# Questions

**Q1.**

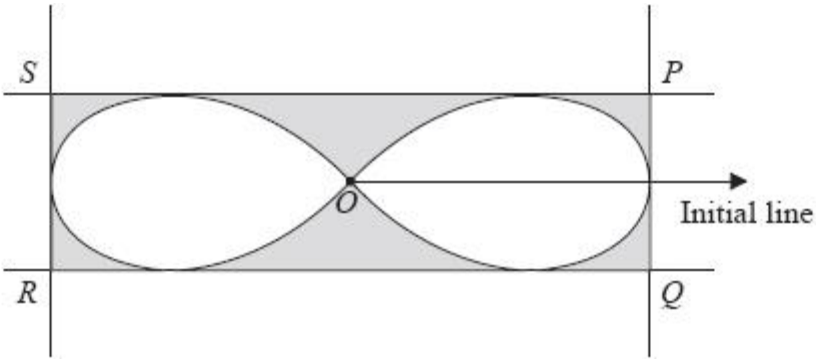



**Figure 1**

Figure 1 shows a sketch of the curve *C* with polar equation

$$r = 4\cos 2\theta, \quad -\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4} \quad \text{and} \quad \frac{3\pi}{4} \leqslant \theta \leqslant \frac{5\pi}{4}$$

The lines *PQ*, *QR*, *RS* and *SP* are tangents to *C*, where *QR* and *SP* are parallel to the initial line and *PQ* and *RS* are perpendicular to the initial line.

(a) Find the polar coordinates of the points where the tangent *SP* touches the curve. Give the values of $\theta$ to 3 significant figures.

**(5)**

(b) Find the exact area of the finite region bounded by the curve *C*, shown unshaded in Figure 1.

**(5)**

(c) Find the area enclosed by the rectangle *PQRS* but outside the curve *C*, shown shaded in Figure 1.

**(5)**

**(Total for question = 15 marks)**

**Q2.**

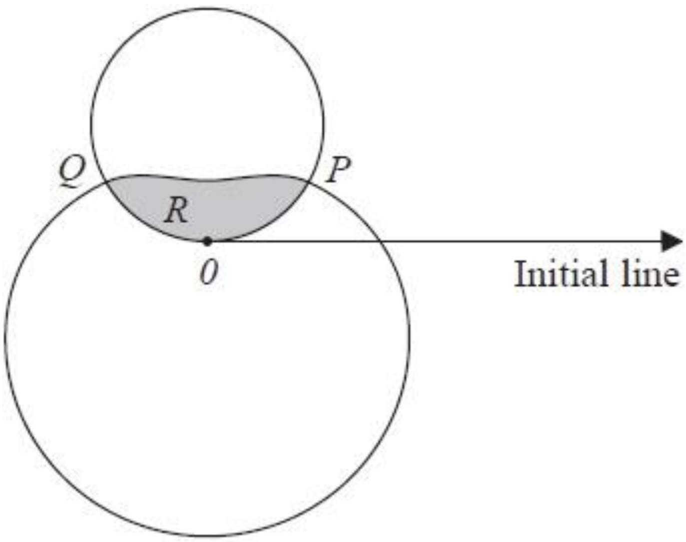



**Figure 1**

Figure 1 shows a sketch of the curves with polar equations

$$r = 2\sin\theta \qquad 0 \leq \theta \leq \pi$$

$$r = 1.5 - \sin\theta \qquad 0 \leq \theta \leq 2\pi$$

The curves intersect at the points $P$ and $Q$.

(a) Find the polar coordinates of the point $P$ and the polar coordinates of the point $Q$.

**(3)**

The region $R$, shown shaded in Figure 1, is enclosed by the two curves.

(b) Find the exact area of $R$, giving your answer in the form $p\pi + q\sqrt{3}$, where $p$ and $q$ are rational numbers to be found.

**(8)**

**(Total for question = 11 marks)**

**Q3.**

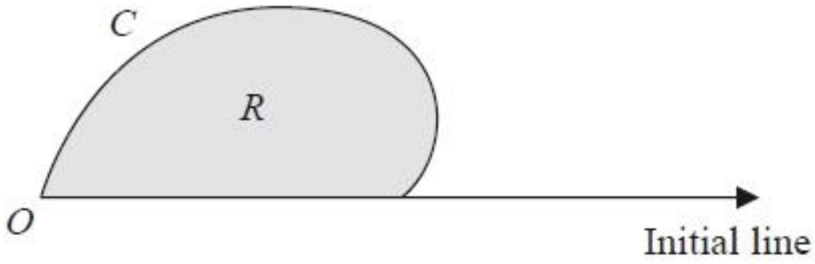



**Figure 1**

The curve *C* shown in Figure 1 has polar equation

$$r = \sin\theta + \cos 2\theta \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

At the point *P* on *C* the tangent to *C* is parallel to the initial line.

Given that *O* is the pole,

(a) find the length of the line of *OP*, giving your answer to 3 significant figures.

**(6)**

The region *R*, shown shaded in Figure 1, is bounded by the curve *C* and the initial line.

(b) Use calculus to find the exact area of *R*.

**(6)**

**(Total for question = 12 marks)**

**Q4.**

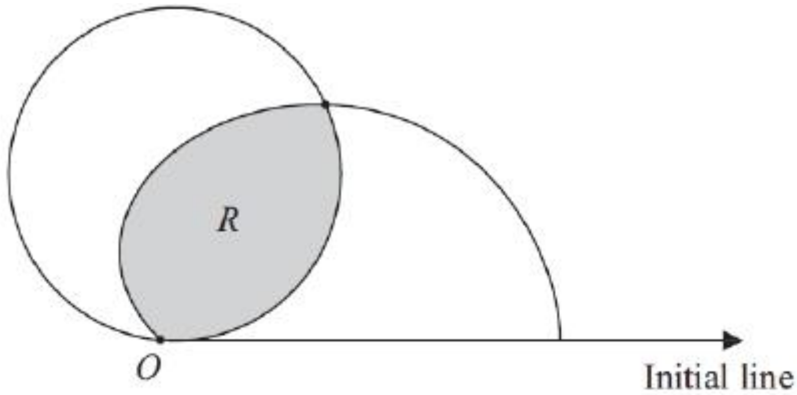



**Figure 1**

Figure 1 shows the 2 curves given by the polar equations

$$r = \sqrt{3}\sin\theta, \qquad 0 \leq \theta \leq \pi$$

$$r = 1 + \cos\theta \qquad 0 \leq \theta \leq \pi$$

(a) Verify that the curves intersect at the point $P$ with polar coordinates $\left(\frac{3}{2}, \frac{\pi}{3}\right)$

**(2)**

The region $R$, bounded by the two curves, is shown shaded in Figure 1.

(b) Use calculus to find the exact area of $R$ giving your answer in the form $a(\pi - \sqrt{3})$, where $a$ is a constant to be found.

**(6)**

**(Total for question = 8 marks)**

**Q5.**

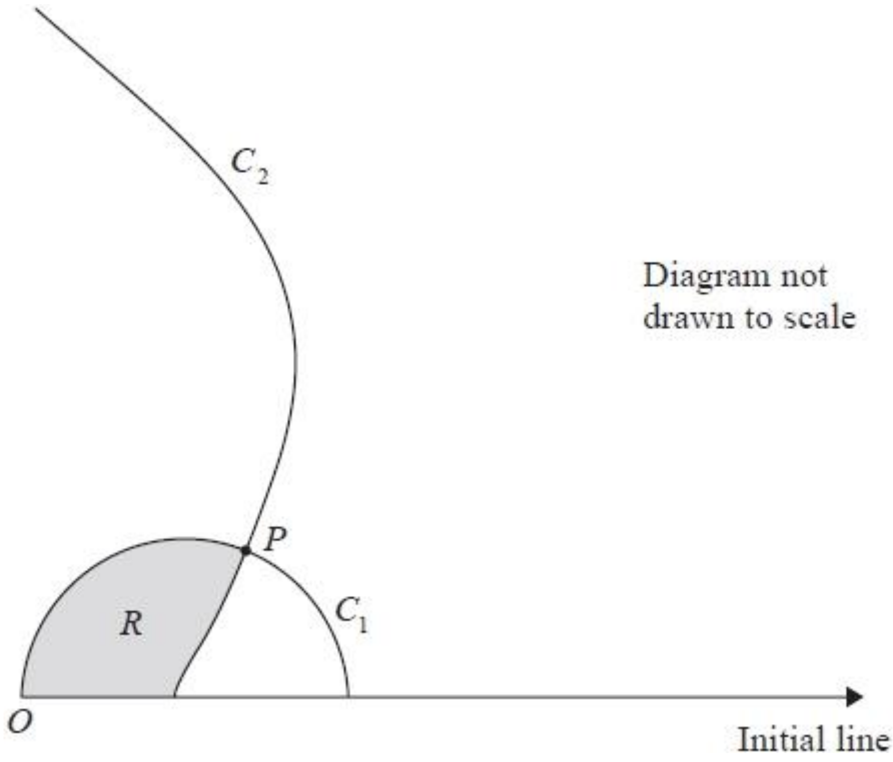



**Figure 1**

Figure 1 shows a sketch of the curves $C_1$ and $C_2$ with polar equations

$$C_1 : r = \frac{3}{2}\cos\theta, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

$$C_2 : r = 3\sqrt{3} - \frac{9}{2}\cos\theta, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

The curves intersect at the point $P$.

(a) Find the polar coordinates of $P$.

**(3)**

The region $R$, shown shaded in Figure 1, is enclosed by the curves $C_1$ and $C_2$ and the initial line.

(b) Find the exact area of $R$, giving your answer in the form $p\pi + q\sqrt{3}$ where $p$ and $q$ are rational numbers to be found.

**(8)**

**(Total for question = 11 marks)**

**Q6.**

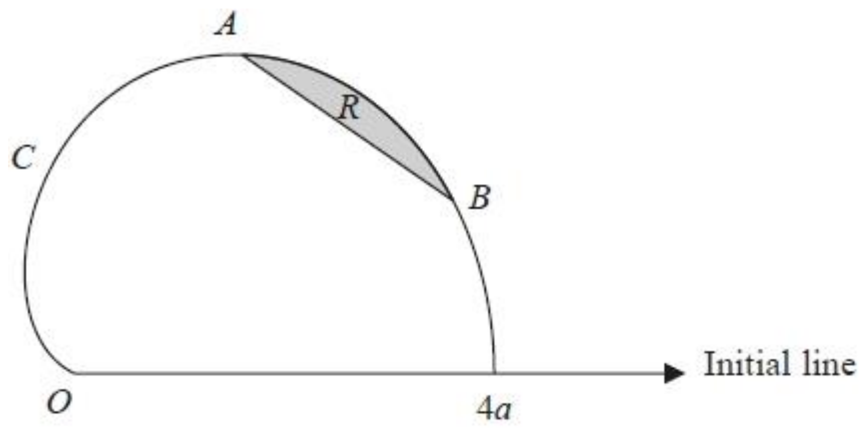



**Figure 1**

The curve $C$, shown in Figure 1, has polar equation

$$r = 2a\,(1 + \cos\theta) \qquad 0 \leq \theta \leq \pi$$

where $a$ is a positive constant.

The tangent to $C$ at the point $A$ is parallel to the initial line.

(a) Determine the polar coordinates of $A$.

**(6)**

The point $B$ on the curve has polar coordinates $\left(a\left(2+\sqrt{3}\right), \dfrac{\pi}{6}\right)$

The finite region $R$, shown shaded in Figure 1, is bounded by the curve $C$ and the line $AB$.

(b) Use calculus to determine the exact area of the shaded region $R$.

Give your answer in the form

$$\frac{a^2}{4}\left(d\pi - e + f\sqrt{3}\right)$$

where $d$, $e$ and $f$ are integers.

**(7)**

**(Total for question = 13 marks)**

## Examiner's Report

Q1.

This question on polar coordinates proved to be quite demanding and fully correct solutions to all parts were not widely seen. In part (a), most knew the initial step of using $r\sin\theta$ and correct differentiation usually followed. Solving $\frac{dy}{d\theta} = 0$ proved challenging to many. Those who had replaced $\cos 2\theta$ with $1 - 2\sin^2\theta$ before differentiation tended to have more success. A variety of approaches were seen to the trigonometric equation but obtaining one of the correct values for $\theta$ was elusive. It was unfortunate to see $2\pi - \theta$ rather than $\pi - \theta$ used for the second value. A common error was to neglect to find the corresponding value for $r$.
Part (b) was a reasonable source of marks for most students and almost all knew that integration of $k \int (4\cos 2\theta)^2$ was required. Almost all were able to write the integrand in terms of $\cos 4\theta$ and integrate correctly. The last method mark was more difficult and inappropriate limits and/or wrong multipliers were widely seen, resulting in various incorrect multiples of $\pi$. Some students thought that the values of $\theta$ from part (a) were required as limits. Students who explicitly showed their method to obtain the area bounded by two loops of the graph were more successful. Many were unaware of how the use of limits outside the range of $\theta$ for which the graph was defined risked incorporating extra loops into their calculation.
Part (c) continued to challenge, although the mark scheme was designed to reward all students who used an appropriate method. A few variations were possible as in part (b) although finding the area of the entire rectangle *PQRS* directly was the usual route. Some students needlessly embarked upon solving $\frac{d}{dx}(r\cos\theta) = 0$. Many were able to write down the length of the rectangle but the width was often incorrect, with values of $2r$ rather than $2r\sin\theta$ a common misconception. Those who obtained a value for the length and width invariably scored the two method marks for an acceptable attempt at the shaded area, but a correct final answer was not common.

**Q2.**

This question on finding an area enclosed by two curves given in polar form was predictably discriminating.

Part (a) required students to find the polar coordinates of the points of intersection of the curves. Most students were able to find a correct value for $\theta$, although a second value was sometimes incorrect or missing. Further values of $\theta$ were occasionally seen. The correct value of $r = 1$ was widely seen although a small number of students gave an additional value of $r$, usually –1, for the point $Q$.

Part (b) asked for the exact area enclosed between the circle and limacon. Those who considered the situation carefully at the start, often by annotating the diagram or drawing their own, were more successful in choosing a correct strategy to find the area of the region enclosed by the curves. The correct formula of $\frac{1}{2}\int r^2 \, d\theta$ was widely used although the $\frac{1}{2}$ was sometimes missing. Use of $\int r \, d\theta$ was very rare. Many were able to score 3 or 4 of the first four marks by obtaining an appropriate integrated expression for the area bounded by the limacon and the straight lines of the segments. Errors included poor squaring, occasional sign errors with the identity for $\sin^2\theta$ and sign and other errors made when integrating the trigonometric terms. Most then used their angles from part (a) as limits, although some chose to double the result of using their smaller angle with an upper limit of $\frac{\pi}{2}$. A wide variety of incorrect limits were seen. The next mark required a correct method to find the area of at least one segment. Most had the correct expression after integration but incorrect limits were again common. Use of the segment area formula $12r^2(\theta - \sin\theta)$ was quite rare. The final method mark required a fully correct method and many attempts did not combine the found areas appropriately. A few students used fully correct limits throughout but mixed up the curves. A very small number attempted $\int(\text{circle} - \text{limacon})^2 \, d\theta$. The final A mark was demanding but it was pleasing to see a good number of students reaching the correct exact value.

**Q3.**

This question on polar coordinates was a good source of marks in part (a) but proved more discriminating in (b).

Part (a) required finding the length of *OP* where point *P* was the point of contact of a tangent parallel to the initial line. Most realised they needed to solve $\frac{\mathrm{d}}{\mathrm{d}\theta}(r\sin\theta) = 0$ although a small number thought solving $\frac{\mathrm{d}}{\mathrm{d}\theta}(r\cos\theta) = 0$ was required. Most obtained and differentiated $r \sin \theta$ correctly but it proved more difficult to obtain the correct three term quadratic in $\sin \theta$. Those who obtained the correct three term quadratic generally proceeded to find the correct value of $r$, usually by finding $\theta$ first, although a few expressed $\cos 2\theta$ in terms of $\sin\theta$.

In part (b), most used the correct area formula and squared correctly but the resulting term of $2\sin\theta\cos 2\theta$ proved challenging to integrate. Most used $\cos 2\theta = 2\cos^2\theta - 1$ and then integrated $4\sin\theta\cos^2\theta$ although a small number used the sum and product formula to obtain $\sin 3\theta - \sin\theta$. The very small number attempting integration by parts, which needed two applications and rearrangement to make $2\sin\theta\cos 2\theta$ the subject, were rarely successful. Some candidates who couldn't integrate this term resorted to evaluating it on their calculator and received no further credit. Those who had integrated successfully, rarely used incorrect limits and the correct exact area was often confidently achieved.

**Q4.**

Students quite often did not appreciate the meaning of the word verify so that what was supposed to be a straightforward start to the question was made into something more complicated as students tried by a variety of means to solve an equation to find $\theta$.

The formula for the area under a polar curve was well known with very few students not being able to express the area as a correct integral although there were occasional problems in getting the limits on to the correct integral.

The method of integrating the powers of sin and cos by expressing them in terms of double angles was well understood and correctly carried out by the majority of students.

Some students did subtract rather than add their integrals but of those who added and had the correct limits on the correct integrals most were able to reach the answer in the required form.

**Q5.**

This question proved more demanding for students although there were many correct solutions.

(a) proved accessible with most students achieving full marks although a significant number of students, having found $\theta$, forgot to find r.

In (b) the formula for the area enclosed by a polar curve was well known as was the idea that a sum of two areas was required. A common error was to use $\pi$ rather than $\frac{\pi}{2}$ as the upper limit of the area bounded by the circle and some students correctly found the area of a segment for the relevant part of the area.

It was pleasing to see how many students could manage to take the quadratic function of $\cos \theta$ and change it into a function involving $\cos 2\theta$ and successfully integrate the resulting function.

The majority of the final answers did involve $\sqrt{3}$ and $\pi$ rather than using decimals. Students should be reminded that it is advisable to show how the limits are being substituted into the results of their integration rather than just produce a numerical result with no working.

**Q6.**

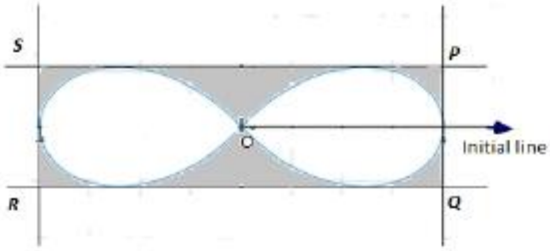
This question was a good source of marks in part (a) but the later marks in (b) were more discriminating.

Part (a) required finding the polar coordinates of the point of contact of a tangent to a polar curve, where the tangent was parallel to the initial line. Most students realised they needed to solve $\frac{\mathrm{d}}{\mathrm{d}\theta}(r\sin\theta)=0$ although a small number thought solving $\frac{\mathrm{d}}{\mathrm{d}\theta}(r\cos\theta)=0$ was required. Occasionally the required $\frac{\mathrm{d}y}{\mathrm{d}\theta}$ was found using $y=r\sin\theta\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}\theta}=\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta+r\cos\theta$. A small number of candidates unnecessarily found $\frac{\mathrm{d}x}{\mathrm{d}\theta}$. However most obtained and differentiated $r\sin\theta$ correctly using the product rule or via $\sin2\theta = 2\sin\theta\cos\theta$ and the correct three term quadratic in $\cos\theta$ was usually reached. Those who obtained the correct 3TQ generally proceeded to find the correct value of $\theta$ and invariably the correct $r$ followed. It was rare to see any candidate who did not reject the $(\pi, 0)$ solution although occasionally $\theta$ was greater than $\frac{\pi}{2}$.

In part (b), most used the correct area formula although a few lost the $\frac{1}{2}$ from $\frac{1}{2}\int r^2\mathrm{d}\theta$. The expression for $r$ was almost always squared correctly. As with the differentiation in part (a), only a few slips were seen with the required use of $\cos^2\theta = \frac{1}{2}\cos2\theta + \frac{1}{2}$ and the subsequent integration. A small number who used the trigonometric identity and also integrated in one line of working forgot to integrate the $\frac{1}{2}\cos2\theta + \frac{1}{2}$ terms. Many errors involved mishandling of the constant $a$ from the curve equation. Those who didn't move the $a^2$ to the left of the integration sign were particularly vulnerable to this. The majority knew to use the limits of their $\frac{\pi}{3}$ and $\frac{\pi}{6}$ and substitution with correct subtraction was usually seen. A small number used the limits $\frac{\pi}{3}$ and 0 followed by subtraction of the result of applying the limits $\frac{\pi}{6}$ and 0. However, the main misconception seen was that many candidates believed that they had now already found the area of the shaded region $R$. A relatively small number knew that they had to subtract the area of the triangle $OAB$ and most who attempted this found the correct expression. The $\frac{1}{2}$ from $\frac{1}{2}ab\sin C$ was occasionally missing. A few more protracted attempts at the area of this triangle were seen – such as dropping perpendiculars to the initial line from $A$ and $B$ to form a trapezium $ABQP$, finding its area and then adding the area of triangle $OAP$ and subtracting that of triangle $OBQ$. Some succumbed to algebraic errors in the last few steps but it was encouraging to see plenty of fully correct and well-organised solutions.

## Mark Scheme

Q1.

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| |  | | |
| **(a)** | $y = r\sin\theta = 4\cos 2\theta\sin\theta$ | Attempts to use $r\sin\theta$ | M1 |
| | $\frac{dy}{d\theta} = 4\cos 2\theta\cos\theta - 8\sin 2\theta\sin\theta$<br>or<br>$y = 4(1-2\sin^2\theta)\sin\theta = 4\sin\theta - 8\sin^3\theta \Rightarrow \frac{dy}{d\theta} = 4\cos\theta - 24\sin^2\theta\cos\theta$<br>A correct expression for $\frac{dy}{d\theta}$ **or any multiple of** $\frac{dy}{d\theta}$ | | B1 |
| | $\frac{dy}{d\theta} = 0 \Rightarrow \theta = ...$ | Set their $\frac{dy}{d\theta} = 0$ and attempt to solve to obtain a value for $\theta$ | M1 |
| | $r = \frac{8}{3},\ \theta = 0.421,\ \theta = 2.72$ | Any one of: $r = \frac{8}{3}$ (or awrt 2.7) or $\theta = 0.421...$ or $\theta = 2.72...$ | A1 |
| | $r = \frac{8}{3}$<br>$\theta = 0.421,\ 2.72$ | Correct value for $r$ and both angles correct. May be seen as $\left(\frac{8}{3}, 0.421\right), \left(\frac{8}{3}, 2.72\right)$. Allow $\left(0.421, \frac{8}{3}\right), \left(2.72, \frac{8}{3}\right)$ but coordinates do not have to be paired and accept awrt 0.421, 2.72 and allow awrt 2.7 for $\frac{8}{3}$. Ignore any other coordinates given once the correct values have been seen. | A1 |
| | | | **(5)** |

| | | | |
|---|---|---|---|
| **(b)** | $A = \ldots\int (4\cos 2\theta)^2\,\mathrm{d}\theta$ | Indication that the integration of $(4\cos 2\theta)^2$ is required. Ignore any limits and ignore any constant factors at this stage. | M1 |
| | $\cos^2 2\theta = \frac{1}{2}(1+\cos 4\theta)$ | A correct identity seen or implied. | A1 |
| | $A = \ldots[\alpha\theta + \beta\sin 4\theta]$ | Integrates to obtain an expression of the form $\alpha\theta + \beta\sin 4\theta$. Ignore any limits and ignore any constant factors. Dependent on the first method mark. | dM1 |
| | $= 16\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{4}}$ | A fully correct method that if evaluated correctly would give the answer $4\pi$. Note that the correct "constant factor" may only be applied at the very last stage of their working and this method mark would only be awarded at that point. **Dependent on all previous method marks.** | ddM1 |
| | **Examples that could score the final M1 (following correct work):**<br>$16\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{4}}$, $8\left[\theta + \frac{1}{4}\sin 4\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$, $8\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$, $16\left[\theta + \frac{1}{4}\sin 4\theta\right]_{\frac{3\pi}{4}}^{\pi}$ | | |
| | $= 4\pi$ | cao | A1 |
| | | | **(5)** |
| **(c)** | $PQ = 2r\sin\theta = \frac{16}{3\sqrt{6}}$ | Correct expression or value for $PQ$ or $PQ/2$.<br>E.g. $2\left(\frac{8}{3}\right)\frac{1}{\sqrt{6}}$, $2\left(\frac{8}{3}\right)\sin 0.421$,<br>$2\left(\frac{8}{3}\right)\sin 2.72$, $\frac{8\sqrt{6}}{9}$ or half of these.<br>May be implied by awrt 2.2 or awrt 1.1 | B1 |
| | $SP = 8$ or $\frac{SP}{2} = 4$ | Correct value for $SP$ or $SP/2$ | B1 |
| | Area $PQRS = \frac{16}{3\sqrt{6}} \times 8 \left(= \frac{64\sqrt{6}}{9}\right)$ | Their $PQ \times SP$. Must be the complete rectangle here. | M1 |
| | Required area $= \frac{128}{3\sqrt{6}} - 4\pi$ | M1: Their rectangle area – their answer to part (b)<br>A1: Correct exact answer or equivalent exact form e.g. $\frac{64\sqrt{6}}{9} - 4\pi$ or allow awrt 4.8 or 4.9 | M1A1 |
| | | | **(5)** |
| | | | **Total 15** |

**Q2.**

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| (a) | $2\sin\theta = 1.5 - \sin\theta \Rightarrow \theta = ...$<br>or<br>$\sin\theta = \frac{r}{2} \Rightarrow r = 1.5 - r \Rightarrow r = ...$ | Equate and attempt to solve for $\theta$<br>or<br>Eliminates $\sin\theta$ and solves for $r$ | M1 |
| | $P\left(1, \frac{\pi}{6}\right)$ | Correct coordinates. Allow the marks as soon as the correct values are seen and allow coordinates the wrong way round and allow awrt 0.524 for $\pi/6$ | A1 |
| | $Q\left(1, \frac{5\pi}{6}\right)$ | Correct coordinates. Allow the marks as soon as the correct values are seen and allow coordinates the wrong way round and allow awrt 2.62 for $5\pi/6$ | A1 |
| | | | (3) |

| | | | |
|---|---|---|---|
| (b) | $\left(\frac{1}{2}\right)\int(1.5-\sin\theta)^2 \mathrm{d}\theta$ or $\left(\frac{1}{2}\right)\int(2\sin\theta)^2 \mathrm{d}\theta$<br>Attempts to use $...\int(\sin\theta)^2 \mathrm{d}\theta$ or $...\int(1.5-\sin\theta)^2 \mathrm{d}\theta$ | | M1 |
| | $(1.5-\sin\theta)^2 = 2.25 - 3\sin\theta + \sin^2\theta = 2.25 - 3\sin\theta + \frac{(1-\cos 2\theta)}{2}$<br>Expands (allow poor squaring e.g. $(1.5-\sin\theta)^2 = 2.25 + \sin^2\theta$ **and** attempts to use $\sin^2\theta = \pm\frac{1}{2} \pm \frac{\cos 2\theta}{2}$ | | M1 |
| | $\frac{1}{2}\int(1.5-\sin\theta)^2 \mathrm{d}\theta = \frac{1}{2}\left[\frac{11}{4}\theta + 3\cos\theta - \frac{1}{4}\sin 2\theta\right]$<br>M1: Attempt to integrate and reaches an expression of the form $\alpha\theta + \beta\cos\theta + \gamma\sin 2\theta$<br>A1: Correct integration (with or without the ½) | | M1A1 |
| | $\frac{1}{2}[\ ]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = \frac{1}{2}\left\{\left(\frac{11}{4}.\frac{5\pi}{6} + 3.\cos\frac{5\pi}{6} - \frac{1}{4}\sin 2.\frac{5\pi}{6}\right) - \left(\frac{11}{4}.\frac{\pi}{6} + 3.\cos\frac{\pi}{6} - \frac{1}{4}\sin 2.\frac{\pi}{6}\right)\right\}$<br>This is a key step and must be the correct method for this part of the area e.g. uses their $\frac{\pi}{6}$ and their $\frac{5\pi}{6}$ (or twice limits of their $\frac{\pi}{6}$ and $\frac{\pi}{2}$) | | M1 |

| | | |
|---|---|---|
| | $\frac{1}{2}\int\left(2\sin\theta\right)^2\,\mathrm{d}\theta=\int\left(1-\cos2\theta\right)\mathrm{d}\theta=\left[\theta-\frac{1}{2}\sin2\theta\right]_0^{\frac{\pi}{6}}=\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)(-0)$<br>Uses the limits 0 and their $\frac{\pi}{6}$ to find at least one segment.<br>If using integration, must have integrated to obtain $p\theta+q\sin2\theta$ with correct use of limits<br>NB can be done as: $\frac{1}{2}(1)^2\left(\frac{\pi}{3}\right)-\frac{1}{2}(1)^2\sin\left(\frac{\pi}{3}\right)$ but must be correct work for their angles | M1 |
| | $\frac{11}{12}\pi-\frac{11\sqrt{3}}{8}+2\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)=\frac{5}{4}\pi-\frac{15}{8}\sqrt{3}$<br>**dd**M1: Adds their two areas to give a numerical value for the shaded area<br>Dependent on the previous 2 M marks and must be a completely correct strategy so needs to be an attempt at:<br>$\frac{1}{2}\int_{"\frac{\pi}{6}"}^{"\frac{5\pi}{6}"}\left(1.5-\sin\theta\right)^2\mathrm{d}\theta$ or $2\times\frac{1}{2}\int_{"\frac{\pi}{6}"}^{\frac{\pi}{2}}\left(1.5-\sin\theta\right)^2\mathrm{d}\theta$<br>$+$<br>$2\times\frac{1}{2}\int_0^{"\frac{\pi}{6}"}\left(2\sin\theta\right)^2\mathrm{d}\theta$ or $\left(\frac{1}{2}\int_0^{"\frac{\pi}{6}"}\left(2\sin\theta\right)^2\mathrm{d}\theta+\frac{1}{2}\int_{"\frac{5\pi}{6}"}^{\pi}\left(2\sin\theta\right)^2\mathrm{d}\theta\right)$<br>A1: Correct answer (allow equivalent fractions) | ddM1A1 |
| | | **(8)** |
| | | **Total 11** |

| | | |
|---|---|---|
| | Note that attempts to use $\left(\frac{1}{2}\right)\int\left(C_1-C_2\right)^2\mathrm{d}\theta$ e.g. $\left(\frac{1}{2}\right)\int\left(2\sin\theta-\left(1.5-\sin\theta\right)\right)^2\mathrm{d}\theta$<br>Will probably only score a maximum of the first 3 marks<br>i.e.<br>M1 for $\left(\frac{1}{2}\right)\int\left(2\sin\theta-\left(1.5-\sin\theta\right)\right)^2\mathrm{d}\theta$<br>M1 for expanding **and** attempting to use $\sin^2\theta=\pm\frac{1}{2}\pm\frac{\cos2\theta}{2}$<br>M1 for attempting to integrate and reaching an expression of the form $\alpha\theta+\beta\cos\theta+\gamma\sin2\theta$ | M1 |

**Q3.**

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| | $r = \sin\theta + \cos 2\theta$ | | |
| **(a)** | $y = r\sin\theta = \sin^2\theta + \sin\theta\cos 2\theta$<br>or e.g.<br>$y = \sin\theta(\sin\theta + \cos 2\theta)$ | Correct expression | B1 |
| | $\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = 2\sin\theta\cos\theta + \cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta$<br>M1: **Correct** use of Chain and Product Rules (allow sign errors only) **on a correct expression**<br>A1: Correct differentiation | | M1A1 |
| | $6\sin^2\theta - 2\sin\theta - 1 = 0$ | Correct quadratic | A1 |
| | $\sin\theta = \dfrac{2 \pm \sqrt{28}}{12} \Rightarrow$ at $P$, $\sin\theta = \dfrac{2+\sqrt{28}}{12} \Rightarrow \theta = \sin^{-1}\dfrac{2+\sqrt{28}}{12} = 0.653067...$<br>$r = \sin\theta + \cos 2\theta = \dfrac{2+\sqrt{28}}{12} + \cos(2\times\text{"}0.653...\text{"})$<br>A complete method to find $OP$. Solves their 3TQ in $\sin\theta$, proceeds to obtain a value for $\theta$ and uses the given expression for $r$ to find $OP$.<br>**or**<br>$\sin\theta = \dfrac{2 \pm \sqrt{28}}{12} \Rightarrow$ at $P$, $\sin\theta = \dfrac{2+\sqrt{28}}{12} \Rightarrow \cos 2\theta = 1 - 2\left(\dfrac{2+\sqrt{28}}{12}\right)^2$<br>$r = \sin\theta + \cos 2\theta = \dfrac{1+\sqrt{7}}{6} + 1 - 2\left(\dfrac{1+\sqrt{7}}{6}\right)^2$<br>A complete method to find $OP$. Solves their 3TQ in $\sin\theta$, proceeds to obtain a value for $\cos 2\theta$ using a correct identity and adds their $\sin\theta$ value to find $OP$.<br>**Dependent on the first method mark.** | | dM1 |
| | $OP = r = 0.8692...$ | awrt 0.869 | A1 |
| | | | **(6)** |

| | | | |
|---|---|---|---|
| | $\left(\sin\theta+\cos 2\theta\right)^2=\sin^2\theta+2\sin\theta\cos 2\theta+\cos^2 2\theta$<br>Attempt to find $r^2$.<br>Allow poor squaring as long as there is the intention to square the bracket. | | M1 |
| | $\int \sin^2\theta \, d\theta=\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta$<br>$\int \cos^2 2\theta \, d\theta=\frac{1}{2}\theta+\frac{1}{8}\sin 4\theta$ | Attempts to integrate $\sin^2\theta$ to obtain $\alpha\theta+\beta\sin 2\theta$ and attempts to integrate $\cos^2 2\theta$ to obtain $\alpha\theta+\beta\sin 4\theta$. This may be implied by an expression of the form $\alpha\theta+\beta\sin 2\theta+\gamma\sin 4\theta$<br>**Dependent on the first method mark.** | dM1 |
| | $\int 2\sin\theta\cos 2\theta \, d\theta=\int 2\sin\theta\left(2\cos^2\theta-1\right) d\theta$<br>$=\int\left(4\sin\theta\cos^2\theta-2\sin\theta\right) d\theta=-\frac{4}{3}\cos^3\theta+2\cos\theta$<br>or<br>$\int 2\sin\theta\cos 2\theta \, d\theta=\int\left(\sin 3\theta-\sin\theta\right) d\theta=-\frac{1}{3}\cos 3\theta+\cos\theta$<br>Fully correct strategy for integrating $(2)\sin\theta\cos 2\theta$<br>**Dependent on the first method mark.** | | dM1 |
| | $\int r^2 d\theta=\frac{1}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)+\cos\theta-\frac{1}{3}\cos 3\theta+\frac{1}{2}\left(\theta+\frac{1}{4}\sin 4\theta\right)$<br>or<br>$\int r^2 d\theta=\frac{1}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)-\frac{4}{3}\cos^3\theta+2\cos\theta+\frac{1}{2}\left(\theta+\frac{1}{4}\sin 4\theta\right)$<br>Fully correct integration | | A1 |
| | $\frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta=\ldots$ | Fully correct method using a correct formula and evidence of use of the limits $\frac{\pi}{2}$ and 0 with subtraction. **Dependent on the <u>first</u> and <u>at least one of the</u> <u>subsequent</u> method marks.** | ddM1 |
| | $=\frac{\pi}{4}-\frac{1}{3}$ | Correct exact area.<br>Allow equivalent exact expressions e.g. $\frac{1}{2}\left(\frac{\pi}{2}-\frac{2}{3}\right)$ | A1 |
| | | | **(6)** |
| | | | **Total 12** |

**Q4.**

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| (a) | $\theta=\frac{\pi}{3}\Rightarrow r=\sqrt{3}\sin\left(\frac{\pi}{3}\right)=\frac{3}{2}$ | Attempt to verify coordinates in at least one of the polar equations | M1 |
| | $\theta=\frac{\pi}{3}\Rightarrow r=1+\cos\left(\frac{\pi}{3}\right)=\frac{3}{2}$ | Coordinates verified in both curves (Coordinate brackets not needed) | A1 |
| | | | **(2)** |
| | **Alternative:** | | |
| | Equate $r$s: $\sqrt{3}\sin\theta=1+\cos\theta$ and verify (by substitution) that $\theta=\frac{\pi}{3}$ is a solution<br>or solve by using $t=\tan\frac{\theta}{2}$<br>or writing $\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}\cos\theta=\frac{1}{2}$ $\quad\sin\left(\theta-\frac{\pi}{6}\right)=\frac{1}{2}\quad\theta=\frac{\pi}{3}$<br>Squaring the original equation allowed as $\theta$ is known to be between 0 and $\pi$ | | M1 |
| | Use $\theta=\frac{\pi}{3}$ in either equation to obtain $r=\frac{3}{2}$ | | A1 |
| | | | |
| (b) | $\frac{1}{2}\int(\sqrt{3}\sin\theta)^2\,\mathrm{d}\theta,\ \ \frac{1}{2}\int(1+\cos\theta)^2\,\mathrm{d}\theta$ | Correct formula used on at least one curve (1/2 may appear later)<br>Integrals may be separate or added or subtracted. | M1 |
| | $=\frac{1}{2}\int 3\sin^2\theta\,\mathrm{d}\theta,\quad\frac{1}{2}\int(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$ | | |
| | $=\left(\frac{1}{2}\right)\int\frac{3}{2}(1-\cos2\theta)\,\mathrm{d}\theta,\ \ \left(\frac{1}{2}\right)\int(1+2\cos\theta+\frac{1}{2}(1+\cos2\theta))\,\mathrm{d}\theta$<br>Attempt to use $\sin^2\theta$ or $\cos^2\theta=\pm\frac{1}{2}\pm\frac{1}{2}\cos2\theta$ on either integral<br>**Not dependent** 1/2 may be missing | | M1 |
| | $=\frac{3}{4}\left[\theta-\frac{1}{2}\sin2\theta\right]_{(0)}^{\left(\frac{\pi}{3}\right)},\quad\frac{1}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin2\theta\right]_{\left(\frac{\pi}{3}\right)}^{(\pi)}$<br>Correct integration (ignore limits) A1A1 or A1A0 | | A1, A1 |
| | $R=\frac{3}{4}\left[\frac{\pi}{3}-\frac{\sqrt{3}}{4}(-0)\right]+\frac{1}{2}\left[\frac{3\pi}{2}-\left(\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right)\right]$ | Correct use of limits for both integrals<br>Integrals must be added.<br>Dep on both previous M marks | ddM1 |
| | $=\frac{3}{4}\left(\pi-\sqrt{3}\right)$ | Cao<br>No equivalents allowed | A1 |
| | | | **(6)** |
| | | | **Total 8** |

**Q5.**

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| | $C_1: \ r=\frac{3}{2}\cos\theta, \quad C_2: \ r=3\sqrt{3}-\frac{9}{2}\cos\theta$ | | |
| (a) | $\frac{3}{2}\cos\theta=3\sqrt{3}-\frac{9}{2}\cos\theta \Rightarrow \theta=...$<br>or<br>$\cos\theta=\frac{2r}{3}\Rightarrow r=3\sqrt{3}-3r\Rightarrow r=...$ | Puts $C_1=C_2$ and attempt to solve for $\theta$<br>or<br>Eliminates $\cos\theta$ and solves for $r$ | M1 |
| | $\theta=\frac{\pi}{6}$ or $r=\frac{3\sqrt{3}}{4}$ | Correct $\theta$ or correct $r$.<br>Allow $\theta=$ awrt 0.524, $r=$ awrt 1.3 | A1 |
| | $r=\frac{3\sqrt{3}}{4}$ and $\theta=\frac{\pi}{6}$ | Correct $r$ and $\theta$ (isw e.g. $\left(\frac{\pi}{6},\frac{3\sqrt{3}}{4}\right)$)<br>Allow $\theta=$ awrt 0.524, $r=$ awrt 1.3 | A1 |
| | | | **(3)** |

| | | |
|---|---|---|
| (b) | $\frac{1}{2}\int\left(3\sqrt{3}-\frac{9}{2}\cos\theta\right)^2 \mathrm{d}\theta$ or $\frac{1}{2}\int\left(\frac{3}{2}\cos\theta\right)^2 \mathrm{d}\theta$ | M1 |
| | Attempts to use correct formula on either curve. The ½ may be implied by later work. | |
| | $\left(3\sqrt{3}-\frac{9}{2}\cos\theta\right)^2=27-27\sqrt{3}\cos\theta+\frac{81}{4}\cos^2\theta=27-27\sqrt{3}\cos\theta+\frac{81}{4}\frac{(\cos 2\theta+1)}{2}$ | M1 |
| | Expands to obtain an expression of the form $a+b\cos\theta+c\cos^2\theta$ and attempts to use $\cos^2\theta=\pm\frac{1}{2}\pm\frac{\cos 2\theta}{2}$ | |
| | $\left(\frac{1}{2}\right)\int\left(3\sqrt{3}-\frac{9}{2}\cos\theta\right)^2 \mathrm{d}\theta=\left(\frac{1}{2}\right)\left[\frac{297}{8}\theta-27\sqrt{3}\sin\theta+\frac{81}{16}\sin 2\theta\right]$ | M1A1 |
| | M1: Attempts to integrate to obtain at least two terms from $\alpha\theta,\ \beta\sin\theta,\ \gamma\sin 2\theta$<br>A1: Correct integration with or without the ½ (NB $\frac{297}{8}=27+\frac{81}{8}$) | |
| | $\left(\frac{1}{2}\right)\left[\frac{297}{8}\theta-27\sqrt{3}\sin\theta+\frac{81}{16}\sin 2\theta\right]_0^{\frac{\pi}{6}}=\left(\frac{1}{2}\right)\left\{\left(\frac{297}{8}.\frac{\pi}{6}-27\sqrt{3}.\sin\frac{\pi}{6}+\frac{81}{16}\sin 2.\frac{\pi}{6}\right)(-0)\right\}$ | M1 |
| | M1: Uses the limits 0 and their $\frac{\pi}{6}$<br>If the substitution for $\theta=0$ evaluates to 0 then the substitution for $\theta=0$ does not need to be seen but if it does not evaluate to 0, the substitution for $\theta=0$ needs to be seen. | |

| | | |
|---|---|---|
| | $\frac{1}{2}\int\left(\frac{3}{2}\cos\theta\right)^2 d\theta=\frac{9}{16}\int(\cos 2\theta+1)\,d\theta=\frac{9}{16}\left[\frac{1}{2}\sin 2\theta+\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}=\frac{9}{16}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)$ | M1 |
| | M1: Uses $\cos^2\theta=\pm\frac{1}{2}\pm\frac{\cos 2\theta}{2}$, integrates to obtain at least $k\sin 2\theta$ and uses the limits of their $\frac{\pi}{6}$ and $\frac{\pi}{2}$ to find the other area<br>NB can be done as a segment : $\frac{1}{2}\left(\frac{3}{4}\right)^2\left(\frac{2\pi}{3}\right)-\frac{1}{2}\left(\frac{3}{4}\right)^2\sin\left(\frac{\pi}{3}\right)$<br>Allow $\frac{1}{2}\left(\frac{3}{4}\right)^2\left(\pi-2\times\text{their}\frac{\pi}{6}\right)-\frac{1}{2}\left(\frac{3}{4}\right)^2\sin\left(\pi-2\times\text{their}\frac{\pi}{6}\right)$ | |
| | $\frac{297}{96}\pi-\frac{351\sqrt{3}}{64}+\frac{9}{16}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)=\frac{105}{32}\pi-\frac{45}{8}\sqrt{3}$ | M1A1 |
| | M1: Adds their two areas both of which are of the form $a\pi+b\sqrt{3}$<br>A1: Correct answer (allow equivalent fractions for $\frac{105}{32}$ and/or $\frac{45}{8}$) | |
| | | **(8)** |
| | | **Total 11** |

Special Case – Uses $\pm(C_1-C_2)$

| | | |
|---|---|---|
| **(b)** | $\frac{1}{2}\int\left(3\sqrt{3}-\frac{9}{2}\cos\theta-\frac{3}{2}\cos\theta\right)^2 d\theta$ | M1 |
| | Attempts to use correct formula on $\pm(C_1-C_2)$. The ½ may be implied by later work. | |
| | $\left(3\sqrt{3}-6\cos\theta\right)^2=27-36\sqrt{3}\cos\theta+36\cos^2\theta=27-36\sqrt{3}\cos\theta+36\frac{(\cos 2\theta+1)}{2}$ | M1 |
| | Expands to obtain an expression of the form $a+b\cos\theta+c\cos^2\theta$ and attempts to use $\cos^2\theta=\pm\frac{1}{2}\pm\frac{\cos 2\theta}{2}$ | |
| | $\left(\frac{1}{2}\right)\int\left(3\sqrt{3}-6\cos\theta\right)^2 d\theta=\left(\frac{1}{2}\right)\left[45\theta-36\sqrt{3}\sin\theta+9\sin 2\theta\right]$ | M1 |
| | Attempts to integrate to obtain at least two terms from $\alpha\theta,\ \beta\sin\theta,\ \gamma\sin 2\theta$ | |
| | **No more marks available** | |
| | | |

**Q6.**

| Question Number | Scheme | Marks |
|---|---|---|
| (a) | $r\sin\theta = 2a\sin\theta + 2a\sin\theta\cos\theta$ OR $r\sin\theta = 2a\sin\theta + a\sin 2\theta$ | B1 |
| | $\dfrac{\mathrm{d}(r\sin\theta)}{\mathrm{d}\theta} = 2a\cos\theta + 2a\cos^2\theta - 2a\sin^2\theta$ OR $\dfrac{\mathrm{d}(r\sin\theta)}{\mathrm{d}\theta} = 2a\cos\theta + 2a\cos 2\theta$ | M1 A1 |
| | $2\cos^2\theta + \cos\theta - 1 = 0$ terms in any order<br>$(2\cos\theta - 1)(\cos\theta + 1) = 0$ | |
| | $\cos\theta = \dfrac{1}{2}$ $\theta = \dfrac{\pi}{3}$ ($\theta = \pi$ need not be seen) | dM1A1 |
| | $r = 2a \times \dfrac{3}{2} = 3a$ | A1 (6) |
| (b) | Area $= \dfrac{1}{2}\int r^2 \mathrm{d}\theta = \dfrac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4a^2(1+\cos\theta)^2 \mathrm{d}\theta$ | |
| | $= 2a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1 + 2\cos\theta + \cos^2\theta)\mathrm{d}\theta$ | M1 |
| | $= 2a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(1 + 2\cos\theta + \dfrac{1}{2}(\cos 2\theta + 1)\right)\mathrm{d}\theta$ | M1 |
| | $= 2a^2\left[\theta + 2\sin\theta + \dfrac{1}{2}\left(\dfrac{1}{2}\sin 2\theta + \theta\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$ | dM1A1 |
| | $= 2a^2\left[\dfrac{\pi}{3} + \sqrt{3} + \dfrac{1}{4}\times\dfrac{\sqrt{3}}{2} + \dfrac{\pi}{6} - \left(\dfrac{\pi}{6} + 1 + \dfrac{1}{4}\times\dfrac{\sqrt{3}}{2} + \dfrac{\pi}{12}\right)\right]$ | M1<br>NB: A1 on e-PEN |
| | $= 2a^2\left(\dfrac{\pi}{4} + \sqrt{3} - 1\right)$ | |
| | Area of $\Delta OAB = \dfrac{1}{2}\times 3a \times (2+\sqrt{3})a \times \sin\dfrac{\pi}{6}$ $\left(= \dfrac{3}{4}a^2(2+\sqrt{3})\right)$ | |
| | Shaded area $= 2a^2\left(\dfrac{\pi}{4} + \sqrt{3} - 1\right) - \dfrac{3}{4}a^2(2+\sqrt{3}) = \dfrac{a^2}{4}(2\pi - 14 + 5\sqrt{3})$ | M1A1cao<br>(7) |
| | | [13] |

| Question Number | Scheme | Marks |
| --- | --- | --- |
| (a) | | |
| B1 | Multiply $r$ by $\sin\theta$<br>Award if not seen explicitly but a correct result following use of double angle formula is seen | |
| M1 | Differentiate $r\sin\theta$ or $r\cos\theta$ ( using product rule or using double angle formula first) | |
| A1 | Correct derivative for $r\sin\theta$ | |
| dM1 | Use $\sin^2\theta+\cos^2\theta=1$ to form a 3TQ in $\cos\theta$ and attempt its solution by a valid method | |
| A1 | Correct value for $\theta$ | |
| A1 | Correct $r$ | |
| (b) | | |
| M1 | Use area $=\frac{1}{2}\int r^2\mathrm{d}\theta$ with $r=2a+2a\cos\theta$, no limits needed, | |
| M1 | Use a double angle formula to obtain a function ready for integrating<br>(Alt method uses integration by parts – may be seen) | |
| dM1 | Attempt the integration $\cos 2\theta\rightarrow\frac{1}{k}\sin 2\theta$ $k=\pm2$ or $\pm1$ | |
| A1 | Correct integration, | |
| M1 | Substitute the limits (need not be simplified). Limits $\frac{\pi}{6}$ and their $\theta$ from (a) provided this is $>\frac{\pi}{6}$<br>NB: A1 on e-PEN | |
| M1 | Obtain the area of $\Delta OAB$ and subtract from their previous area | |
| A1 | Correct answer | |