Questions

Q1.

The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{4} = 1$$

The line l is a tangent to H at the point P ($4 \cosh \alpha$, $2 \sinh \alpha$), where α is a constant, $\alpha \neq 0$

(a) Using calculus, show that an equation for l is

$$2y \sinh \alpha - x \cosh \alpha + 4 = 0$$

(4)

The line l cuts the y-axis at the point A.

(b) Find the coordinates of A in terms of α.

(2)

The point B has coordinates (0, $10 \sinh \alpha$) and the point S is the focus of H for which $x > 0$

(c) Show that the line segment AS is perpendicular to the line segment BS.

(5)

(Total for question = 11 marks)

Q2.

The hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a and b are positive constants.

The hyperbola H has eccentricity $\frac{\sqrt{21}}{4}$ and passes through the point $(12, 5)$.

Find

(a) the value of a and the value of b,

(4)

(b) the coordinates of the foci of H

(1)

(Total for question = 5 marks)

Q3.

The ellipse E has equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$

The line L has equation $y = mx + c$, where m and c are constants.

Given that L is a tangent to E,

(a) show that

$$c^2 - 25m^2 = 9$$

(4)

(b) find the equations of the tangents to E which pass through the points (3, 4).

(5)

(Total for question = 9 marks)

Q4.

The line with equation $x = 9$ is a directrix of an ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{8} = 1$$

where a is a positive constant.

Find the two possible exact values of the constant a.

(6)

(Total for question = 6 marks)

Q5.

An ellipse has equation

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

The point P lies on the ellipse and has coordinates $(5\cos\theta, 2\sin\theta)$, $0 < \theta < \frac{\pi}{2}$

The line L is a normal to the ellipse at the point P.

(a) Show that an equation for L is

$$5x\sin\theta - 2y\cos\theta = 21\sin\theta\cos\theta$$

(5)

Given that the line L crosses the y-axis at the point Q and that M is the midpoint of PQ,

(b) find the exact area of triangle OPM, where O is the origin, giving your answer as a multiple of $\sin 2\theta$

(6)

(Total for question = 11 marks)

Q6.

The hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The line l is a normal to H at the point P ($a\sec\theta$, $b\tan\theta$), $0 < \theta < \frac{\pi}{2}$

(a) Using calculus, show that an equation for l is

$$ax\sin\theta + by = (a^2 + b^2)\tan\theta$$

(5)

The line l meets the x-axis at the point Q, and the point M is the midpoint of PQ.

(b) Find the coordinates of M.

(3)

(c) Hence find the cartesian equation of the locus of M as θ varies, giving your answer in the form $y^2 = \mathrm{f}(x)$.

(4)

(Total for question = 12 marks)

Q7.

A hyperbola H has equation

$\frac{x^2}{a^2} - \frac{y^2}{9} = 1$ where a is a positive constant

The foci of H are at the points with coordinates (6, 0) and (–6, 0)

Find

(a) the exact value of the constant a,

(3)

(b) the equations of the directrices of H.

(3)

(Total for question = 6 marks)

Q8.

An ellipse has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a > 0,\ b > 0$$

The line l has equation $y = mx + c$ where $m < 0$ and $c > 0$

Given that l is a tangent to the ellipse,

(a) show that $c^2 = b^2 + a^2m^2$

(4)

The tangent l meets the positive x-axis at the point A and the positive y-axis at the point B.
The origin is O.

Given that m varies,

(b) show that the minimum area of triangle OAB is ab.

(7)

(Total for question = 11 marks)

Q9.

The hyperbola H has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

and the ellipse E has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

where $a > b > 0$

The line l is a tangent to **hyperbola** H at the point P ($a \sec \theta$, $b \tan \theta$), where $0 < \theta < \frac{\pi}{2}$

(a) Using calculus, show that an equation for l is

$$bx \sec \theta - ay \tan \theta = ab$$

(4)

Given that the point F is the focus of **ellipse**E for which $x > 0$ and that the line l passes through F,

(b) show that l is parallel to the line $y = x$

(5)

(Total for question = 9 marks)

Examiner's Report

Q1.

Most tried to find the equation for the tangent to the hyperbola by using the parametric form of x and y. This proved to be easier than differentiating implicitly or explicitly. In establishing the printed result, some candidates jumped from (y – 2sinhα) = $\frac{\cosh\alpha}{2\sinh\alpha}$ (x – 4coshα) to 2$y$sinhα – $x$coshα + 4 = 0 with no intermediate work and thereby lost the final accuracy mark in this part.
Part (b) was very straightforward for the majority of candidates although there were a few who found where the tangent crossed the x-axis instead of the y-axis.
In (c), provided that the candidate used the correct eccentricity equation, it was relatively straightforward to find the correct value for ae although some multiplied e by 16 rather than 8. It was then relatively easy to find the gradients of AS and BS and prove these lines were perpendicular or to use vector **AS** and **BS** and prove the scalar product to be zero. The only problem was that some forgot to give a conclusion.

Q2.

This was a good source of 5 marks for many students. In part (a), almost all students used the correct eccentricity formula and substituted x = 12 and y = 5 into the hyperbola and solved the resulting simultaneous equations in a and b. There were occasional processing errors and some gave the positive and negative values for a and b. The majority of students knew how to find the foci in part (b) and a follow through mark was available for those with earlier errors. A significant number of students gave only the positive focus.

Q3.

In part (a) the majority of students substituted the general equation of the straight line into the equation of the ellipse and then used the property of equal roots for the resulting quadratic in x. The second most popular method was to find a general tangent to the ellipse using the parametric form (5 cos θ, 3 sin θ) to give $y=-\frac{3\cos\theta}{5\sin\theta}x+\frac{3}{\sin\theta}$ and then to show $\left(\frac{3}{\sin\theta}\right)^2-25\left(-\frac{3\cos\theta}{5\sin\theta}\right)^2$. A small number of students took more laborious routes involving implicit differentiation and simultaneous equations and such methods were met with varying degrees of success. It is worth pointing out here that students cannot simply quote the general result $a^2m^2 + b^2 = c^2$ and expect to gain any marks. This was, after all, the result they were being asked to prove.

Many fully correct solutions were seen in part (b) and the main method was to substitute the given point into the general straight line and then solve this simultaneously with the result achieved in part (a). There were a large number of students who thought the point (3, 4) was on the ellipse and proceeded to find the equation of the tangent to the ellipse at that point.

Q4.

Most used the correct equations for the directrix and the eccentricity and were then able to proceed to eliminate either a or e to obtain a quadratic in one variable. Most then succeed in finding the two values of a although there were a number of candidates who multiplied their value of e by 81 and not 9.

Q5.

Most got (a) fully correct although the occasional student lost an *x* somewhere in the middle but "found" it again at the end as the answer was given.

In (b) most correctly found the coordinates of *Q* and many then went on to find the coordinates of *M* although some made errors in calculating the *y* coordinate. There were then many different ways of finishing the question. The first method was to find the coordinates of the point (which can be called *L*) where the line given in (a) cuts the *x*-axis and then find the areas of the triangles *OPL* and *OML* and add them together. Alternatively some used the determinant method to find the area of the triangle required. The next method did not need the coordinates of *M* but used triangle *OPQ* and then used the fact that area of triangle *OPM* was half of this due to the fact that distance *PM* = ½ × distance *QM* – this was quite popular. Very few used the method of finding distance *OP* and then the distance from the point *M* to line *OP*. The most common error was to work out the area of triangle *OQM* but not explain why this area was the same as area of triangle *OPM* (due to distance *PM* = distance *MQ*)

Q6.

Differentiation of the Cartesian equation and the parametric equations to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ were equally popular and few errors were made. Students who simplified $\frac{b\sec^2\theta}{a\sec\theta\tan\theta}$ initially to obtain $\frac{b}{a\sin\theta}$ generally found the later algebra easier. Writing down the equation of the normal was well done though a small number of students wrote down the tangent equation. Most solutions managed to manipulate their normal equation correctly to reach the printed answer.

Calculation of the intersection of this normal with the *x*-axis (point *Q*) generally caused no problems. There was no requirement to simplify the trigonometry but students who did generally found later parts of the question easier. Finding the mid-point, *M*, of the line from a point on the hyperbola to *Q* was generally successful though a few solutions subtracted the *x* coordinates. Final answers for *M* were of varying degrees of complexity thus causing problems later.

Part (c) was a discriminator with a high proportion of students not attempting or making little progress in converting the parametric coordinates of *M* into a Cartesian equation. Making $\sec\theta$ and $\tan\theta$ the subject of the equations was the easiest method but using an incorrect version of $1 + \tan^2\theta = \sec^2\theta$ was a costly error for a number of students. A few solutions used $\sin^2\theta + \cos^2\theta = 1$ though this resulted in more work. Making y^2 the subject of this equation was challenging (or ignored) for many and final answers varied in the amount of simplification shown.

Q7.

Although a few slips were seen the majority of candidates emerged with all six marks here. In the first part, $\pm\sqrt{27}$ was occasionally given for the value of *a*. In part (b), a few candidates did not give their answer as equations for the directrices. Weaker responses included those using incorrect identities such as the eccentricity formula for an ellipse. A very small number confused *e* with the base of natural logarithms.

Q8.

This question on an ellipse was fairly discriminating, particularly in part (b), but a substantial number of candidates scored most of the marks here.

In part (a), most eliminated *y* and expanded $(mx + c)^2$ correctly but then some did not realise the need to use the discriminant. Those that did, often went on to achieve the given answer with no errors although some got bogged down in the algebra. It was rare to see the alternative verification of the result using a parameter. Some weaker responses merely found $\frac{\mathrm{d}y}{\mathrm{d}x}$ or attempted the equation of the tangent but made no further progress.

Some candidates did not offer an attempt for part (b) but it was relatively common to see the correct coordinates for the intercepts and an appropriate expression for the area. Some could not progress from this point but many did use the result in part (a) to establish an expression for the area in one variable, usually *m*. Those who did further work usually differentiated their expression correctly, set it equal to 0 and solved for *m*. The last mark proved quite hard to score since many did not correctly handle the fact that *m* was negative and that the minimum area occurred when $m = -\frac{b}{a}$. A significant number of candidates were successful using parametric coordinates. Those who obtained the correct $A = \frac{ab}{\sin 2\theta}$ often went on to score all the marks provided they had a valid argument to show that this produced a minimum area of *ab*.

Q9.

Part (a) was answered well with most students scoring full marks. Few errors were made differentiating implicitly to reach a gradient of $\frac{b\sec^2\theta}{a\sec\theta\tan\theta}$ or $\frac{b}{a\sin\theta}$. Most students applied the formula $y - y_1 = m(x - x_1)$ to write down the equation of the tangent and then rearranged using $\sec^2\theta - \tan^2\theta = 1$ to reach the printed answer. There were occasional careless errors. A few solutions preferred the $y = mx + c$ approach to produce a tangent equation though this did require a lot more manipulation.

Part (b) proved to be a major challenge for students and few correct solutions were seen. The approach for many seemed to be to write down any formulae they knew and then play around without any obvious coherent strategy. A simple diagram may well have helped understand the overall direction in which to proceed. Often a maximum of two marks were scored by writing the focus as $(ae,0)$ and applying these coordinates in the tangent equation to obtain $e\sec\theta = 1$ or $\cos\theta = e$. Occasionally $\pm ae$ was written down and it was quite a search to see that only the positive one was being used.

There were few errors writing down a formula for the eccentricity of the ellipse though it was not always applied in a convincing way. The easiest way to the answer was to write the answer to the gradient from part (a) as $\frac{b}{a\sin\theta} = \frac{b}{a\sqrt{1-\cos^2\theta}} = \frac{b}{a\sqrt{1-e^2}}$. The application of the eccentricity formula soon reaches a gradient of value 1. Solutions which used the eccentricity formula at an earlier stage to reach $\sec\theta = \frac{a}{\sqrt{a^2-b^2}}$ often stopped as $\tan\theta$ was not calculated. The approach in which the equation of the tangent was simplified to $y = x - \sqrt{a^2-b^2}$ was the least popular, but when done this way was generally successful

Mark Scheme

Q1.

Question	Scheme	Marks
(a)	$\frac{dx}{d\theta} = 4\sinh\alpha$ and $\frac{dy}{d\theta} = 2\cosh\alpha$ so $\frac{dy}{dx} = \frac{2\cosh\alpha}{4\sinh\alpha}$	
	M1: Differentiates x and y and divides correctly A1: Correct derivative in terms of α	
	OR $\frac{2x}{16} - \frac{2yy'}{4} = 0 \Rightarrow y' = \frac{x}{4y} = \frac{4\cosh\alpha}{8\sinh\alpha}$	
	M1: Differentiates implicitly to obtain $px - qyy' = 0$ and makes y' the subject A1: Correct derivative in terms of α	M1A1
	OR $y = \frac{\sqrt{x^2-16}}{2} \Rightarrow y' = \frac{x}{2\sqrt{x^2-16}} = \frac{4\cosh\alpha}{2\sqrt{16\cosh^2\alpha - 16}} \left(= \frac{4\cosh\alpha}{8\sinh\alpha}\right)$	
	M1: Differentiates explicitly to obtain $y' = \frac{kx}{\sqrt{x^2-16}}$ A1: Correct derivative in terms of α	
	Equation of tangent is $(y - 2\sinh\alpha) = \frac{\cosh\alpha}{2\sinh\alpha}(x - 4\cosh\alpha)$ **(I)**	M1
	Correct straight line method using their gradient in terms of α	
	$2y\sinh\alpha - 4\sinh^2\alpha = x\cosh\alpha - 4\cosh^2\alpha$ **(II)**	
	$2y\sinh\alpha + 4\left(\cosh^2\alpha - \sinh^2\alpha\right) - x\cosh\alpha = 0 \Rightarrow 2y\sinh\alpha - x\cosh\alpha + 4 = 0$*	A1*
	See use of $\cosh^2\alpha - \sinh^2\alpha = 1$ to give printed answer – **there must be some working to establish the printed answer: (I) to * is A0, (II) to * is A1**	
		(4)
(b)	Puts $x = 0$ to give A is $\left(0, \frac{-2}{\sinh\alpha}\right)$ — M1: Uses $x = 0$ in the given equation to find y A1: $y = \frac{-2}{\sinh\alpha}$ or $y = \frac{-4}{2\sinh\alpha}$	M1A1
		(2)
(c)	$b^2 = a^2\left(e^2 - 1\right) \Rightarrow a^2e^2 = 20$ — Uses the **correct** eccentricity formula to obtain a value for a^2e^2 or ae Or finds a value for e and multiplies by a. Or finds a value for e^2 and multiplies by a^2.	M1
	$ae = \sqrt{20}$ or $2\sqrt{5}$ — Correct value for ae **Allow correct answer only**	A1
	Gradient $AS = \frac{\frac{2}{\sinh\alpha}}{2\sqrt{5}}$ or Gradient $BS = -\frac{10\sinh\alpha}{2\sqrt{5}}$ Or $\overline{AS} = \begin{pmatrix} 2\sqrt{5} \\ \frac{\frac{2}{\sinh\alpha}}{2\sqrt{5}} \end{pmatrix}$ or $\overline{BS} = \begin{pmatrix} 2\sqrt{5} \\ -10\sinh\alpha \end{pmatrix}$	B1
	At least one correct gradient or vector (allow as "coordinates") in terms of sinhα (allow if also in terms of a and or e) E.g Gradient $AS = \frac{\frac{2}{\sinh\alpha}}{ae \text{ or } 4e \text{ or } a\frac{\sqrt{5}}{2}}$ or Gradient $BS = -\frac{10\sinh\alpha}{ae \text{ or } 4e \text{ or } a\frac{\sqrt{5}}{2}}$	
	$\frac{\frac{2}{\sinh\alpha}}{2\sqrt{5}} \times -\frac{10\sinh\alpha}{2\sqrt{5}} = -1$ so AS and BS are perpendicular — M1: Multiplies their AS and BS gradients or uses scalar product e.g. $\overline{\mathbf{SB}}.\overline{\mathbf{SA}}$ in terms of sinhα only and must be seen explicitly. A1: Product = -1 or scalar product = 0 with no errors and conclusion	M1A1
		(5)
		Total 11

Q2.

Question Number	Scheme	Notes	Marks
	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$		
(a)	$\frac{12^2}{a^2}-\frac{5^2}{b^2}=1$ **and** $b^2=a^2\left(\left(\frac{\sqrt{21}}{4}\right)^2-1\right)$	Substitutes the given point into the hyperbola with the 12 and 5 correctly positioned and substitutes the given value of e into the **correct** eccentricity equation.	M1
	$b^2=\frac{5}{16}a^2 \Rightarrow \frac{144}{a^2}-\frac{80}{a^2}=1 \Rightarrow a \text{ or } a^2=...$ or $\frac{45}{b^2}-\frac{25}{b^2}=0 \Rightarrow b \text{ or } b^2=...$	Solves simultaneously to obtain a value for a or a^2 or b or b^2	M1
	$a=8,\ b=\sqrt{20}$	Allow equivalents for $\sqrt{20}$ e.g. $2\sqrt{5}$ or awrt 4.47. **Do not allow ± in either case**	A1, A1
			(4)
(b)	$(\pm ae, 0)=(\pm 2\sqrt{21}, 0)$	Both (follow through their a). **Must be coordinates.**	B1ft
			(1)
			Total 5
	Alternative to (a):		
	$12=a\sec\theta$, $5=b\tan\theta$ **and** $b^2=a^2\left(\left(\frac{\sqrt{21}}{4}\right)^2-1\right)$; $\frac{b}{a}=\frac{\sqrt{5}}{4},\ \frac{b}{a}=\frac{5}{12}\operatorname{cosec}\theta \Rightarrow \operatorname{cosec}\theta=\frac{5}{3\sqrt{5}}$	Substitutes the given value of e into the **correct** eccentricity equation and substitutes the given point into the correct parametric form and eliminates a and b	M1
	$\operatorname{cosec}\theta=\frac{5}{3\sqrt{5}} \Rightarrow a=...$ or $b=...$	Solves to obtain a value for a or b	M1
	$a=8,\ b=\sqrt{20}$	Allow equivalents for $\sqrt{20}$ e.g. $2\sqrt{5}$ or awrt 4.47. **Do not allow ± in either case**	A1, A1

Q3.

Question Number	Scheme	Notes	Marks
(a)	$\frac{x^2}{25}+\frac{(mx+c)^2}{9}=1$	Uses E and L to obtain an equation in one variable	M1
	$9x^2+25(m^2x^2+2cmx+c^2)=225$		
	$(25m^2+9)x^2+50cmx+25c^2-225=0$	Correct quadratic with terms collected	A1
	$b^2=4ac \Rightarrow (50cm)^2=4(25m^2+9)(25c^2-225)$	Use of $b^2=4ac$	M1
	$2500c^2m^2=4(625c^2m^2-5625m^2+225c^2-2025)$		
	$225c^2-5625m^2=2025$		
	$c^2-25m^2=9$ *	Achieves printed answer with no errors	A1*
	Use of the unproved general case $a^2m^2+b^2=c^2$ scores no marks		
			(4)
	See end of scheme for alternatives		
(b)	$c=4-3m \Rightarrow (4-3m)^2-25m^2=9$	Uses the point (3, 4) and solves simultaneously to obtain an equation in one variable	M1
	$16m^2+24m-7=0$		
	$(4m-1)(4m+7)=0 \Rightarrow m=\frac{1}{4},\ -\frac{7}{4}$	M1: Solves their quadratic to obtain 2 values for m A1: Correct values	M1A1
	$m=\frac{1}{4} \Rightarrow c=\frac{13}{4},\ m=-\frac{7}{4} \Rightarrow c=\frac{37}{4}$	Finds at least one value for c	M1
	$y=\frac{1}{4}x+\frac{13}{4},\ y=-\frac{7}{4}x+\frac{37}{4}$	Correct equations	A1
			(5)

(b) Way 2	$m=\frac{4-c}{3} \Rightarrow c^2-25\left(\frac{4-c}{3}\right)^2=9$	Uses the point (3, 4) and solves simultaneously to obtain an equation in one variable	M1
	$16c^2-200c+481=0$		
	$(4c-37)(4c-13)=0 \Rightarrow c=\frac{37}{4},\ \frac{13}{4}$	M1: Solves their quadratic to obtain 2 values for c A1: Correct values	M1A1
	$c=\frac{13}{4} \Rightarrow m=\frac{1}{4},\ c=\frac{37}{4} \Rightarrow m=-\frac{7}{4}$	Finds at least one least one value for m	M1
	$y=\frac{1}{4}x+\frac{13}{4},\ y=-\frac{7}{4}x+\frac{37}{4}$	Correct equations	A1
	Generally if candidates assume (3, 4) lies on the ellipse they score no marks		
			(5)
			Total 9

Alternatives to (a)

Way 2	Tangent at $\left(5\cos\theta, 3\sin\theta\right)$ is $y=-\frac{3\cos\theta}{5\sin\theta}x+\frac{3}{\sin\theta}$	M1: Full attempt at a general tangent A1: Correct tangent	M1A1
	$c^2-25m^2=\frac{9}{\sin^2\theta}-25\frac{9\cos^2\theta}{25\sin^2\theta}$	Substitutes their c and m into c^2-25m^2	M1
	$c^2-25m^2=9*$	Achieves printed answer with no errors	A1*

Way 3	$\frac{x^2}{25}+\frac{y^2}{9}=1\Rightarrow\frac{2x}{25}+\frac{2y}{9}\frac{\mathrm{d}y}{\mathrm{d}x}=0$		
	$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{9x}{25y}\Rightarrow m=-\frac{9x}{25y}$		
	$y=-\frac{9x}{25m}=mx+c\Rightarrow x=-\frac{25mc}{9+25m^2}$ and $y=\frac{9c}{9+25m^2}$		
	M1: Differentiates implicitly, uses $\frac{\mathrm{d}y}{\mathrm{d}x}=m$ and $y=mx+c$ to obtain x and y in terms of m and c A1: $x=-\frac{25mc}{9+25m^2}$ and $y=\frac{9c}{9+25m^2}$		M1A1
	$\frac{x^2}{25}+\frac{y^2}{9}=1\Rightarrow\frac{25m^2c^2}{\left(9+25m^2\right)^2}+\frac{9c^2}{\left(9+25m^2\right)^2}=1$	Substitutes their x and y in terms of m and c into E	M1
	$25m^2c^2+9c^2=\left(9+25m^2\right)^2\Rightarrow c^2-25m^2=9*$	Achieves printed answer with no errors	A1*

Q4.

Question Number	Scheme		Marks
	$\pm\frac{a}{e}=\pm 9$ **and** $a^2(1-e^2)=8$	Both equations correct	B1
	$a^4-81a^2+648=0$ or $81e^4-81e^2+8=0$	M1: Eliminates an unknown to produce a quadratic in a^2 or e^2 A1: Correct three term quadratic in any form with terms collected	M1A1
	$(a^2-72)(a^2-9)=0 \Rightarrow a^2=...$ or $(9e^2-8)(9e^2-1)=0 \Rightarrow e^2=...$	Uses a standard method (see notes) to solve quadratic as far as $a^2=...$ or $e^2=...$ (Must be $a^2=...$ or $e^2=...$ at this stage not $a=...$ or $e=....$ but this may be implied by later work) **May be implied by correct answers only.**	M1
	$a=3$ and $a=6\sqrt{2}$	M1: Complete method to find a. Either square roots from $a^2=...$ or square roots from $e^2=....$ and uses $a=9e$ at least once A1: cao (both answers correct). Do not accept $\pm$ for either of the answers unless the negative is rejected later.	M1A1
			(6)
			Total 6

Q5.

Question Number	Scheme	Notes	Marks
	$\frac{x^2}{25}+\frac{y^2}{4}=1, \quad P(5\cos\theta, 2\sin\theta)$		
(a)	$\frac{dx}{d\theta}=-5\sin\theta,\ \frac{dy}{d\theta}=2\cos\theta$ or $\frac{2x}{25}+\frac{2y}{4}\frac{dy}{dx}=0$	Correct derivatives or correct implicit differentiation	B1
	$\frac{dy}{dx}=\frac{2\cos\theta}{-5\sin\theta}$	Divides their derivatives correctly or substitutes and rearranges	M1
	$M_N=\frac{5\sin\theta}{2\cos\theta}$	Correct perpendicular gradient rule	M1
	$y-2\sin\theta=\frac{5\sin\theta}{2\cos\theta}(x-5\cos\theta)$	Correct straight line method (any complete method) **Must** use their gradient of the normal.	M1
	$5x\sin\theta-2y\cos\theta=21\sin\theta\cos\theta$*	cso	A1*
			(5)
(b)	At Q, $x=0 \Rightarrow y=-\frac{21}{2}\sin\theta$		B1
	M is $\left(\frac{0+5\cos\theta}{2},\frac{2\sin\theta-\frac{21}{2}\sin\theta}{2}\right)$ $\left(=\left(\frac{5}{2}\cos\theta,-\frac{17}{4}\sin\theta\right)\right)$	Correct mid-point method for at least one coordinate Can be implied by a correct x coordinate	M1
	L cuts x-axis at $\frac{21}{5}\cos\theta$		B1
	Area $OPM = OLP + OLM$ $\frac{1}{2}.\frac{21}{5}\cos\theta.2\sin\theta+\frac{1}{2}.\frac{21}{5}\cos\theta.\frac{17}{4}\sin\theta$	M1: Correct triangle area method using their coordinates A1: Correct expression	M1A1
	$=\frac{105}{16}\sin 2\theta$	Or $6.5625\sin 2\theta$ must be positive	A1(6)
			Total 11
	ALTs for (b)		

1	**Using Area *OPM***		
	See above for B1M1		B1M1
	Area $\Delta OPM = \frac{1}{2}\begin{vmatrix} 0 & 5\cos\theta & \frac{5}{2}\cos\theta & 0 \\ 0 & 2\sin\theta & -\frac{17}{4}\sin\theta & 0 \end{vmatrix}$	M1: Correct determinant with their coords, with 2 or 3 points. $\begin{matrix}0\\0\end{matrix}$ should be at both or neither end. A1: Correct determinant (There are more complicated determinants using the 3 points.)	M1A1
	$= \frac{1}{2}\left(0 + 5\sin\theta\cos\theta + 0 - 0 + \frac{85}{4}\sin\theta\cos\theta - 0\right)$	A1	A1
	$= \frac{105}{4}\sin\theta\cos\theta$		
	$= \frac{105}{16}\sin 2\theta$		A1

2	**Using Area *OPQ*:**		
	At Q, $x = 0 \Rightarrow y = -\frac{21}{2}\sin\theta$		B1
	Area $\Delta OPQ = \frac{1}{2}\begin{vmatrix} 5\cos\theta & 0 \\ 2\sin\theta & -\frac{21}{2}\sin\theta \end{vmatrix}$	Can be implied by the following line	M1A1
	$= \frac{1}{2} \times \frac{105}{2}\sin\theta\cos\theta$	OQ is base, x coord of P is height	A1
	$= \frac{105}{8}\sin 2\theta$		
	Area $OPM = \frac{1}{2}$ Area OPQ		M1
	$= \frac{105}{16}\sin 2\theta$		A1

3	At Q, $x = 0 \Rightarrow y = -\frac{21}{2}\sin\theta$		B1
	M is $\left(\frac{0 + 5\cos\theta}{2}, \frac{2\sin\theta - \frac{21}{2}\sin\theta}{2}\right)$ $\left(= \left(\frac{5}{2}\cos\theta, -\frac{17}{4}\sin\theta\right)\right)$		M1
	$OP = \sqrt{4\sin^2\theta + 25\cos^2\theta}$ $\left(= \sqrt{4 + 21\cos^2\theta}\right)$		B1
	$d = \dfrac{\frac{5}{2}\cos\theta \times \frac{2\sin\theta}{5\cos\theta} + \frac{17}{4}\sin\theta}{\sqrt{\frac{4\sin^2\theta}{25\cos^2\theta} + 1}} = \dfrac{\frac{21}{4}\sin\theta}{\sqrt{\frac{4 + 21\cos^2\theta}{25\cos^2\theta}}}$		
	Area $= \frac{1}{2} \times \dfrac{\frac{21}{4}\sin\theta}{\sqrt{\frac{4 + 21\cos^2\theta}{25\cos^2\theta}}} \times \sqrt{4 + 21\cos^2\theta}$		M1A1
	$= \frac{105}{16}\sin 2\theta$		A1

Q6.

Question Number	Scheme	Notes	Marks
	$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$		
(a)	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{b\sec^2\theta}{a\sec\theta\tan\theta}$ or $\dfrac{b^2x}{a^2y}$ or $\dfrac{bx}{a^2}\left(\dfrac{x^2}{a^2}-1\right)^{-\frac{1}{2}}$	Correct tangent gradient in any form	B1
	$m_N=-\dfrac{a\sec\theta\tan\theta}{b\sec^2\theta}\left(=-\dfrac{a}{b}\sin\theta\right)$	Use parametric forms and the correct perpendicular rule	M1
	$y-b\tan\theta=-\dfrac{a}{b}\sin\theta\left(x-a\sec\theta\right)$	M1: Correct straight line method using their m_N Use of $y=mx+c$ must include finding a value for c A1: Correct equation any equivalent to that shown.	M1A1
	$by-b^2\tan\theta=-ax\sin\theta+a^2\tan\theta$		
	$ax\sin\theta+by=\left(a^2+b^2\right)\tan\theta$ *	Completes to printed answer with at least one intermediate step	A1*
			(5)
(b)	$y=0\Rightarrow x=\dfrac{\left(a^2+b^2\right)\tan\theta}{a\sin\theta}\left(=\dfrac{\left(a^2+b^2\right)}{a}\sec\theta\right)$	Correct x coordinate	B1
	M is $\left(\dfrac{1}{2}\left(\dfrac{a^2+b^2}{a}\sec\theta+a\sec\theta\right),\dfrac{b}{2}\tan\theta\right)$ $=\left(\dfrac{2a^2+b^2}{2a}\sec\theta,\dfrac{b}{2}\tan\theta\right)$ oe	M1: Correct midpoint method for their x coordinate A1: Correct coordinates for M, any equivalent accepted. Need not be in coordinate brackets.	M1A1
			(3)
(c)	$\sec\theta=\dfrac{2ax}{2a^2+b^2},\tan\theta=\dfrac{2y}{b}\Rightarrow 1+\left(\dfrac{2y}{b}\right)^2=\left(\dfrac{2ax}{2a^2+b^2}\right)^2$	M1: Correct attempt to eliminate θ using coordinates of M A1: Correct equation	M1A1
	$y^2=\dfrac{b^2}{4}\left(\dfrac{4a^2x^2}{\left(2a^2+b^2\right)^2}-1\right)$ oe	dM1: Makes y^2 the subject A1: Correct equation in the required form	dM1A1
			(4)
			Total 12

Q7.

Question Number	Scheme		Marks
(a)	$ae = 6, \quad a^2\left(e^2 - 1\right) = 9$	Both correct equations needed but need not be shown explicitly	B1
	$e = \frac{6}{a} \Rightarrow 36 - a^2 = 9 \Rightarrow a = ...$ Or $a = \frac{6}{e} \Rightarrow 36 - \frac{36}{e^2} = 9 \Rightarrow e = \frac{2\sqrt{3}}{3}$ $\Rightarrow a = \frac{6}{e} = ...$	Eliminates e from their 2 equations to obtain an equation in a and solves for a or a^2 **or** Eliminates a from their 2 equations to find e and then finds a	M1
	$a = \sqrt{27}$ or $3\sqrt{3}$	Correct exact value. $a = \pm\sqrt{27}$ is A0 unless $-\sqrt{27}$ is rejected.	A1
			(3)
(b)	$e^2 - 1 = \frac{9}{27} \quad e^2 = \frac{36}{27} \quad e = \frac{2}{\sqrt{3}}$ or $\frac{2\sqrt{3}}{3}$ Finds a numerical value for e **<u>using a correct identity</u>**. This mark can be awarded if e has been found in part (a) or may be seen as part of their calculation to find $\frac{a}{e}$		M1
	$(x =)(\pm)\frac{a}{e} = ...$	Obtains a numerical value for x using a correct equation for at least one directrix with their a and e (signs can be ignored)	M1
	$x = \pm\frac{9}{2}$	**Two** correct **equations**. Allow equivalents but must be **simplified**. So allow $x = \pm\frac{9}{2}$, $x = \pm 4.5$	A1
			(3)
			[Total 6]

Note: Use of $b^2 = a^2\left(1 - e^2\right)$ can score (a) B0 M1 A0 and (b) M0 (if used again) M1 A0

Q8.

Question Number	Scheme	Marks
(a) Way 1	$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1=\ldots$$ Eliminates y from the equation of the ellipse and attempts to expand $(mx+c)^2$	M1
	$$\frac{x^2}{a^2}+\frac{m^2x^2+2cmx+c^2}{b^2}=1$$ Correct equation with the $(mx+c)^2$ expanded correctly	A1
	$4a^4m^2c^2=4a^2\left(b^2+a^2m^2\right)\left(c^2-b^2\right)$ Uses discriminant = 0 or equivalent **Dependent on the first method mark**	dM1
	$a^2m^2c^2=b^2c^2-b^4+a^2m^2c^2-a^2m^2\times b^2$ $c^2=b^2+a^2m^2$ * Complete to obtain the GIVEN result with no errors seen. At least one intermediate step should be shown.	A1 cso
		(4)
(a) Way 2	$$m=\frac{b\cos\theta}{-a\sin\theta}\Rightarrow y-b\sin\theta=\frac{b\cos\theta}{-a\sin\theta}(x-a\cos\theta)$$ $$\Rightarrow y=\frac{b\cos\theta}{-a\sin\theta}x+\frac{b}{\sin\theta}\Rightarrow m=\frac{b\cos\theta}{-a\sin\theta},\quad c=\frac{b}{\sin\theta}$$ M1: Forms the equation of a general tangent and "extracts" c and m A1: Correct c and m	M1A1
	$b^2+a^2m^2=b^2+a^2\left(\frac{b\cos\theta}{-a\sin\theta}\right)^2$ — Substitutes their m into $b^2+a^2m^2$ or equivalent work. **Dependent on the first method mark**	dM1
	$$b^2+a^2m^2=\frac{b^2\cos^2\theta+b^2\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}=c^2$$ $b^2+a^2m^2=c^2$ * Completes correctly with conclusion e.g. tick, QED etc.	A1

(b) Way 1	$x=0\Rightarrow y=c,\quad y=0\Rightarrow x=-\dfrac{c}{m}$ *or* $(A \text{ is })\left(-\dfrac{c}{m},\ 0\right)\ (B \text{ is })(0,\ c)$ Correct values for the intercepts or correct coordinates.		B1
	Area $\Delta OAB=-\dfrac{c^2}{2m}\left(\text{or } \dfrac{c^2}{2m}\right)$	Correct expression for the area (allow + or − here)	B1
	$=-\dfrac{b^2+a^2m^2}{2m}\left(\text{or } \dfrac{b^2+a^2m^2}{2m}\right)$	Uses $c^2=b^2+a^2m^2$ in the area expression to eliminate c.	M1
		Correct expression (may be unsimplified) (allow + or − here)	A1
	$\dfrac{\mathrm{d}A}{\mathrm{d}m}=\dfrac{b^2}{2}m^{-2}-\dfrac{a^2}{2}$ or $\dfrac{\mathrm{d}A}{\mathrm{d}m}=-\dfrac{2m\times 2a^2m-2\left(b^2+a^2m^2\right)}{4m^2}$ Differentiates wrt m (must be correct differentiation for their A). **Dependent on the previous M**		dM1
	At min $m^2=\dfrac{b^2}{a^2}\ \ m=(\pm)\dfrac{b}{a}$	Equate their derivative to 0 and solves for m^2 or m. **Dependent on all the previous M's**	ddM1
	Min area $=-\dfrac{b^2+b^2}{-\dfrac{2b}{a}}=ab\ \ \left(\text{units}^2\right)$	Correct completion with no errors.	A1
			(7)
			[Total 11]
(b) Way 2	$x=0\Rightarrow y=c,\quad y=0\Rightarrow x=-\dfrac{c}{m}$ *or* $(A \text{ is })\left(-\dfrac{c}{m},\ 0\right)\ (B \text{ is })(0,\ c)$ Correct values for the intercepts or correct coordinates.		B1
	Area $\Delta OAB=-\dfrac{c^2}{2m}\left(\text{or } \dfrac{c^2}{2m}\right)$	Correct expression for the area (allow + or − here)	B1
	$=-\dfrac{b^2+a^2m^2}{2m}\left(\text{or } \dfrac{b^2+a^2m^2}{2m}\right)$	Uses $c^2=b^2+a^2m^2$ in the area expression to eliminate c.	M1
		Correct expression (may be unsimplified) (allow + or − here)	A1
	$A=-\dfrac{(am+b)^2-2amb}{2m}$	Correct completion of the square in the numerator. **Dependent on the previous M**	dM1
	$=ab-\dfrac{(am+b)^2}{2m}$ is minimum when $am+b=0$ Correct argument for establishing the minimum **Dependent on all the previous M's**		ddM1
	Min area $=ab\ \ \left(\text{units}^2\right)$	Correct completion with no errors.	A1

(b) Way 3	$x=0 \Rightarrow y=\dfrac{ab\cos^2\theta}{a\sin\theta}+b\sin\theta\left(=\dfrac{b}{\sin\theta}\right)$ **or** $y=0 \Rightarrow y=\dfrac{ab\sin^2\theta}{b\cos\theta}+a\cos\theta\left(=\dfrac{a}{\cos\theta}\right)$ Correct value for one of the intercepts or correct coordinates.		B1
	$x=0 \Rightarrow y=\dfrac{ab\cos^2\theta}{a\sin\theta}+b\sin\theta\left(=\dfrac{b}{\sin\theta}\right)$ **and** $y=0 \Rightarrow y=\dfrac{ab\sin^2\theta}{b\cos\theta}+a\cos\theta\left(=\dfrac{a}{\cos\theta}\right)$ Correct values for both intercepts or correct coordinates.		B1
	Area ΔOAB $A=\dfrac{1}{2}\dfrac{a}{\cos\theta}\dfrac{b}{\sin\theta}$	Correct method for the area using their intercepts	M1
		Correct area (may be unsimplified)	A1
	$\dfrac{dA}{d\theta}=\dfrac{-2ab\left(\cos^2\theta-\sin^2\theta\right)}{4\sin^2\theta\cos^2\theta}=0 \Rightarrow \theta=\dfrac{\pi}{4}$ Adopts a correct strategy for finding θ at the minimum **Dependent on the previous M**		dM1
	$A_{\min}=\dfrac{ab}{2\sin\frac{\pi}{4}\cos\frac{\pi}{4}}$	Uses their value for θ to find the minimum value. **Dependent on all the previous M's**	ddM1
	$=ab \ \left(\text{units}^2\right)$	Cso	A1
	Alternative for last 3 marks:		
	Area ΔOAB $A=\dfrac{ab}{2\sin\theta\cos\theta}=\dfrac{ab}{\sin 2\theta}$ And the minimum will occur when sin 2θ is maximum i.e. when sin 2θ = 1 Score this mark for a valid argument for determining the minimum **Dependent on the previous M**		dM1
	$A_{\min}=\dfrac{ab}{1}$	Completes the process of finding the minimum. **Dependent on all the previous M's**	ddM1
	$=ab \ \left(\text{units}^2\right)$	Correct completion with no errors.	A1

(b) Way 4	$x=0 \Rightarrow y=c, \quad y=0 \Rightarrow x=-\dfrac{c}{m}$ *or* $(A \text{ is })\left(-\dfrac{c}{m},\ 0\right)$ $(B \text{ is })(0, c)$ Correct values for the intercepts or correct coordinates.		B1
	Area $\Delta OAB = -\dfrac{c^2}{2m}\left(\text{or } \dfrac{c^2}{2m}\right)$	Correct expression for the area (allow + or − here)	B1
	$=\dfrac{ac^2}{2\sqrt{c^2-b^2}}\left(=-\dfrac{ac^2}{2\sqrt{c^2-b^2}}\right)$	Uses $c^2=b^2+a^2m^2$ in the area expression to eliminate m.	M1
		Correct expression. (allow + or − here)	A1
	$\dfrac{\mathrm{d}A}{\mathrm{d}c}=\dfrac{2ac\times 2\sqrt{c^2-b^2}-2ac^3\left(c^2-b^2\right)^{-\frac{1}{2}}}{4\left(c^2-b^2\right)}$ Differentiates wrt c. **Dependent on the previous M**		dM1
	At min $2c^2-2b^2-c^2=0 \Rightarrow c^2=2b^2$ Equate their derivative to 0 and solves to obtain c in terms of b **Dependent on all the previous M's**		dM1
	Min area $=\dfrac{2ab^2}{2\sqrt{2b^2-b^2}}=ab \ \left(\text{units}^2\right)$	Correct completion with no errors.	A1

There will be other valid methods in part (b). Generally, the first 4 marks are for obtaining an expression for the area of AOB and then applying the result in part (a) to enable progress to be made in establishing the minimum or for using the general tangent in terms of θ to find the intercepts and hence the area of AOB. The final 3 marks are for selecting and implementing a correct strategy for proving that the minimum area is ab.

There may also be other valid methods in part (a).

If you are in any doubt whether a particular method is valid then please seek advice from your Team Leader.

Q9.

Question Number	Scheme	Notes	Marks
	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$		
(a)	$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{b^2}{a^2}\frac{a\sec\theta}{b\tan\theta}\left(=\frac{b}{a\sin\theta}\right)$ or $\frac{\mathrm{d}y}{\mathrm{d}\theta}=b\sec^2\theta,\frac{\mathrm{d}x}{\mathrm{d}\theta}=a\sec\theta\tan\theta\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}$ or $y=b\sqrt{\frac{x^2}{a^2}-1}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{b}{2}.\frac{2x}{a^2}\left(\frac{x^2}{a^2}-1\right)^{-\frac{1}{2}}=\frac{ab\sec\theta}{a^2}\left(\sec^2\theta-1\right)^{-\frac{1}{2}}$	M1: Attempts to differentiate implicitly or parametrically or directly to obtain $\frac{\mathrm{d}y}{\mathrm{d}x}=$ A1: Correct derivative as a function of θ Any equivalent form	M1A1
	$y-b\tan\theta=\frac{b}{a\sin\theta}(x-a\sec\theta)$	Correct straight line method Must be complete ie $y=mx+c$ used needs an attempt at c	dM1
	$ay\tan\theta-ab\tan^2\theta=b\sec\theta(x-a\sec\theta)$		
	$bx\sec\theta-ay\tan\theta=ab$*	Reaches printed answer with at least one intermediate line of working	A1* cso
			(4)
(b)	F is $(ae,\ 0)$	Correct focus (may be implied but if y coordinate = 0 not used give B0	B1
	$abe\sec\theta=ab(\Rightarrow e=\cos\theta)$	Substitute the coordinates of the focus into l	M1
	$m=\frac{b}{a\sin\theta}=\frac{b}{a\sqrt{1-\cos^2\theta}}=\frac{b}{a\sqrt{1-e^2}}$	Uses the gradient of l to obtain an expression for m in terms of a, b and e	M1
	For an ellipse $b^2=a^2(1-e^2)$	Use of the correct eccentricity formula for an ellipse in their expression for m	M1
	So $m=\frac{a\sqrt{1-e^2}}{a\sqrt{1-e^2}}=1$ so l is parallel to $y=x$	Correct completion with no errors and conclusion	A1 cso **(5)**
			Total 9

	(b) Way 2		
	$\frac{b}{a\sin\theta}=1\Rightarrow\sin\theta=\frac{b}{a}$	$\sin\theta=\frac{b}{a}$	B1
	$\sec\theta=\frac{1}{\cos\theta}=\frac{1}{\sqrt{1-\frac{b^2}{a^2}}}=\frac{a}{\sqrt{a^2-b^2}}$	Attempt $\sec\theta$	M1
	$bx\sec\theta-ay\tan\theta=ab\Rightarrow bx\frac{a}{\sqrt{a^2-b^2}}=ab\Rightarrow x=...$	Substitute for $\sec\theta$ and uses $y=0$ and makes x the subject	M1
	$\Rightarrow x=\sqrt{a^2-b^2}$		
	For an ellipse $b^2=a^2\left(1-e^2\right)\Rightarrow ae=\sqrt{a^2-b^2}$	Use of the correct eccentricity formula for an ellipse	M1
	So tangent passes through $(ae, 0)$ which is F	Correct completion with no errors and conclusion	A1 cso
	(b) Way 3		
	Focus is $\left(ae,0\right)$		B1
	$=\left(\sqrt{\left(a^2-b^2\right)},0\right)$	Use of the correct eccentricity formula for the ellipse	M1
	Eqn of line: $bx\sec\theta-ay\tan\theta=ab$		
	So $b\sqrt{\left(a^2-b^2\right)}\sec\theta-0=ab$	Line passes thro' the focus	M1
	$\sec\theta=\frac{a}{\sqrt{a^2-b^2}}\Rightarrow\tan\theta=\frac{b}{\sqrt{a^2-b^2}}$	Attempts $\sec\theta$ and $\tan\theta$	M1
	$x-y=\sqrt{a^2-b^2}$ OR Sub $\sec\theta$ and $\tan\theta$ into gradient to get 1	Correct completion with no errors and conclusion	
	$\therefore$ Parallel to $y=x$		A1 cso **(5)**