

Questions

Q1.

Obtain the general solution of the equation

$$x^2 \frac{dy}{dx} + (x \cot x + 2)xy = 4 \sin x \quad 0 < x < \pi$$

Give your answer in the form $y = f(x)$

(8)

(Total for question = 8 marks)

Q2.

(a) Show that the substitution $z = y^{-2}$ transforms the differential equation

$$\frac{dy}{dx} + 2xy = xe^{-x^2}y^3 \quad (I)$$

into the differential equation

$$\frac{dz}{dx} - 4xz = -2xe^{-x^2} \quad (II)$$

(4)

(b) Solve differential equation (II) to find z as a function of x .

(5)

(c) Hence find the general solution of differential equation (I), giving your answer in the form $y^2 = f(x)$.

(1)

(Total for question = 10 marks)

Q3.

$$x \frac{dy}{dx} + (1 - 2x)y = x, \quad x > 0$$

Find the general solution of the differential equation, giving your answer in the form $y = f(x)$.

(9)

(Total for question = 9 marks)

Q4.

Find, in terms of k , where k is a positive integer, the general solution of the differential equation

$$(1+x)\frac{dy}{dx} + ky = x^{\frac{1}{2}}(1+x)^{2-k}, \quad x > 0$$

giving your answer in the form $y = f(x)$.

(6)

(Total for question = 6 marks)

Q5.

Find the general solution of the differential equation

$$\cos x \frac{dy}{dx} + y \sin x = (\cos^2 x) \ln x, \quad 0 < x < \frac{\pi}{2}$$

Give your answer in the form $y = f(x)$.

(8)

(Total for question = 8 marks)

Q6.

(a) Find the general solution of the differential equation

$$(x^2 + 1)\frac{dy}{dx} + xy - x = 0$$

giving your answer in the form $y = f(x)$.

(6)

(b) Find the particular solution for which $y = 2$ when $x = 3$

(2)

(Total for question = 8 marks)

Q7.

$$(\cos x) \frac{dy}{dx} + (\sin x) y = 2 \cos^3 x \sin x - 3 \quad 0 \leq x < \frac{\pi}{2}$$

(a) Find the general solution of this differential equation. Give your answer in the form $y = f(x)$.

(7)

(b) Find the particular solution of this differential equation for which $y = 3\sqrt{3}$ at $x = \frac{\pi}{3}$.
Give your answer in the form $y = f(x)$.

(3)

(Total for question = 10 marks)

Examiner's Report

Q1.

This question discriminated fairly well between candidates with around half gaining full marks. Some candidates did not fully divide the equation by x^2 , and so lost the first mark. Most tried to find an integrating factor (IF) of the correct form, with some gaining the correct IF. It was disappointing to see so many incorrect IFs, which resulted in these candidates only gaining one more mark by multiplying through by their IF, but then trying to integrate some complicated functions. The majority of those with the correct IF finished the question very well, with correct integration and treatment of the constant.

Q2.

Students were well prepared for the demands of the first part of this question and they were able to transform the original differential equation by a variety of valid methods.

It was the second part of the question that caused problems. The idea of how to use their integrating factor was understood by most but the integration of $-2xe^{3x^2}$ caused problems. Students at this level should not just take the presence of a product in an integral to mean that integration by parts must be needed. The most common mark for this part was 3/5.

Of those who did obtain a final solution for z , most were able to get their answer in the form $y^2 = \dots$ although it was also disappointing to see how many students thought that taking the inverse of a sum of terms could be done by inverting the separate terms.

Q3.

Some candidates forgot to divide the right hand side by x , leaving it as x while others forgot to multiply their integrating factor on the right hand side and so integrated 1. However most candidates were able to determine an integrating factor of $e^{\ln x - 2x}$, but some candidates could not get any further. Most who got this far were able to write their integrating factor as xe^{-2x} and multiply through to produce a correct statement involving $\int xe^{-2x} dx$. Most candidates then attempted integration by parts, many correctly, but errors were often seen here, mostly in the form of incorrect coefficients. Those candidates who obtained an integrating factor which did not give rise to an integral that required integration by parts could gain no further marks. Candidates who got as far as an expression which was the result of an integration usually completed the final step to gain a general solution for y which included a constant although a few incorporated e^{2x} into the constant, creating a new "constant".

Q4.

Students were usually well prepared for the demands of this question with the majority of students obtaining one of the versions of the correct answer. It was perhaps surprising seeing students integrating $x^{\frac{1}{2}}(1+x)$ using parts rather than, more simply, expanding the brackets. However this longer method was executed successfully in most cases.

The constant of integration was introduced at the correct stage by nearly all students.

Q5.

This first order differential equation question saw good scoring by the vast majority. It was rare to see any student fail to divide through by $\cos x$ before forming the integrating factor. A small number produced $e^{\sec 2x}$ instead of $e^{\ln \sec x}$ or $e^{-\ln \cos x}$ after integrating $\tan x$. The method of multiplying both sides by the integrating factor was well-known, although not all could obtain a right hand side of $\ln x$ or $\int \ln x$, usually the result of copying errors. The last two marks were more discriminating. $\int \ln x$ was widely given as $\frac{1}{x}$. Those who applied integration by parts were almost always successful although a few forgot to include the constant of integration in their final answer or had it incorrectly placed.

Q6.

This question on a first order differential equation was more discriminating, with most marks being lost as a result of integration errors. Most students successfully converted the differential equation into the form $\frac{dy}{dx} + P(x)y = Q(x)$ although a few made $\frac{dy}{dx}$ the subject and invariably abandoned their attempt. The formula for the integrating factor was well known although the integration of $\frac{x}{1+x^2}$ was not always correct or errors simplifying $e^{\frac{1}{2}\ln(1+x^2)}$ occurred. Students who proceeded directly to $y' = \int Q' dx$ tended to be more successful than those who needlessly tried to write the left hand side as the derivative of a product. The resulting integration of $\frac{1}{(1+x^2)^{\frac{1}{2}}}$ also caused problems with some doomed attempts at integration by parts seen. Unusually, this differential equation could be solved by separation of variables although sign errors in integration were common for those who chose this route.

The method mark for the required particular solution in part (b) was widely scored. The few who had an exponential constant from separation of variables often found themselves unable to determine the value of c .

Q7.

This first order differential equation question was a little more discriminating. Most knew to divide through by $\cos x$ although this was not always carried out successfully, in particular with the terms on the right hand side leading to a common error of failing to divide the final term by $\cos x$ and arriving at $\int \sec x$ instead of $\int \sec^2 x$ for the final term. Correctly obtaining $\sec x$ as the integrating factor was common. A small number of candidates remain confused about how to correctly apply the integrating factor – the most common error being the failure to multiply $Q(x)$ by I . Those who had obtained $\int (2 \cos x \sin x - 3 \sec^2 x) dx$ were usually able to integrate successfully. Most candidates recognised $\int 2 \sin x \cos x$ as $\int \sin 2x$ and obtained $-\frac{1}{2} \cos 2x$ and it was rare to see this evaluated as $-\cos^2 x$ or $\sin^2 x$. Only a small number failed to divide through by $\sec x$ (or multiply by $\cos x$) to obtain y as a function of x at the end.

In part (b), the method mark was often scored, but there were a few slips obtaining the constant. A few obtained the correct constant but then failed to place it correctly into the particular solution. A common error was to assert that $3\sqrt{3} + 3\frac{\sqrt{3}}{2} = 3\frac{\sqrt{3}}{2}$. The simplification of $\frac{-3 \tan x}{\sec x} = -3 \sin x$ caused problems for some when attempting to evaluate their constant of integration

Mark Scheme

Q1.

Question Number	Scheme	Marks
	$\frac{dy}{dx} + \frac{(x \cot x + 2)}{x} y = \frac{4 \sin x}{x^2}$ $\text{IF} = e^{\int \frac{(x \cot x + 2)}{x} dx}$ $= e^{(\ln \sin x + 2 \ln x)}$ $= x^2 \sin x$ $\frac{d}{dx} (\text{their IF} \times y) = \text{their IF} \times \frac{4 \sin x}{x^2}$ $yx^2 \sin x = \int 4 \sin^2 x dx = 4 \int \frac{1 - \cos 2x}{2} dx = 4 \left(\frac{x}{2} - \frac{1}{4} \sin 2x \right) (+C)$ $y = \frac{2x - \sin 2x + C}{x^2 \sin x} \quad \text{oe}$	B1 M1 A1 A1 M1 dM1A1 A1cao [8]
B1 M1 A1 A1 M1 dM1 A1 A1	Divide through by x^2 Attempt an IF of the form $e^{\int \frac{k(x \cot x + 2)}{x} dx}$ $(\ln \sin x + 2 \ln x)$ Correct IF Multiply through by their IF and write LHS in form shown – can be implied by next line. Allow if IF is seen instead of their function provided an IF has been attempted. Allow use of their RHS Attempt to integrate $\sin^2 x$, including using $\sin^2 x = \frac{1}{2}(1 \pm \cos 2x)$ $\cos 2x \rightarrow k \sin 2x$ depends on previous M mark Correct integration, constant not needed Include the constant and treat it correctly. Must have $y = \dots$	

Q2.

Question Number	Scheme	Notes	Marks
	$\frac{dy}{dx} + 2xy = xe^{-x^2} y^3$		
(a)	$z = y^{-2} \Rightarrow y = z^{-\frac{1}{2}}$		
	$\frac{dy}{dx} = -\frac{1}{2} z^{-\frac{3}{2}} \frac{dz}{dx}$	M1: $\frac{dy}{dx} = kz^{-\frac{3}{2}} \frac{dz}{dx}$ A1: Correct differentiation	M1A1
	$-\frac{1}{2} z^{-\frac{3}{2}} \frac{dz}{dx} + \frac{2x}{z^{\frac{1}{2}}} = xe^{-x^2} z^{-\frac{3}{2}}$	Substitutes for dy/dx	M1
	$\frac{dz}{dx} - 4xz = -2xe^{-x^2} *$	Correct completion to printed answer with no errors seen	A1cso
			(4)
(a) Alternative 1			
	$\frac{dz}{dy} = -2y^{-3} \text{ oe}$	M1: $\frac{dz}{dy} = ky^{-3}$ A1: Correct differentiation	M1A1
	$-\frac{1}{2} y^3 \frac{dz}{dx} + 2xy = xe^{-x^2} y^3$	Substitutes for dy/dx	M1
	$\frac{dz}{dx} - 4xz = -2xe^{-x^2} *$	Correct completion to printed answer with no errors seen	A1
(a) Alternative 2			
	$\frac{dz}{dx} = -2y^{-3} \frac{dy}{dx}$	M1: $\frac{dz}{dx} = ky^{-3} \frac{dy}{dx}$ inc chain rule A1: Correct differentiation	M1A1
	$-\frac{1}{2} y^3 \frac{dz}{dx} + 2xy = xe^{-x^2} y^3$	Substitutes for dy/dx	M1
	$\frac{dz}{dx} - 4xz = -2xe^{-x^2} *$	Correct completion to printed answer with no errors seen	A1
(b)	$I = e^{\int -4x dx} = e^{-2x^2}$	M1: $I = e^{\int \pm 4x dx}$ A1: e^{-2x^2}	M1A1
	$ze^{-2x^2} = \int -2xe^{-3x^2} dx$	$z \times I = \int -2xe^{-x^2} I dx$	dM1
	$\frac{1}{3} e^{-3x^2} (+c)$	$\int xe^{qx^2} dx = pe^{qx^2} (+c)$	M1
	$z = ce^{2x^2} + \frac{1}{3} e^{-x^2}$	Or equivalent	A1
			(5)
(c)	$\frac{1}{y^2} = ce^{2x^2} + \frac{1}{3} e^{-x^2} \Rightarrow y^2 = \frac{1}{ce^{2x^2} + \frac{1}{3} e^{-x^2}}$	$y^2 = \frac{1}{(b)} \left(= \frac{3e^{x^2}}{1 + ke^{3x^2}} \right)$	B1ft
			(1)
			Total 10

Q3.

Question Number	Scheme		Marks
	$x \frac{dy}{dx} + (1 - 2x)y = x$		
	$\frac{dy}{dx} + \frac{(1 - 2x)}{x} y (= 1)$	Divides by x - may be implied by subsequent working	M1
	Integrating factor $I = e^{\int \frac{1-2x}{x} dx}$	Correct attempt at I , including an attempt at the integration. $\ln$ must be seen if not fully correct.	dM1
	$= e^{\ln(x) - 2x}$	Correct expression	A1
	$= xe^{-2x}$	No errors in working allowed	A1
	$xye^{-2x} = \int xe^{-2x} dx$	Multiply through by their IF and integrate LHS. Can be given if $y \times$ their IF is seen	M1
	$= -\frac{1}{2} xe^{-2x} + \frac{1}{2} \int e^{-2x} dx$	M1: Correct application by parts ie differentiate x and attempt to integrate e^{-2x}	M1A1
		A1: Correct expression	
	$= -\frac{1}{2} xe^{-2x} - \frac{1}{4} e^{-2x} (+c)$	A1: Complete the integration to a correct result. Constant not required.	A1
	$y = \frac{ce^{2x}}{x} - \frac{1}{4x} - \frac{1}{2}$	Oe Must have $y = \dots$ Must include a constant. fit their previous line	A1ft
			(9)
			Total 9

Q4.

	Scheme	Notes	Marks
	$(1+x)\frac{dy}{dx} + ky = x^{\frac{1}{2}}(1+x)^{2-k}$		
	$\frac{dy}{dx} + \frac{ky}{(1+x)} = \frac{x^{\frac{1}{2}}(1+x)^{2-k}}{(1+x)}$	Divides by $(1+x)$ including the ky term	M1
	$I = e^{\int \frac{k}{1+x} dx} = (1+x)^k$	dM1: Attempt integrating factor. $I = e^{\int \frac{k}{1+x} dx}$ is sufficient for this mark but must include the k . Condone omission of “dx”. A1: $(1+x)^k$	dM1A1
	$y(1+x)^k = \int x^{\frac{1}{2}}(1+x) dx$	Reaches $y \times (\text{their } I) = \int x^{\frac{1}{2}}(1+x)^{1-k} \times (\text{their } I) dx$	M1
	$\int x^{\frac{1}{2}}(1+x) dx = \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} (+ c)$ or by parts $\int x^{\frac{1}{2}}(1+x) dx = \frac{2}{3}x^{\frac{3}{2}}(1+x) - \frac{4}{15}x^{\frac{5}{2}} (+ c)$	Correct integration	A1
	$y = \frac{\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + c}{(1+x)^k}$ or e.g. $y = \frac{\frac{2}{3}x^{\frac{3}{2}}(1+x) - \frac{4}{15}x^{\frac{5}{2}} + c}{(1+x)^k}$ or e.g. $y = \frac{2}{3}x^{\frac{3}{2}}(1+x)^{1-k} - \frac{4}{15}x^{\frac{5}{2}}(1+x)^{-k} + c(1+x)^{-k}$ or e.g. $y = \frac{10x^{\frac{3}{2}}(1+x) - 4x^{\frac{5}{2}} + c}{15(1+x)^k}$ or e.g. $y = \frac{2}{3}x^{\frac{3}{2}}(1+x)^{-k} + \frac{2}{5}x^{\frac{5}{2}}(1+x)^{-k} + c(1+x)^{-k}$ Correct answer with the constant correctly placed. Allow any equivalent correct answer.		A1
			(6)
			Total 6

Q5.

Question Number	Scheme	Notes	Marks
	$\cos x \frac{dy}{dx} + y \sin x = (\cos^2 x) \ln x$		
	$\frac{dy}{dx} + y \frac{\sin x}{\cos x} = \cos x \ln x$	Attempt to divide through by $\cos x$. If the intention is not clear, must see at least 2 terms divided by $\cos x$.	M1
	$I = e^{\int \frac{\sin x}{\cos x} dx} = e^{-\ln \cos x}$	M1: $e^{\int \pm \text{their } P(x) (dx)}$. Dependent on the first method mark. A1: $e^{-\ln \cos x}$ or $e^{\ln \sec x}$	dM1A1
	$= \frac{1}{\cos x}$	$\frac{1}{\cos x}$ or $(\cos x)^{-1}$ or $\sec x$	A1
	$\frac{y}{\cos x} = \int \ln x \, dx$ or $\frac{d}{dx} \left(\frac{y}{\cos x} \right) = \ln x$	M1: $y \times \text{their } I = \int Q(x) \times \text{their } I \, dx$ or $\frac{d}{dx} (y \times \text{their } I) = Q(x) \times \text{their } I$ A1: $\frac{y}{\cos x} = \int \ln x \, dx$ or $\frac{d}{dx} \left(\frac{y}{\cos x} \right) = \ln x$	M1A1
	$\frac{y}{\cos x} = x \ln x - x + C$	Attempts $\int \ln x \, dx$ by parts correctly (correct sign needed unless correct formula quoted and used).	M1
	$y = (x \ln x - x + C) \cos x$	Any equivalent with the constant correctly placed and “ $y = \dots$ ” must appear at some stage.	A1
			Total 8
	Note: Failure to divide by $\cos x$ at the start would mean that only the 3rd Method mark is available.		

Q6.

Question Number	Scheme	Notes	Marks
(a)	$(x^2 + 1) \frac{dy}{dx} + xy - x = 0$		
	$\frac{dy}{dx} + \frac{xy}{(1+x^2)} = \frac{x}{(1+x^2)}$	Correct form.	B1
	$I = e^{\int \frac{x}{1+x^2} dx} = e^{\frac{1}{2} \ln(1+x^2)} = (1+x^2)^{\frac{1}{2}}$	M1: $I = e^{\int \frac{x}{1+x^2} dx} = e^{k \ln(1+x^2)}$ where k is a constant. (Condone missing brackets around the $x^2 + 1$) A1: Correct integrating factor of $(1+x^2)^{\frac{1}{2}}$	M1A1
	$y(1+x^2)^{\frac{1}{2}} = \int \frac{x}{(1+x^2)^{\frac{1}{2}}} dx$	Uses their integration factor to reach the form $yI = \int QI dx$	M1
	$= (1+x^2)^{\frac{1}{2}} (+c)$	Correct integration (+ c not needed)	A1
	$y = 1 + c(1+x^2)^{-\frac{1}{2}}$ oe	Cao with the constant correctly placed. (The “ $y =$ ” must appear at some point)	A1
			(6)

Way 2	Alternative by separation of variables:		
	$\int \frac{dy}{1-y} = \int \frac{x}{x^2+1} dx$	Separates variables correctly	B1
	$\int \frac{x}{x^2+1} dx = \frac{1}{2} \ln(x^2+1)$	M1: $\int \frac{x}{x^2+1} dx = k \ln(x^2+1)$ where k is a constant. (Condone missing brackets around the $x^2 + 1$) A1: Correct integration $\frac{1}{2} \ln(x^2+1)$	M1A1
	$\int \frac{dy}{1-y} = -\ln(1-y)$	$\int \frac{dy}{1-y} = k \ln(1-y)$ or e.g. $\int \frac{dy}{y-1} = k \ln(y-1)$	M1
	$-\ln(1-y) = \frac{1}{2} \ln(x^2+1) (+c)$	Fully correct integration	A1
	$y = 1 + c(1+x^2)^{-\frac{1}{2}}$ oe	Cao and isw if necessary.	A1
			(6)
(b)	$2 = 1 + c(1+3^2)^{-\frac{1}{2}} \Rightarrow c = \dots$	Substitutes $x = 3$ and $y = 2$ and attempts to find a value for c .	M1
	$(y =) 1 + \sqrt{10}(1+x^2)^{-\frac{1}{2}}$ oe	Cao. (“ $y =$ ” not needed for this mark) and apply isw if necessary.	A1
			(2)
			Total 8

Q7.

Question Number	Scheme	Notes	Marks
	$(\cos x) \frac{dy}{dx} + (\sin x)y = 2 \cos^3 x \sin x - 3$		
(a)	$\frac{dy}{dx} + (\tan x)y = 2 \cos^2 x \sin x - 3 \sec x$	Attempt to divide through by $\cos x$. If the intention is not clear must see at least 2 terms divided by $\cos x$.	M1
	Integrating Factor: $I = e^{\int \tan x \, dx}$	$I = e^{\int \pm \text{their } P(x) \, (dx)}$ from $\frac{dy}{dx} + Py' = \dots$ Dependent on the first method mark. May be implied by use of $\sec x$ as the integrating factor.	dM1
	$I = \sec x$	$\frac{1}{\cos x}$ or $(\cos x)^{-1}$ or $\sec x$	A1
	$y \sec x = \int \sec x (2 \cos^2 x \sin x - 3 \sec x) (dx)$ or $\frac{d}{dx}(y \sec x) = \sec x (2 \cos^2 x \sin x - 3 \sec x)$ $y \times \text{their } I = \int Q(x) \times \text{their } I (dx)$ or $\frac{d}{dx}(y \times \text{their } I) = Q(x) \times \text{their } I$ If there is any doubt, must multiply at least one of their $Q(x)$ terms by I		M1
	$\int 2 \cos x \sin x \, dx = \sin^2 x$ or $-\cos^2 x$ or $-\frac{1}{2} \cos 2x$ Must follow the previous method mark		A1 M1 on ePEN
	$\int -3 \sec^2 x \, dx = -3 \tan x$ Must follow the previous method mark		A1
	Examples of a correct answer: $y = \cos x \sin^2 x - 3 \sin x + k \cos x$ or $y = -\frac{1}{2} \cos x \cos 2x - 3 \sin x + k \cos x$ or $y = -\cos^3 x - 3 \sin x + k \cos x$ Follow through their integration and their integrating factor but must be $y = \dots$ with the constant dealt with correctly and depends on the third method mark.		A1ft
			(7)

(b)	$3\sqrt{3} = \frac{1}{2} \cdot \left(\frac{\sqrt{3}}{2}\right)^2 - 3\frac{\sqrt{3}}{2} + k\frac{1}{2} \Rightarrow k = \left(9\sqrt{3} - \frac{3}{4}\right)$ $3\sqrt{3} = -\frac{1}{2} \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right) - 3\frac{\sqrt{3}}{2} + k\frac{1}{2} \Rightarrow k = \left(9\sqrt{3} - \frac{1}{4}\right)$ $3\sqrt{3} = \left(-\frac{1}{2}\right)^3 - 3\frac{\sqrt{3}}{2} + k\frac{1}{2} \Rightarrow k = \left(9\sqrt{3} + \frac{1}{4}\right)$ Substitutes the given conditions into their $y = f(x)$ and attempts to find their constant	M1	
	$k = 9\sqrt{3} - \frac{3}{4}$ or $9\sqrt{3} - \frac{1}{4}$ or $9\sqrt{3} + \frac{1}{4}$	Correct constant for their method	A1
	$y = \cos x \sin^2 x - 3 \sin x + \left(9\sqrt{3} - \frac{3}{4}\right) \cos x$ or $y = -\frac{1}{2} \cos x \cos 2x - 3 \sin x + \left(9\sqrt{3} - \frac{1}{4}\right) \cos x$ or $y = -\cos^3 x - 3 \sin x + \left(9\sqrt{3} + \frac{1}{4}\right) \cos x$ Or equivalent correct answer. Must be $y = \dots$		A1
			(3)
			Total 10