Questions

Q1.

$$y = e^{\cos 2x}$$

(a) Show that

$$\frac{d^2y}{dx^2} = e^{\cos^2 x}(\sin^2 2x - 2\cos 2x)$$

(4)

(b) Hence find the Maclaurin series expansion of $e^{\cos 2x}$ up to and including the term in x^2

(3)

(Total for question = 7 marks)

Q2.

Given that $y = \cot x$,

(a) Show that

$$\frac{d^2y}{dx^2} = 2\cot x + 2\cot^3 x$$

(3)

(b) Hence show that

$$\frac{d^3y}{dx^3} = p\cot^4 x + q\cot^2 x + r$$

where p, q and r are integers to be found.

(3)

(c) Find the Taylor series expansion of $\cot x$ in ascending powers of $\left(x - \frac{\pi}{3}\right)$ up to and including the term in $\left(x - \frac{\pi}{3}\right)^3$.

(3)

(Total for question = 9 marks)

Q3.

$$f(x) = \sin\left(\frac{3}{2}x\right)$$

(a) Find the Taylor series expansion for f(x) about $\frac{\pi}{3}$ in ascending powers of $\left(x - \frac{\pi}{3}\right)$ up to and including the term in $\left(x - \frac{\pi}{3}\right)^4$

(6)

(b) Hence obtain an estimate of sin $\frac{1}{2}$, giving your answer to 4 decimal places.

(2)

(Total for question = 8 marks)

Q4.

Find the Taylor series expansion about x = 1 of $\frac{1}{\sqrt{1 + x^2}}$ in ascending powers of (x – 1), up to and including
the term in $(x – 1)^2$, simplifying each term.

(Total for question = 8 marks)

Q5.

$$2\frac{d^2y}{dx^2} + \frac{dy}{dx} - xy = 1$$

(a) Show that

$$\frac{d^4y}{dx^4} = \frac{1}{2}\left(a\frac{dy}{dx} + bx\frac{d^2y}{dx^2} + c\frac{d^3y}{dx^3}\right)$$

where a, b and c are constants to be found.

(4)

Given that $y = 1$ and $\frac{dy}{dx} = 1$ when $x = 2$

(b) find a series solution for y in ascending powers of $(x - 2)$, up to and including the term in $(x - 2)^4$. Write each term in its simplest form.

(4)

(c) Use the solution to part (b) to find an approximate value for y when $x = 2.1$, giving your answer to 3 decimal places.

(2)

(Total for question = 10 marks)

Examiner's Report

Q1.

A Maclaurin series expansion of $e^{\cos 2x}$ was required here and many fully correct responses were seen. In part (a), most students were able to obtain the correct $\frac{dy}{dx}$. Those that then replaced the 2sin$x$cos$x$ with sin2x usually proceeded correctly. Use of $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ prior to differentiation helped some to follow this route. A few students took natural logarithms of both sides before differentiating.
The method to produce the series expansion in part (b) was not always known. Attempts were seen which tried to use the series expansions of e^x and cos x. Those who used the correct formula were usually able to obtain full marks although f ′(0) was occasionally evaluated as e. A few students failed to calculate their trigonometric expressions or gave their answer in a decimal form.

Q2.

This again was a question that students were well prepared for. The main problems in parts (a) and (b) centred around incorrect signs in differentiation. However for the most part students could get to the correct answers by the most efficient means. There were few solutions where students reverted to sin and cos to work with.

The Taylor expansion in powers of (x – pi/3) was well known. It should be stressed to students the importance of writing down the formula before using it although they must be careful about the use of f(x) as in the formula rather than the correct y or cot x.

It was pleasing to see students working in exact fractions rather than resorting to decimals.

Q3.

This was another question where students clearly understood what was required of them. The requisite derivatives and their evaluation at $\frac{\pi}{3}$ were correctly done by the majority although there were some poor attempts at the differentiation in some cases.
There were some attempts at Maclaurin series rather than the required Taylor series.

Weaker solutions to (b) substitute $\frac{1}{2}$ rather than $\frac{1}{3}$ into their expansion.

Q4.

This question on a Taylor series expansion produced a surprisingly mixed response. The general method was well known but many succumbed to errors with the required differentiations, with some failing to deploy the chain and product rules correctly. The second derivative proved more problematic with a significant minority of candidates incorrectly producing an $\alpha x(1+x^2)^{-5/2}$ term. Those who obtained the correct derivatives were usually able to then produce correct numerical expressions for the values at *x* = 1. A very small number of errors were seen in applying a correct Taylor series. Candidates who quoted the correct expansion formula usually obtained the M mark when substituting in their values, but it was evident that an over reliance on calculators left many unable to correctly simplify the resulting expression for the third coefficient which left them with an inexact form. The final mark was occasionally withheld for an unsimplified answer or in cases where candidates had reverted to decimals.

It was interesting to see the common error amongst candidates of interpreting $-(2)^{-\frac{3}{2}}$ as $-\frac{1}{\sqrt[3]{2}}$ instead of $\frac{-1}{(\sqrt{2})^3}$.

Q5.

A substantial number of students scored full marks with little obvious difficulty with this question which required a Taylor series solution to a differential equation.

In part (a), most differentiated implicitly and dealt with the products correctly. A smaller number made $\frac{d^2y}{dx^2}$ the subject first but were more prone to slips. An "*x*" was occasionally missing from the final answer. A few students failed to differentiate the "1" at the start of their working.

Part (b) required a series solution for *y* and most attempted the values of the necessary derivatives at *x* = 2, although slips were seen with less organised work. A correct Taylor expansion for their values usually resulted although a small number produced a series in powers of *x* rather than *x* – 2 or had 2 in place of f(2).

In part (c) the method mark was widely scored but the accuracy mark was often lost due to miscalculation. The final answer was required to the correct number of decimal places.

Mark Scheme

Q1.

Question Number	Scheme	Notes	Marks
	$y=e^{\cos^2 x}$		
(a)	$\frac{dy}{dx}=-2\sin x\cos x e^{\cos^2 x}=-\sin 2x e^{\cos^2 x}$ M1: Differentiates using the chain rule to obtain an expression of the form $\alpha\sin x\cos x e^{\cos^2 x}$ or $\beta\sin 2x e^{\cos^2 x}$ A1: Correct derivative Note that candidates may use $\frac{1}{2}(1+\cos 2x)$ instead of $\cos^2 x$		M1A1
	$\frac{d^2y}{dx^2}=-\sin 2x(-2\sin x\cos x e^{\cos^2 x})-2\cos 2x e^{\cos^2 x}$ Correct use of the Product Rule on their first derivative **Dependent on the first method mark.**		dM1
	$\frac{d^2y}{dx^2}=e^{\cos^2 x}(\sin^2 2x-2\cos 2x)$ *	Achieves the printed answer with no errors.	A1*
	Alternative using logs: $y=e^{\cos^2 x}\Rightarrow \ln y=\cos^2 x\Rightarrow \frac{1}{y}\frac{dy}{dx}=-2\sin x\cos x$ M1: $\frac{1}{y}\frac{dy}{dx}=k\sin x\cos x$ or $k\sin 2x$ A1: $\frac{1}{y}\frac{dy}{dx}=-2\sin x\cos x$ $\frac{dy}{dx}=-y\sin 2x\Rightarrow \frac{d^2y}{dx^2}=-\frac{dy}{dx}\sin 2x-2y\cos 2x$ M1: Correct use of product rule $\frac{d^2y}{dx^2}=e^{\cos^2 x}(\sin^2 2x-2\cos 2x)$ * A1: Achieves the printed answer with no errors.		
			(4)
(b)	$\frac{dy}{dx}=0,\ \frac{d^2y}{dx^2}=-2e$	Both seen, can be implied by subsequent work.	B1
	$f(x)=f(0)+xf'(0)+\frac{x^2}{2}f''(0)+\ldots$ $=e^{\cos^2 0}-\sin 0e^{\cos^2 0}x+\frac{1}{2}e^{\cos^2 0}(\sin^2 0-2\cos 0)x^2+\ldots$ Applies the **correct** Maclaurin expansion, the "$\frac{1}{2}$" is required and there must be no x's in the derivatives. This can be implied by their expansion but if the expansion in incorrect for their values and the formula is not quoted, score M0.		M1
	$\left(e^{\cos^2 x}\right)=e(1-x^2+\ldots)$	Or exact equivalent e.g. $e-ex^2$ (i.e. all trig. evaluated)	A1
			(3)
			Total 7

Q2.

Question Number	Scheme	Notes	Marks
	$y = \cot x$		
(a)	$\frac{dy}{dx} = -\text{cosec}^2 x$		
	$\frac{d^2y}{dx^2} = (-2\text{cosec}\, x)(-\text{cosec}\, x \cot x)$	M1: Differentiates using the chain rule or product/quotient rule A1: Correct derivative	M1A1
	$= 2\text{cosec}^2 x \cot x = 2\cot x + 2\cot^3 x$ *	A1: Correct completion to printed answer $1 + \cot^2 x = \text{cosec}^2 x$ or $\cos^2 x + \sin^2 x = 1$ must be used Full working must be shown	A1cso*
			(3)
	Alternative:		
	$y = \frac{\cos x}{\sin x} \rightarrow \frac{dy}{dx} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$		
	$\frac{d^2y}{dx^2} = -\left(-2\sin^{-3} x \cos x\right) = \ldots$		M1A1
	A1: Correct completion to printed answer see above		A1
(b)	$\frac{d^3y}{dx^3} = -2\text{cosec}^2 x - 6\cot^2 x\,\text{cosec}^2 x$	Correct third derivative	B1
	$= -2\left(1+\cot^2 x\right) - 6\cot^2 x\left(1+\cot^2 x\right)$	Uses $1 + \cot^2 x = \text{cosec}^2 x$	M1
	$= -6\cot^4 x - 8\cot^2 x - 2$	cso	A1
			(3)

(c)	$\text{f}\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}, \text{f}'\left(\frac{\pi}{3}\right) = -\frac{4}{3}, \text{f}''\left(\frac{\pi}{3}\right) = \frac{8}{3\sqrt{3}}, \text{f}'''\left(\frac{\pi}{3}\right) = -\frac{16}{3}$ M1: Attempts all 4 values at $\frac{\pi}{3}$ No working need be shown		M1
	$(y =)\frac{1}{\sqrt{3}} - \frac{4}{3}\left(x - \frac{\pi}{3}\right) + \frac{4}{3\sqrt{3}}\left(x - \frac{\pi}{3}\right)^2 - \frac{8}{9}\left(x - \frac{\pi}{3}\right)^3$ M1: Correct application of Taylor using their values. Must be up to and including $\left(x - \frac{\pi}{3}\right)^3$ A1: Correct expression Must start $y = \ldots$ or $\cot x$ f(x) allowed provided defined here or above as $\text{f}(x) = \cot x$ or y Decimal equivalents allowed (min 3 sf apart from 0.77), 0.578, 1.33, 0.770, (0.7698.., so accept 0.77) 0.889		M1A1
			(3)
			Total 9

Q3.

<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td rowspan="4">(a)</td><td rowspan="2">$f(x) = \sin\left(\frac{3}{2}x\right)$
$f'(x) = \frac{3}{2}\cos\left(\frac{3}{2}x\right)$
$f''(x) = -\frac{9}{4}\sin\left(\frac{3}{2}x\right)$
$f'''(x) = -\frac{27}{8}\cos\left(\frac{3}{2}x\right)$
$f''''(x) = \frac{81}{16}\sin\left(\frac{3}{2}x\right)$</td><td>M1: Attempt first 4 derivatives. Should be $\sin \to \cos \to \sin \to \cos \to \sin$. I.e. ignore signs and coefficients.</td><td rowspan="2">M1A2</td></tr>
<tr><td>A1: $f' = \frac{3}{2}\cos\left(\frac{3}{2}x\right)$ and $f'' = -\frac{9}{4}\sin\left(\frac{3}{2}x\right)$
A1: $f''' = -\frac{27}{8}\cos\left(\frac{3}{2}x\right)$ and $f'''' = \frac{81}{16}\sin\left(\frac{3}{2}x\right)$
Allow un-simplified e.g. $f'' = -\frac{3}{2}\cdot\frac{3}{2}\sin\left(\frac{3}{2}x\right)$</td></tr>
<tr><td colspan="2">$y\left(\frac{\pi}{3}\right) = 1, y'\left(\frac{\pi}{3}\right) = 0, y''\left(\frac{\pi}{3}\right) = -\frac{9}{4}, y'''\left(\frac{\pi}{3}\right) = 0, y''''\left(\frac{\pi}{3}\right) = \frac{81}{16}$
Attempts at least 1 derivative at $x = \frac{\pi}{3}$</td><td>M1</td></tr>
<tr><td>$f(x) = 1 - \frac{9}{8}\left(x - \frac{\pi}{3}\right)^2 + \frac{27}{128}\left(x - \frac{\pi}{3}\right)^4$</td><td>dM1: Correct use of Taylor series. I.e.
$f(x) = f\left(\frac{\pi}{3}\right) + \left(x - \frac{\pi}{3}\right)f'\left(\frac{\pi}{3}\right) + \left(x - \frac{\pi}{3}\right)^2 \frac{f''\left(\frac{\pi}{3}\right)}{2!} + \ldots$
Evidence of at least one term of the correct structure i.e. $\left(x - \frac{\pi}{3}\right)^n \frac{f^n\left(\frac{\pi}{3}\right)}{n!}$ and not a Maclaurin series. **Dependent on the previous method mark.**

A1: Correct expansion. Allow equivalent **single** fractions for $\frac{9}{8}$ and/or $\frac{27}{128}$ and allow decimal equivalents i.e. 1.125 and 0.2109375. Ignore any extra terms.</td><td>dM1A1</td></tr>
<tr><td></td><td></td><td></td><td>**(6)**</td></tr>
<tr><td rowspan="2">(b)</td><td>$f\left(\frac{1}{3}\right) = 0.4815$</td><td>M1: Attempts $f\left(\frac{1}{3}\right)$ or states $x = \frac{1}{3}$

A1: 0.4815 cao</td><td>M1A1</td></tr>
<tr><td colspan="2"></td><td>**(2)**</td></tr>
<tr><td></td><td></td><td></td><td>**Total 8**</td></tr>
</table>

Q4.

Question Number	Scheme	Notes	Marks
	$y=\dfrac{1}{\sqrt{1+x^2}}$		
	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=-x(1+x^2)^{-\frac{3}{2}}$	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=kx\left(1+x^2\right)^{-\frac{3}{2}}$	M1
		$\dfrac{\mathrm{d}y}{\mathrm{d}x}=-x\left(1+x^2\right)^{-\frac{3}{2}}$. Allow in any correct unsimplified form and isw if necessary.	A1
	$\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=1}=-\dfrac{1}{2\sqrt{2}}$ $\left(-(2)^{-\frac{3}{2}},\ -\dfrac{\sqrt{2}}{4}\text{ are common}\right)$	Correct value for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $x=1$. Allow for any correct **exact** numerical possibly unsimplified expression and isw if necessary.	A1
	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=-x(1+x^2)^{-\frac{3}{2}}\Rightarrow\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-(1+x^2)^{-\frac{3}{2}}+\dfrac{3}{2}x.2x(1+x^2)^{-\frac{5}{2}}\left(=\dfrac{2x^2-1}{\left(1+x^2\right)^{\frac{5}{2}}}\right)$ or $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{x}{(1+x^2)^{\frac{3}{2}}}\Rightarrow\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{-(1+x^2)^{\frac{3}{2}}+\frac{3}{2}x.2x(1+x^2)^{\frac{1}{2}}}{(1+x^2)^3}\left(=\dfrac{2x^2-1}{\left(1+x^2\right)^{\frac{5}{2}}}\right)$ dM1: Product rule: $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\alpha(1+x^2)^{-\frac{3}{2}}+\beta x^2(1+x^2)^{-\frac{5}{2}}$ Quotient rule: $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{\alpha(1+x^2)^{\frac{3}{2}}+\beta x^2(1+x^2)^{\frac{1}{2}}}{(1+x^2)^3}$ **Dependent on the first method mark** A1: Fully correct second derivative. Allow in any correct unsimplified form and isw if necessary.		dM1A1
	$\left(\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_{x=1}=\dfrac{1}{4\sqrt{2}}$ $\left(-(2)^{-\frac{3}{2}}+3(2)^{-\frac{5}{2}},\ \dfrac{\sqrt{2}}{8}\text{ are common}\right)$	Correct value for $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at $x=1$. Allow for any correct **exact** numerical possibly unsimplified expression and isw if necessary.	A1
	$\mathrm{f}(x)=\mathrm{f}(1)+(x-1)\mathrm{f}'(1)+\dfrac{(x-1)^2}{2}\mathrm{f}''(1)+\ldots$ $\left(\dfrac{1}{\sqrt{1+x^2}}\right)=\dfrac{1}{\sqrt{2}}-\dfrac{1}{2\sqrt{2}}(x-1)+\dfrac{1}{8\sqrt{2}}(x-1)^2$ M1: Attempts f (1) and applies the correct Taylor series using their values. Must see an attempt at the **first 3** terms. **If the general series is not quoted and their series does not follow their values score M0** A1: Correct expansion (allow equivalent simplified (single fraction) coefficients		M1A1
			(8)
			Total 8

Q5.

Question Number	Scheme	Notes	Marks
	$2\frac{d^2y}{dx^2}+\frac{dy}{dx}-xy=1$		
(a)	$2\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-x\frac{dy}{dx}-y=0$	B1: $2\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}$ or equivalent correct terms if they rearrange the given equation. M1: Attempt product rule on xy. Allow sign errors only so need to see $\pm x\frac{dy}{dx}\pm y$	B1M1
	$2\frac{d^4y}{dx^4}+\frac{d^3y}{dx^3}-x\frac{d^2y}{dx^2}-\frac{dy}{dx}-\frac{dy}{dx}=0$	Differentiates again to obtain an expression that contains the fourth derivative including product rule on $x\frac{dy}{dx}$ to give $\pm x\frac{d^2y}{dx^2}\pm\frac{dy}{dx}$. (Allow terms to be "listed")	M1
	$\frac{d^4y}{dx^4}=\frac{1}{2}\left(2\frac{dy}{dx}+x\frac{d^2y}{dx^2}-\frac{d^3y}{dx^3}\right)$	If the "1" is not dealt with correctly e.g. if it "disappears" at the wrong time, this mark should be withheld.	A1
			(4)
(b)	$y''(2)=1,\ y'''(2)=1,\ y''''(2)=\frac{3}{2}$	M1: Attempt $y''(2)$, $y'''(2)$ and $y''''(2)$ A1: Correct values	M1A1
	$y=f(2)+(x-2)f'(2)+\frac{(x-2)^2f''(2)}{2!}+\frac{(x-2)^3f'''(2)}{3!}+\frac{(x-2)^4f''''(2)}{4!}$ Attempt correct Taylor expansion **with their values.** Allow the terms to be "listed" for this mark.		M1
	$(y=)1+(x-2)+\frac{(x-2)^2}{2}+\frac{(x-2)^3}{6}+\frac{(x-2)^4}{16}$	Correct simplified expression.	A1
			(4)
(c)	$x=2.1\Rightarrow y=1+(0.1)+\frac{(0.1)^2}{2}+\frac{(0.1)^3}{6}+\frac{(0.1)^4}{16}$	Substitutes $x=2.1$ into an expansion involving $(x-2)$	M1
	$y=1.105$ **only** Note this is **not** awrt.	Cao (Note that this mark must follow the final A1 in (b) i.e. 1.105 must come from a correct expansion). Incorrect answer with no working scores M0. Correct answer following a correct expansion scores M1A1.	A1
			(2)
			Total 10