e^{-3x}+\frac{3}{2}e^x-\frac{3}{2}e^{-x}$ $=\frac{e^{3x}-e^{-3x}}{2}=\sinh 3x$*		A1*
			(2)
(b)	$\sinh 3x=19\sinh x\Rightarrow 4\sinh^3 x+3\sinh x=19\sinh x$ $\Rightarrow 4\sinh^3 x-16\sinh x=0$ Uses the result from (a) and combines terms		M1
	$(\sinh x=0$ or$)$ $\sinh^2 x=4$	$\sinh^2 x=4$ or $\sinh x=(\pm)2$	A1
	$(0, 0)$	States the origin as one intersection	B1
	$\ln\left(2+\sqrt{5}\right)$ **and** $-\ln\left(2+\sqrt{5}\right)$	Two correct non-zero x values(allow e.g. $\ln\left(-2+\sqrt{5}\right)$ for $-\ln\left(2+\sqrt{5}\right)$)	A1
	$\left(\ln\left(2+\sqrt{5}\right),38\right)$ **and** $\left(-\ln\left(2+\sqrt{5}\right),-38\right)$	Two correct **points** (allow e.g. $\ln\left(-2+\sqrt{5}\right)$ for $-\ln\left(2+\sqrt{5}\right)$)	A1
			(5)
	Alternative for (b) using exponentials		
	$\sinh 3x=19\sinh x\Rightarrow\frac{e^{3x}-e^{-3x}}{2}=\frac{19\left(e^x-e^{-x}\right)}{2}\Rightarrow\ldots$ Substitutes the correct exponential forms and collects terms to one side		M1
	$\Rightarrow e^{6x}-19e^{4x}+19e^{2x}-1=0$	Correct equation (or equivalent)	A1
	$(0, 0)$	States the origin as one intersection	B1
	$\frac{1}{2}\ln\left(9+4\sqrt{5}\right)$ **or** $\frac{1}{2}\ln\left(9-4\sqrt{5}\right)$	Two correct non-zero x values (oe)	A1
	$\left(\frac{1}{2}\ln\left(9+4\sqrt{5}\right),38\right)$ **and** $\left(\frac{1}{2}\ln\left(9-4\sqrt{5}\right),-38\right)$	Two correct **points** (oe)	A1
			Total 7

Q8.

Question Number	Scheme		Marks
(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh\frac{x}{3}$	Correct expression for d*y*/d*x* seen explicitly or used	B1
	$\int \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int \sqrt{1+\sinh^2\left(\frac{x}{3}\right)}\mathrm{d}x = \int \cosh\left(\frac{x}{3}\right)\mathrm{d}x$ Uses the correct formula for arc length and reaches: $k\int \pm\cosh\left(\frac{x}{3}\right)\mathrm{d}x$		M1
	$= 3\sinh\left(\frac{x}{3}\right)$	Correct integration	A1
	$\text{length} = \left[3\sinh\left(\frac{x}{3}\right)\right]_{-3a}^{3a} = 3\left(\sinh a - \sinh(-a)\right) = \ldots$ **or** $\text{length} = 2\left[3\sinh\left(\frac{x}{3}\right)\right]_{0}^{3a} = 2\times 3\left(\sinh a - \sinh(0)\right) = \ldots$ Correct use of limits – in the second case, the lower (0) limit does not have to been seen used		M1
	$= 6\sinh a$	Correct expression	A1
	Do not be overly concerned if a "sinh" becomes "sin" or there is a missing "d*x*" along the way but the final answer must be correct.		
			(5)
(b)	$6\sinh a = 12 \Rightarrow \sinh a = 2 \Rightarrow x_Q = 3a = 3\text{arsinh}2$ Uses their arc length in terms of sinh *a* and the 12 to find *a* in terms of arsinh or ln and multiplies by 3		M1
	$= 3\ln\left(2+\sqrt{5}\right)$	Correct answer **including brackets** and no other answers.	A1
			(2)
(c)	$y_Q = 3\cosh a = 3\sqrt{1+\sinh^2 a} = 3\sqrt{1+"2"^2}$ or $y_Q = 3\cosh a = 3\cosh\left(\ln\left(2+\sqrt{5}\right)\right) = 3\left(\frac{\mathrm{e}^{\ln(2+\sqrt{5})}+\mathrm{e}^{-\ln(2+\sqrt{5})}}{2}\right) = \ldots$ Use the curve equation with $x = 3a$ or $x = -3a$ and their value for *a* or sinh *a* to obtain a numerical **value** for y_Q i.e. no e's or ln's or cosh's etc.		M1
	$= 3\sqrt{5}$	Cao	A1
			(2)
			[Total 9]

Q9.

Question Number	Scheme	Notes	Marks
	$y = x - \operatorname{artanh}\left(\dfrac{2x}{1+x^2}\right)$		
(a)	$\dfrac{\mathrm{d}\left\{\operatorname{artanh}\left(\dfrac{2x}{1+x^2}\right)\right\}}{\mathrm{d}x} = \dfrac{1}{1-\left(\dfrac{2x}{1+x^2}\right)^2} \times \left(\dfrac{2\left(1+x^2\right)-4x^2}{\left(1+x^2\right)^2}\right)$ M1: Correct attempt to differentiate artanh using the chain and quotient (or product) rule, obtaining an expression in terms of x A1: Correct expression in any form ie no simplification required Quotient rule must have denominator $\left(1+x^2\right)^2$ and numerator to be $\lambda\left(1+x^2\right)-\mu x^2,\ \lambda,\mu>0$ Product rule must reach $\dfrac{\lambda}{\left(1+x^2\right)} - \dfrac{\mu x^2}{\left(1+x^2\right)^2}\quad \lambda,\mu>0$		M1A1
	$1-\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1-\left(1+\dfrac{2}{x^2-1}\right) = \dfrac{2}{1-x^2}$	dM1: Attempt $1-\dfrac{\mathrm{d}y}{\mathrm{d}x}$ A1: cao and cso	dM1 A1cso
			(4)

	(a) Way 2		
	$y = x - \operatorname{artanh}\left(\frac{2x}{1+x^2}\right) = x - \frac{1}{2}\ln\left(\dfrac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right)$		
	$= x - \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)^2 = x - \ln(x+1) + \ln(x-1)$		
	$\dfrac{\mathrm{d}\{-\ln(x+1)+\ln(x-1)\}}{\mathrm{d}x} = \dfrac{-1}{x+1} + \dfrac{1}{x-1}$	M1: Attempt to differentiate A1: Correct derivative	M1A1
	$1-\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1-\left(1+\dfrac{2}{x^2-1}\right) = \dfrac{2}{1-x^2}$	dM1: Attempt $1-\frac{\mathrm{d}y}{\mathrm{d}x}$ A1: cao and cso	dM1 A1cso

	(a) Way 3		
	$y = x - \ln\left(\dfrac{x+1}{x-1}\right)$	Obtained as Way 2	
	$\dfrac{\mathrm{d}\left(\ln\left(\dfrac{x+1}{x-1}\right)\right)}{\mathrm{d}x} = \left(\dfrac{x-1}{x+1}\right) \times \dfrac{(x-1)-(x+1)}{(x-1)^2}$	M1: Attempt to differentiate A1: Correct derivative	M1A1
	$1-\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1-\left(1+\dfrac{2}{x^2-1}\right) = \dfrac{2}{1-x^2}$	dM1: Attempt $1-\frac{\mathrm{d}y}{\mathrm{d}x}$ A1: cao and cso	dM1 A1 cso

	(a) Way 4		
	$u = \operatorname{artanh}\left(\frac{2x}{1+x^2}\right) \Rightarrow \tanh u = \frac{2x}{1+x^2}$		
	$\left(1-\tanh^2 u\right)\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$		
	$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2} \times \frac{1}{1-\frac{4x^2}{\left(1+x^2\right)^2}}$	M1: Attempt to differentiate to obtain d(artanh(...))/dx as a function of x A1: Correct (unsimplified) derivative	
	Reduces to $\frac{2}{1-x^2}$		
	Then as main scheme	dM1A1cso	
(b)	$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2\left(1-x^2\right)^{-2} \times -2x \left(= \frac{-4x}{\left(1-x^2\right)^2}\right)$	M1: Attempt second derivative (quotient/product rule as in (a))	M1A1ft
		A1ft: Follow through their k or leave as k	
	$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{-4x}{\left(1-x^2\right)^2} + x\left(\frac{2}{1-x^2}\right)^2 = 0$	M1: Attempt $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ with their value for k from (a)	M1A1cso
		A1: cso	
			(4)
			Total 8