Questions

Q1.

(a) Show that $r^3 - (r - 1)^3 \equiv 3r^2 - 3r + 1$

(1)

(b) Hence prove by the method of differences that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

[*You may use* $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ *without proof.*]

(5)

(Total for question = 6 marks)

Q2.

(a) Express $\frac{4r+2}{r(r+1)(r+2)}$ in partial fractions.

(3)

(b) Hence, using the method of differences, prove that

$$\sum_{r=1}^{n} \frac{4r+2}{r(r+1)(r+2)} = \frac{n(an+b)}{2(n+1)(n+2)}$$

where *a* and *b* are constants to be found.

(5)

(Total for question = 8 marks)

Q3.

(a) Express $\dfrac{2}{r^2 - 1}$ in partial fractions.

(1)

(b) Hence, using the method of differences, show that, for $n \in \mathbb{Z}, n > 2$

$$\sum_{r=2}^{n} \frac{2}{r^2 - 1} = \frac{(3n+2)(n-1)}{2n(n+1)}$$

(5)

(c) Hence show that, for $n > 1$

$$\sum_{r=n}^{3n} \frac{2}{r^2 - 1} = \frac{2(an-1)(bn+1)}{3n(cn+1)(n-1)}$$

where a, b and c are integers to be found.

(4)

(Total for question = 10 marks)

Q4.

(a) Express $\dfrac{1}{(r+6)(r+8)}$ in partial fractions.

(1)

(b) Hence show that

$$\sum_{r=1}^{n} \frac{2}{(r+6)(r+8)} = \frac{n(an+b)}{56(n+7)(n+8)}$$

where a and b are integers to be found.

(4)

(Total for question = 5 marks)

Q5.

(a) Show that

$$\frac{1}{(r+1)(r+2)(r+3)} \equiv \frac{1}{2(r+1)(r+2)} - \frac{1}{2(r+2)(r+3)}$$

(2)

(b) Hence, or otherwise, find

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)(r+3)}$$

giving your answer as a single fraction in its simplest form.

(4)

(Total for question = 6 marks)

Q6.

(a) Express $\frac{1}{4r^2-1}$ in partial fractions.

(1)

(b) Hence prove that

$$\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{n}{2n+1}$$

(3)

(c) Find the exact value of

$$\sum_{r=9}^{25} \frac{5}{4r^2-1}$$

(2)

(Total for question = 6 marks)

Examiner's Report

Q1.

This question required use of the method of differences and summation algebra to prove the standard result for the sum of the squares. A wide range of mark profiles were seen here. Almost all students were able to prove the identity in part (a) without error, but part (b) proved rather more demanding.
Some students did not use the result in part (a) and instead used the standard result for the sum of the cubes or proof by induction. The method of differences was usually correctly applied and with sufficient terms included. Errors were seen in handling the three term summation although the mark for replacing $\Sigma 1$ with n was commonly awarded. Those who had obtained the correct algebraic expression for Σr^2 tended to reach the printed answer convincingly.

Q2.

This question on the method of differences saw good scoring in part (a) but less marks were awarded for part (b).

Part (a) required a rational expression to be written in partial fractions and it was rare to see an inappropriate method such as the wrong form of fractions, although a few succumbed to sign or numerical slips.

It seemed that many students were poorly prepared to deal with applying the method of differences to groups of three fractions, with many unable to deduce how the cancelling took place. Some found terms for at least r = 1, 2 and 3 but neglected to find any terms at the upper end of the summation. Those who presented their work carefully were more likely to identify the six terms that remained and were less likely to miss any out. Those with appropriate terms usually scored the final method mark for combining their fractions but fully correct solutions were not widely seen. A few attempts involved rewriting the fractions from part (a) into groups of two and and then applying the method of differences separately – these produced mixed results.

Q3.

This question involved using the method of differences. It was very rare to see any errors in obtaining the correct partial fractions in part (a). The scoring in part (b) was slightly more mixed. The general method was widely known (and for the most part well executed) but some candidates failed to notice that the summation started at r = 2, usually leading to the indeterminate form of $\frac{1}{0}$ (computed as 0 or 1) appearing in their non-cancelling terms. Some of these candidates were able to recover by subtracting the r = 1 terms later. Those who had obtained terms of the correct form usually scored the second method mark with an appropriate attempt to combine these terms. The full five marks were commonly awarded. Part (c) had the command word "Hence" and required the given result from part (b) to be used. Unfortunately, some candidates chose to repeat the differences work from earlier and some went on to achieve the correct result but did not receive any credit as they hadn't used the 'hence' requested in the question. The correct expression for S_{3n} was widely seen although some then subtracted S_n rather than S_{n-1}. A few found the resulting algebra rather intimidating – usually making the mistake of needlessly multiplying out all the brackets.

Q4.

In part (a) almost all students obtained the correct partial fractions expression with any errors caused by numerical slips. In part (b) most students attempted to sum the given expression by the method of differences, and most also made the connection between (a) and (b) by multiplying their sum by 2 either at the beginning or at the end. Students who were unsuccessful tended to forget to multiply the series by 2, while some resorted to using known formulae to try to obtain the result, despite the question being specific about using the method of differences. Such attempts generally scored zero marks.

Q5.

This question was successfully attempted by the majority of candidates, with most adopting a partial fractions approach to part (a)and attempting to write the left hand side as a sum or difference of two or three fractions. Many variations were seen, with those candidates who used the given right hand side to produce two fractions with denominators $2(r + 1)(r + 2)$ and $2(r + 2)(r + 3)$ generally more successful. Of those candidates who attempted to find three partial fractions, although the majority successfully found the numerators, many did not produce adequate evidence to progress from this to the given result.

Writing $\frac{1}{2(r+1)} - \frac{1}{(r+2)} + \frac{1}{2(r+3)}$ and moving straight to the printed result without splitting the middle term and combining to two terms was an incomplete method and so was awarded no marks.

Surprisingly few candidates took the simpler route of combining the RHS over a common denominator, but where this approach was used it almost invariably led to a correct result.

Part (b) was well attempted, with many candidates scoring all the marks. Most were able to write the first and last pairs of terms and use a method of differences to eliminate terms resulting in two correct fractions. Where errors were seen, these were usually a failure to combine the remaining two fractions or manipulation errors when attempting to do so.

Q6.

The students found this an accessible start to the paper. There were very few errors in determining the numerators of the partial fractions. The method of differences was well known and enough work was presented to be convincing as to how the terms cancelled. Sufficient intermediate steps were seen to lead to the printed result.
(c) was usually well done but there were cases where the 5 was omitted, where n = 9 was used rather than n = 8 and where the original fraction was used rather than its sum.

Mark Scheme

Q1.

Question Number	Scheme	Notes	Marks
(a)	$r^3-(r-1)^3 \equiv r^3-(r^3-3r^2+3r-1)$ or $r^3-\left(r^3+\binom{3}{1}r^2(-1)+\binom{3}{2}r(-1)^2+(-1)^3\right)$ $\equiv 3r^2-3r+1$ * or $r^3-(r-1)^3 \equiv (r^2+r(r-1)+(r-1)^2)$ $\equiv 3r^2-3r+1$ *	Shows a correct expansion of $(r-1)^3$ or uses $a^3-b^3=(a-b)(a^2+ab+b^2)$ and achieves the printed answer with no errors.	B1*
			(1)
(b)	$n^3-(n-1)^3$ $(n-1)^3-(n-2)^3$ $(n-2)^3-(n-3)^3$ … … 3^3-2^3 2^3-1^3 1^3-0^3	Uses the method of differences. Must include at least $r=1,2,\ldots,n$ or $r=1,\ldots,(n-1),\ n$. But may implied by sight of $\sum r^3-(r-1)^3=n^3$ if insufficient terms shown. If method is clearly other than differences (see note below), then score M0.The final A mark can be witheld if differences not shown i.e. just writes down n^3.	M1
	$n^3=\sum_{r=1}^{n}\left(3r^2-3r+1\right)=\sum_{r=1}^{n}3r^2-\sum_{r=1}^{n}3r+\sum_{r=1}^{n}1$ Sets $n^3=\sum_{r=1}^{n}\left(3r^2-3r+1\right)$ and attempts to expand RHS		M1
	$\sum_{r=1}^{n}1=n$	$\sum_{r=1}^{n}1=n$ seen or implied	B1
	$3\sum_{r=1}^{n}r^2=n(n-1)(n+1)+\frac{3}{2}n(n+1)$	Rearranges to make $k\sum_{r=1}^{n}r^2$ the subject and substitutes for $\sum_{r=1}^{n}r$. **Dependent on the first method mark.**	dM1
	$\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1)$ **	Completely correct solution with no errors seen.	A1*
	Allow e.g. $\frac{2n^3+3n^2+n}{6}=\frac{1}{6}n(n+1)(2n+1)$		
			(5)
	Note: May be seen in (b): $\sum_{r=1}^{n}r^3-(r-1)^3=\frac{1}{4}n^2(n+1)^2-\frac{1}{4}n^2(n-1)^2=n^3$ etc. Scores a maximum M0M1B1dM0A0 (not using differences)		
	Generally, there are no marks for proof by induction		
			Total 6

Q2.

Question Number	Scheme	Notes	Marks
(a)	$\dfrac{4r+2}{r(r+1)(r+2)}$		
	$\dfrac{1}{r}+\dfrac{2}{(r+1)}-\dfrac{3}{(r+2)}$	M1: Correct partial fractions method e.g. substitution or compares coefficients to obtain one of A, B or C for $\dfrac{A}{r}, \dfrac{B}{(r+1)}, \dfrac{C}{(r+2)}$	M1A1 A1
		A1: 2 Correct fractions (or values) A1: All correct (fractions or values)	
	Correct answer with no working scores full marks in (a)		
			(3)

Question Number	Scheme	Notes	Marks
(b)	**Must have partial fractions of the form $\dfrac{A}{r}, \dfrac{B}{(r+1)}, \dfrac{C}{(r+2)}$ $A, B, C \neq 0$ to score the <u>first</u> M mark in (b)**		
	$\sum_{r=1}^{n} = \left(\dfrac{1}{1}+\dfrac{2}{2}-\dfrac{3}{3}\right)+\left(\dfrac{1}{2}+\dfrac{2}{3}-\dfrac{3}{4}\right)+\ldots$ $\ldots+\left(\dfrac{1}{n-1}+\dfrac{2}{n}-\dfrac{3}{n+1}\right)+\left(\dfrac{1}{n}+\dfrac{2}{n+1}-\dfrac{3}{n+2}\right)$ Attempts at least the first 2 groups of terms and the last 2 groups of terms **which may be implied by their fractions identified below**. Allow other letters for n (most likely to be r) except for the final mark – see below If terms are found beyond the limits of the summation e.g. $r = 0$, $r = n + 1$, these can be ignored for this mark as long as at least the terms for $r = 1, 2, n-1$ and n are seen		M1
	$=\dfrac{1}{1}+\dfrac{2}{2}+\dfrac{1}{2}-\dfrac{3}{n+1}+\dfrac{2}{n+1}-\dfrac{3}{n+2}$	A1: $\dfrac{1}{1}+\dfrac{2}{2}+\dfrac{1}{2}\left(=\dfrac{5}{2}\right)$ identified as the only constant terms	A1 A1
		A1: $-\dfrac{3}{n+1}+\dfrac{2}{n+1}-\dfrac{3}{n+2}$ oe e.g $-\dfrac{1}{n+1}-\dfrac{1}{n+2}-\dfrac{2}{n+2}$ identified as the only algebraic terms	
	$=\dfrac{5(n^2+3n+2)-2(n+2)-6(n+1)}{2(n+1)(n+2)}$	Attempt common denominator from terms of the form $A, \dfrac{B}{n+1}, \dfrac{C}{n+2}$ only. Must see $(n + 1)(n + 2)$ in the denominator and an unsimplified polynomial of order 2 in the numerator.	M1
	$\dfrac{n(5n+7)}{2(n+1)(n+2)}$	Must be in terms of n for this mark.	A1
			(5)
			Total 8

	Alternative for (b)		
	$\frac{1}{r}+\frac{2}{(r+1)}-\frac{3}{(r+2)}=\left(\frac{1}{r}-\frac{1}{r+2}\right)+2\left(\frac{1}{r+1}-\frac{1}{r+2}\right)$ $\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+2}\right)=\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\ldots+\frac{1}{n-1}-\frac{1}{n+1}+\frac{1}{n}-\frac{1}{n+2}=1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$ $2\sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)=\frac{1}{2}-\frac{1}{3}+\ldots+\frac{1}{n}-\frac{1}{n+2}=\frac{1}{2}-\frac{1}{n+2}$ Re-writes their partial fractions correctly and attempts at least 2 groups of terms at start and end for first sum **and** 1 group at the start and end for the second sum		M1
	$\sum_{r=1}^{n}=\frac{5}{2}-\frac{1}{n+1}-\frac{3}{n+2}$	A1: $\frac{1}{1}+\frac{2}{2}+\frac{1}{2}\left(=\frac{5}{2}\right)$ identified as the only constant terms A1: A1: $-\frac{3}{n+1}+\frac{2}{n+1}-\frac{3}{n+2}$ oe e.g $-\frac{1}{n+1}-\frac{1}{n+2}-\frac{2}{n+2}$ identified as the only algebraic terms	A1A1
	$=\frac{5\left(n^2+3n+2\right)-2(n+2)-6(n+1)}{2(n+1)(n+2)}$	Attempt common denominator from terms of the form $A, \frac{B}{n+1}, \frac{C}{n+2}$ only. Must see $(n+1)(n+2)$ in the denominator and an unsimplified polynomial of order 2 in the numerator.	M1
	$\frac{n(5n+7)}{2(n+1)(n+2)}$	Must be in terms of n for this mark.	A1

Q3.

Question Number	Scheme	Notes	Marks
(a)	$\frac{2}{(r-1)(r+1)} \equiv \frac{A}{(r-1)} + \frac{B}{(r+1)}$		
	$\frac{2}{(r-1)(r+1)} \equiv \frac{1}{(r-1)} - \frac{1}{(r+1)}$	Oe e.g. allow $\frac{1}{(r-1)} + \frac{-1}{(r+1)}$ **Must be in terms of r.**	B1
	Do not allow this mark for just finding their constants e.g. $A = 1, B = -1$ but allow this mark to be recovered in (b) if the correct partial fractions are seen or used.		
			(1)

(b)	**To score in (b) they must be using partial fractions of the form $\frac{A}{(r-1)} + \frac{B}{(r+1)}$**		
	$\sum_{r=2}^{n} = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \ldots\ldots\ldots\ldots + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right)$ Attempts at least the first 2 groups of terms and the last 2 groups of terms **which may be implied by their non-cancelling fractions identified below** Allow other letters for n (most likely to be r) except for the final mark – see below If terms are found beyond the limits of the summation e.g. $r = 0$, $r = 1$, these can be ignored for this mark as long as at least the terms for $r = 2$, 3, $n - 1$ and n are seen		M1
	$\sum_{r=2}^{n} \frac{2}{r^2 - 1} = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{(n+1)}$	$1, \frac{1}{2}\left(\text{or} \frac{3}{2}\right)$ identified as the only constant term(s). Follow through their partial fractions so allow $\frac{\textit{their } A}{1}, \frac{\textit{their } A}{2}$	A1ft **M1 on ePEN**
		$-\frac{1}{n}, -\frac{1}{(n+1)}$ identified as the only algebraic terms. Follow through their partial fractions so allow $\frac{\textit{their } B}{n}, \frac{\textit{their } B}{n+1}$	A1ft
	$= \frac{3n(n+1) - 2(n+1) - 2n}{2n(n+1)}$	Attempts common denominator from terms of the form $A, \frac{B}{n}, \frac{C}{n+1}$ only. Must see $n(n + 1)$ in the denominator and an unsimplified quadratic expression in the numerator. **Dependent on the first method mark.**	dM1
	$= \frac{3n^2 - n - 2}{2n(n+1)} = \frac{(3n+2)(n-1)}{2n(n+1)}$ *	Cso. No errors seen but see note below.	A1cso
	Note: If extra terms were considered for the first M mark, (usually those for r = 1), allow a full recovery if the correct constant terms are 'extracted'. Some candidates attempt $\sum_{r=1}^{n} \frac{2}{r^2 - 1} - \sum_{r=1}^{1} \frac{2}{r^2 - 1}$ and the same ruling applies as the term for $r = 1$ effectively cancels out, leaving the correct non-cancelling terms.		
			(5)

(c)	$$S_{3n}=\sum_{r=2}^{3n}\frac{2}{(r-1)(r+1)}=\frac{(3\times 3n+2)(3n-1)}{2\times 3n(3n+1)}$$ Correct, possibly unsimplified, expression for S_{3n} using the **given** result in (b)		B1
	$$S_{3n}-S_{n-1}=\frac{(3\times 3n+2)(3n-1)}{2\times 3n(3n+1)}-\frac{(3(n-1)+2)(n-2)}{2(n-1)n}$$ Attempts $S_{3n}-S_{n-1}$ using the **given** result in (b) If there is any doubt about the "S_{n-1}", at least 3 of the n's should be replaced by n - 1		M1
	$$=\frac{(9n+2)(3n-1)(n-1)-3(3n+1)(3n-1)(n-2)}{6n(3n+1)(n-1)}$$ $$=\frac{(3n-1)(9n^2-7n-2-3(3n^2-5n-2))}{6n(3n+1)(n-1)}$$ Attempts common denominator involving n, $3n+1$ and $n-1$ and attempts a factor of $3n-1$ in the numerator or vice versa. Note that the numerator may be expanded completely to give $24n^2+4n-4$ and the $3n-1$ then attempted as a factor. **Dependent on the previous mark.**		dM1
	$$=\frac{2(3n-1)(2n+1)}{3n(3n+1)(n-1)}$$	Cao	A1
			(4)
	If (c) is attempted using the method of differences (i.e. repeating the work in (b)) then this scores 0/4 in part (c).		
			Total 10

Q4.

Question Number	Scheme	Notes	Marks
	$\dfrac{1}{(r+6)(r+8)}$		
(a)	$\dfrac{1}{2(r+6)}-\dfrac{1}{2(r+8)}$ oe	Correct partial fractions, any equivalent form	B1
			(1)
(b)	$=\left(2\times\dfrac{1}{2}\right)\left(\dfrac{1}{7}-\dfrac{1}{9}+\dfrac{1}{8}-\dfrac{1}{10}+\dfrac{1}{9}-\dfrac{1}{11}\ldots+\dfrac{1}{n+5}-\dfrac{1}{n+7}+\dfrac{1}{n+6}-\dfrac{1}{n+8}\right)$ Expands at least 3 terms at start and 2 at end (may be implied) The partial fractions obtained in (a) can be used without multiplying by 2. Fractions may be $\dfrac{1}{2}\times\dfrac{1}{7}-\dfrac{1}{2}\times\dfrac{1}{9}$ etc These comments apply to both M1 and A1		M1
	$=\dfrac{1}{7}+\dfrac{1}{8}-\dfrac{1}{n+7}-\dfrac{1}{n+8}$	Identifies the terms that do not cancel	A1
	$=\dfrac{15(n+7)(n+8)-56(2n+15)}{56(n+7)(n+8)}$	Attempt common denominator Must have multiplied the fractions from (a) by 2 now	M1
	$=\dfrac{n(15n+113)}{56(n+7)(n+8)}$		A1cso
			(4)
			Total 5

Q5.

Question Number	Scheme		Marks
(a)	$\frac{1}{2(r+1)(r+2)}-\frac{1}{2(r+2)(r+3)}$ $=\frac{r+3-(r+1)}{2(r+1)(r+2)(r+3)}$	Attempt common denominator	M1
	$=\frac{2}{2(r+1)(r+2)(r+3)}$ $=\frac{1}{(r+1)(r+2)(r+3)}$	Correct proof	A1
			(2)
(a) Way 2	$\frac{1}{(r+1)(r+2)(r+3)}=\frac{1}{r+2}\left(\frac{1}{(r+1)(r+3)}\right)=\frac{1}{r+2}\left(\frac{1}{2(r+1)}-\frac{1}{2(r+3)}\right)$		M1
	$\frac{1}{2(r+1)(r+2)}-\frac{1}{2(r+2)(r+3)}$		A1
	M1: Factor of $\frac{1}{r+2}$ and attempt partial fractions A1: Correct proof		
	Other methods: **Complete** method scores M1 All work correct inc final answer reached A1		
(b)	$\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)(r+3)}=$ $=\frac{1}{12}-\frac{1}{24}+\ldots\ldots$ $+\frac{1}{2(n+1)(n+2)}-\frac{1}{2(n+2)(n+3)}$	Attempt at least the first pair and the last pair of terms as shown. Must start at 1 and end at n	M1
	$=\frac{1}{12}-\frac{1}{2(n+2)(n+3)}$	Identifies that the first and last terms do not cancel.	M1
	$=\frac{n^2+5n+6-6}{12(n+2)(n+3)}$	Correctly combined fractions	A1
	$=\frac{n^2+5n}{12(n+2)(n+3)}$ or $\frac{n(n+5)}{12(n+2)(n+3)}$ *	Allow either form isw attempts to multiply out the denominatot	A1
			(4)
			Total 6

Q6.

Question Number	Scheme	Notes	Marks
(a)	$\frac{1}{4r^2-1}$		
	$\frac{1}{2(2r-1)}-\frac{1}{2(2r+1)}$ or $\frac{\frac{1}{2}}{(2r-1)}-\frac{\frac{1}{2}}{(2r+1)}$ or equivalent or $\frac{1}{4r^2-1}\equiv\frac{A}{2r-1}+\frac{B}{2r+1}\Rightarrow A=\frac{1}{2}, B=-\frac{1}{2}$	Correct partial fractions or correct values of 'A' and 'B'. Isw if possible so if correct values of 'A' and 'B' are found, award when seen even if followed by incorrect partial fractions.	B1
			(1)
(b)	$\sum_{r=1}^{n}\frac{1}{4r^2-1}=\frac{1}{2}\left(1-\frac{1}{3}+......+\frac{1}{2n-1}-\frac{1}{2n+1}\right)$ Attempt at least first and last terms using their partial fractions. May be implied by e.g. $\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$		M1
	$\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$ or $\frac{1}{2}-\frac{1}{2(2n+1)}$ or $\frac{1}{2}-\frac{1}{4n+2}$	Correct expression	A1
	$\frac{n}{2n+1}$*	Correct completion with no errors	A1*
	Allow a different variable to be used in (a) and (b) but final answer in (b) must be as printed i.e. in terms of n.		
			(3)
(c)	$\sum_{r=9}^{25}\frac{5}{4r^2-1}=(5)(\text{f}(25)-\text{f}(8))$	$\text{f}(25)-\text{f}(8)$ where $\text{f}(n)=\frac{n}{2n+1}$	M1
	$=5\left(\frac{25}{51}-\frac{8}{17}\right)=\frac{5}{51}$	cao	A1
	Correct answer with no working in (c) scores both marks.		
			(2)
			Total 6