Questions

Q1.

(a) Use de Moivre's theorem to show that

$$\cos^5\theta \equiv p\cos 5\theta + q\cos 3\theta + r\cos\theta$$

where p, q and r are rational numbers to be found.

(6)

(b) Hence, showing all your working, find the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^5\theta \, d\theta$$

(4)

(Total for question = 10 marks)

Q2.

(a) Use de Moivre's theorem to show that

$$\cos 7\theta \equiv 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

(6)

(b) Hence find the four distinct roots of the equation

$$64x^7 - 112x^5 + 56x^3 - 7x + 1 = 0$$

giving your answers to 3 decimal places where necessary.

(5)

(Total for question = 11 marks)

Q3.

(a) Express the complex number $-8 - 8i\sqrt{3}$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$

(3)

(b) Hence solve the equation $z^4 = -8 - 8i\sqrt{3}$, giving the roots in the form $z = a + ib$ where a and b are real.

(5)

(Total for question = 8 marks)

Q4.

(a) Express the complex number $18\sqrt{3} - 18\text{i}$ in the form

$$r(\cos\theta + \text{i}\sin\theta) \qquad -\pi < \theta \leqslant \pi$$

(3)

(b) Solve the equation

$$z^4 = 18\sqrt{3} - 18\text{i}$$

giving your answers in the form $re^{\text{i}\theta}$ where $-\pi < \theta \leqslant \pi$

(5)

(Total for question = 8 marks)

Q5.

Solve the equation

$$z^5 = 32$$

Give your answers in the form $r(\cos\theta + \text{i}\sin\theta)$, where $r > 0$ and $0 \leq \theta < 2\pi$

(5)

(Total for question = 5 marks)

Q6.

(a) Use de Moivre's theorem to

(i) show that

$$\cos 5\theta \equiv \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

(ii) find an expression for $\sin 5\theta$ in terms of $\cos\theta$ and $\sin\theta$

(4)

(b) Hence show that

$$\tan 5\theta = \frac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1}$$

where $t = \tan\theta$ and $\cos 5\theta \neq 0$

(2)

(c) Hence find a quadratic equation whose roots are $\tan^2\frac{\pi}{5}$ and $\tan^2\frac{2\pi}{5}$

Give your answer in the form $ax^2 + bx + c = 0$ where a, b and c are integers to be found.

(4)

(d) Deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$

(2)

(Total for question = 12 marks)

Examiner's Report

Q1.

In (a) the most popular method was to use the expansion of $\left(z+\frac{1}{z}\right)^5$ and the use of $\frac{z^n+1}{z^n}=2\cos n\theta$ to achieve the required result. Errors in the expansion were rare but the omission of the factor of 2 in 2cos $n\theta$ was seen more often. An alternative method was to expand $(\cos\theta + i\sin\theta)^5$ and then compare the real part with $\cos 5\theta$. Although the expansion was usually completed successfully the method often failed when students were unable to deal with the resulting $\cos^3\theta$ term.
In (b) the integration was well attempted, even by those with wrong values for p, q and r. A correct final exact answer was seen frequently but students are advised to show the substitution of the limits into the result of integration rather than just writing down an answer.

Q2.

Part (a) required the use of de Moivre's theorem to prove the identity for cos 7θ in terms of powers of cos θ and was generally well-answered. The method of expanding $(\cos\theta + \mathrm{i}\sin\theta)^7$ was well known and although occasional slips, usually with signs, were seen, most students were able to correctly express cos 7θ in terms of cos θ and sin θ. The majority proceeded to replace $\sin^2\theta$ with $1 - \cos^2\theta$ but many students did not explicitly show use of the expansions of $(1 - \cos^2\theta)^2$ or $(1 - \cos^2\theta)^3$. A small number wrote down the given answer when it was not consistent with their previous work. Attempts using $\left(z+\frac{1}{z}\right)^7$ were rare and usually ended abruptly.

Part (b) required the use of part (a) to solve a heptic equation in x. Some students got confused with the variables here or thought that cos 7θ = 0 or +1 rather than –1. Most who had the correct equation were able to obtain at least one correct solution for 7θ although $\pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi$... was occasionally seen instead of $\pm\pi, \pm 3\pi, \pm 5\pi, \pm 7\pi$. A very common error was the failure to calculate the cosine of the angles after division by seven. Many merely offered the decimal versions of their angles in radians.

Q3.

This question involved finding the fourth roots of a complex number. In part (a) the correct value for r was almost always achieved but many had an incorrect strategy in obtaining the argument. Some gave a correct equivalent value for θ but it was not in the specified range. A very small number went straight to an $r(\cos\theta - \mathrm{i}\sin\theta)$ form. Candidates who obtained the incorrect argument in part (a) would be advised to sketch a diagram with the point correctly positioned so that they can interpret the argument correctly.

Part (b) saw good scoring on the whole. A few candidates did not introduce $\pm 2k\pi$ at any stage and could only obtain one root. Others divided their angle by four first and then introduced the $\pm 2k\pi$ while other candidates got into a muddle when attempting $\pm\frac{k\pi}{2}$ to their arguments. Those proceeding correctly were generally able to obtain all four roots in trigonometric form but occasional slips were made converting into a + ib form, for example writing things like $-1+\sqrt{3}$ and $-i-\sqrt{3}$ instead of $-i+\sqrt{3}$.

Q4.

Part (a) very was accessible to most candidates. They were able to find the modulus easily but a small minority did not find the correct principal argument and hence lost the final accuracy mark. Some final responses came with the trigonometric form incorrectly written as $36\left(\cos\frac{\pi}{6}-\sin\frac{\pi}{6}\right)$ which again lost the final accuracy mark.

Part (b) was also very well answered by the majority of candidates. Most were able to apply de Moivre's theorem and correctly apply a method to find the four roots, usually gaining all the marks. A common error was not finding $\sqrt[4]{36}$ correctly – 3 was seen far too frequently.

Q5.

This question required the fifth roots of 32 to be determined and was a good source of marks for most students, with many scoring the full five marks very confidently. However, some students could not use de Moivre's theorem correctly, leading to errors such as $z = 2(\cos 2k\pi + i\sin 2k\pi)$. Occasionally the incorrect $\arg(32) = \frac{\pi}{2}$ was used. A more common error was to provide the correct five solutions but with arguments of $-\frac{2\pi}{5}$ and $-\frac{4\pi}{5}$ used instead of $\frac{8\pi}{5}$ and $\frac{6\pi}{5}$ respectively. Some solutions were seen in the form $r(\cos\theta - i\sin\theta)$. The $k = 0$ solution of $2(\cos 0 + i\sin 0)$ or just $z = 2$ was occasionally missing. A very small minority of candidates gave answers in degrees.

Q6.

The final question on using de Moivre's theorem for trigonometric identities provided most students with marks in parts (a) and (b), but the last two question parts proved very discriminating. In part (a), the method of expanding $(\cos\theta + i\sin\theta)^5$ was well-known and often fully correct. Some students were clearly rushing their working here and left themselves more vulnerable to sign errors and other slips. Some expressions for $\sin 5\theta$ were offered without "i" being removed. Use of $\left(z+\frac{1}{z}\right)^5$ was not common and most attempts via this route became bogged down in awkward algebra.

The first mark in part (b) was widely scored but many were unable to achieve the printed result. Those who identified that the numerator and denominator had to be divided by $\cos^5\theta$ usually produced the answer with little effort. Others attempted to use various identities and other manipulations and found reaching the given answer elusive.

Many students offered no response to the final two parts. The key in part (c) was to take note of the "Hence". Solutions from multiplying out $\left(x-\tan^2\frac{\pi}{5}\right)\left(x-\tan^2\frac{2\pi}{5}\right)$ could receive no credit. Those that attempted to use the result from part (b) almost always arrived at the correct quartic equation from which they could usually derive the required quadratic.

Part (d) was a deduction and it was essential that use of the product of the roots from the quadratic was clearly evident. Attempts that purely consisted of $\tan^2\frac{\pi}{5}\tan^2\frac{2\pi}{5} = 5$ followed by the given answer could not score here. Acceptable evidence included an explanation or sight of $x_1x_2 = 5$, $\alpha\beta = 5$ or use of $\frac{c}{a}$. An alternative by calculating the exact roots as surds and then multiplying them together was also acceptable and a successful route for some students.

Mark Scheme

Q1.

Question Number	Scheme	Notes	Marks
(a) WAY 1	$\left(z+\frac{1}{z}\right)^5 = z^5+5z^3+10z+\frac{10}{z}+\frac{5}{z^3}+\frac{1}{z^5}$	M1: Attempt to expand $\left(z\pm\frac{1}{z}\right)^5$ A1: Correct expansion with correct powers of z.	M1A1
	$z=\cos\theta+\mathrm{i}\sin\theta \Rightarrow z+\frac{1}{z}=2\cos\theta$	May be implied	B1
	$=z^5+\frac{1}{z^5}+5\left(z^3+\frac{1}{z^3}\right)+10\left(z+\frac{1}{z}\right)=2\cos 5\theta+10\cos 3\theta+20\cos\theta$ Uses at least one of $z^5+\frac{1}{z^5}=2\cos 5\theta$ or $z^3+\frac{1}{z^3}=2\cos 3\theta$		M1
	$\left(z+\frac{1}{z}\right)^5=32\cos^5\theta$		B1
	$\cos^5\theta=\frac{1}{16}\cos 5\theta+\frac{5}{16}\cos 3\theta+\frac{5}{8}\cos\theta$	Correct expression	A1
			(6)
	WAY 2 (Using $\mathrm{e}^{\mathrm{i}\theta}$)		
	$\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}\right)^5=\mathrm{e}^{5\mathrm{i}\theta}+5\mathrm{e}^{3\mathrm{i}\theta}+10\mathrm{e}^{\mathrm{i}\theta}+10\mathrm{e}^{-\mathrm{i}\theta}+5\mathrm{e}^{-3\mathrm{i}\theta}+\mathrm{e}^{-5\mathrm{i}\theta}$	M1: Attempt to expand $\left(\mathrm{e}^{\mathrm{i}\theta}\pm\mathrm{e}^{-\mathrm{i}\theta}\right)^5$ A1: Correct expansion	M1A1
	$2\cos\theta=\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}$	May be implied	B1
	$=\mathrm{e}^{5\mathrm{i}\theta}+\mathrm{e}^{-5\mathrm{i}\theta}+5\left(\mathrm{e}^{3\mathrm{i}\theta}+\mathrm{e}^{-3\mathrm{i}\theta}\right)+10\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}\right)=2\cos 5\theta+10\cos 3\theta+20\cos\theta$ Uses one of $\mathrm{e}^{5\mathrm{i}\theta}+\mathrm{e}^{-5\mathrm{i}\theta}=2\cos 5\theta$ or $\mathrm{e}^{3\mathrm{i}\theta}+\mathrm{e}^{-3\mathrm{i}\theta}=2\cos 3\theta$		M1
	$\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}\right)^5=32\cos^5\theta$		B1
	$\cos^5\theta=\frac{1}{16}\cos 5\theta+\frac{5}{16}\cos 3\theta+\frac{5}{8}\cos\theta$	Correct expression	A1
	WAY 3 (Using De Moivre on cos 5θ and identity for cos 3θ)		
	$(\cos\theta+\mathrm{i}\sin\theta)^5=c^5+5\mathrm{i}c^4s+10c^3\mathrm{i}^2s^2+10c^2\mathrm{i}^3s^3+5c\mathrm{i}^4s^4+\mathrm{i}^5s^5$ M1: Attempts to expand. NB may only consider real parts here. A1: Correct real terms (may include i's) (Ignore imaginary parts for this mark)		M1A1
	$\cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$	Correct real terms with no i's	B1
	$=\cos^5\theta-10\cos^3\theta\left(1-\cos^2\theta\right)+5\cos\theta\left(1-\cos^2\theta\right)^2$	Uses $\sin^2\theta=1-\cos^2\theta$ to eliminate $\sin\theta$	M1
	$16\cos^5\theta=\cos 5\theta+20\cos^3\theta-5\cos\theta$		
	$\cos 3\theta=4\cos^3\theta-3\cos\theta$	Correct identity for cos 3θ	B1
	$16\cos^5\theta=\cos 5\theta+5\cos 3\theta+10\cos\theta$		
	$\cos^5\theta=\frac{1}{16}\cos 5\theta+\frac{5}{16}\cos 3\theta+\frac{5}{8}\cos\theta$	Correct expression	A1
			(6)

(b)	$\int\left(\frac{1}{16}\cos 5\theta+\frac{5}{16}\cos 3\theta+\frac{5}{8}\cos\theta\right)\mathrm{d}\theta=\frac{1}{80}\sin 5\theta+\frac{5}{48}\sin 3\theta+\frac{5}{8}\sin\theta$ M1: Attempt to integrate – Evidence of $\cos n\theta\to\pm\frac{1}{n}\sin n\theta$ where $n=5$ or 3 A1ft: Correct integration (ft their p, q, r)		M1A1ft
	$\left[\frac{1}{80}\sin 5\theta+\frac{5}{48}\sin 3\theta+\frac{5}{8}\sin\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\left(\frac{1}{80}\sin\frac{5\pi}{3}+\frac{5}{48}\sin\pi+\frac{5}{8}\sin\frac{\pi}{3}\right)-\left(\frac{1}{80}\sin\frac{5\pi}{6}+\frac{5}{48}\sin\frac{\pi}{2}+\frac{5}{8}\sin\frac{\pi}{6}\right)$ Substitutes the given limits into a changed function and subtracts the right way round. There should be evidence of the substitution of $\frac{\pi}{3}$ and $\frac{\pi}{6}$ into their changed function for at least 2 of their terms and subtraction. If there is no evidence of substitution and the answer is incorrect, score M0 here.		M1
	$=\frac{49\sqrt{3}}{160}-\frac{203}{480}$	Allow exact equivalents e.g. $=\frac{1}{16}\left(4.9\sqrt{3}-\frac{203}{30}\right)$	A1
	If they use the letters p, q and r or their values of p, q and r, even from no working, the M marks are available in (b) but <u>not</u> the A marks.		
			(4)
			Total 10

Q2.

Question Number	Scheme	Notes	Marks
(a)	$(\cos\theta+\mathrm{i}\sin\theta)^7=\cos^7\theta+\binom{7}{1}\cos^6\theta\mathrm{i}\sin\theta+\binom{7}{2}\cos^5\theta(\mathrm{i}\sin\theta)^2+\ldots$ Attempts to expand $(\cos\theta+i\sin\theta)^7$ including a recognisable attempt at binomial coefficients (May only see real terms)		M1
	$(\cos 7\theta=)c^7+{}^7\mathrm{C}_2c^5\mathrm{i}^2s^2+{}^7\mathrm{C}_4c^3\mathrm{i}^4s^4+{}^7\mathrm{C}_6c\mathrm{i}^6s^6$ Identifies real terms with $\cos 7\theta$		M1
	$=c^7-21c^5s^2+35c^3s^4-7cs^6$	Correct expression with coefficients evaluated and i's dealt with correctly	A1
	$=c^7-21c^5(1-c^2)+35c^3(1-c^2)^2-7c(1-c^2)^3$	Replaces $\sin^2\theta$ with $1-\cos^2\theta$ used anywhere in their expansion.	M1
	$=22c^7-21c^5+35c^3(1-2c^2+c^4)-7c(1-3c^2+3c^4-c^6)$ Applies the expansions of $(1-\cos^2\theta)^2$ and $(1-\cos^2\theta)^3$ to their expression		M1
	$=64\cos^7\theta-112\cos^5\theta+56\cos^3\theta-7\cos\theta$*	Correct expression obtained with no errors	A1
	Useful intermediate expression: $=22c^7-21c^5+35c^3-70c^5+35c^7-7c+21c^2-21c^5+7c^7$		
			(6)

	Alternative 1 for (a):	
	$\left(z+\frac{1}{z}\right)^7=z^7+\binom{7}{1}z^6\frac{1}{z}+\binom{7}{2}z^5\frac{1}{z^2}+\ldots$ Attempts to expand $\left(z+\frac{1}{z}\right)^7$ including binomial coefficients	M1
	$\left(z+\frac{1}{z}\right)^7=z^7+\frac{1}{z^7}+7\left(z^5+\frac{1}{z^5}\right)+21\left(z^3+\frac{1}{z^3}\right)+35\left(z+\frac{1}{z}\right)$ $(2\cos\theta)^7=2\cos 7\theta+7(2\cos 5\theta)+21(2\cos 3\theta)+35(2\cos\theta)$ M1: Uses $z^n+\frac{1}{z^n}=2\cos n\theta$ at least once (including $n=1$) A1: Correct expression in terms of cos	M1A1
	$128\cos^7\theta=2\cos 7\theta+14(16\cos^5\theta-20\cos^3\theta+5\cos\theta)+42(4\cos^3\theta-3\cos\theta)+70\cos\theta$ M1: Correct method to find $\cos 5\theta$ in terms of $\cos\theta$ and applies this to their expression M1: Correct method to find $\cos 3\theta$ in terms of $\cos\theta$ and applies this to their expression	M1M1
	$\cos 7\theta=64\cos^7\theta-112\cos^5\theta+56\cos^3\theta-7\cos\theta$*	A1

	Alternative 2 for (a):	
	$\left(z+\frac{1}{z}\right)^7=z^7+\binom{7}{1}z^6\frac{1}{z}+\binom{7}{2}z^5\frac{1}{z^2}+\ldots$ Attempts to expand $\left(z+\frac{1}{z}\right)^7$ including binomial coefficients	M1
	$\left(z+\frac{1}{z}\right)^7=z^7+\frac{1}{z^7}+7\left(z^5+\frac{1}{z^5}\right)+21\left(z^3+\frac{1}{z^3}\right)+35\left(z+\frac{1}{z}\right)$	
	$z^7+\frac{1}{z^7}=2\cos 7\theta=\left(z+\frac{1}{z}\right)^7-7\left(z^5+\frac{1}{z^5}\right)-21\left(z^3+\frac{1}{z^3}\right)-35\left(z+\frac{1}{z}\right)$ M1: Identifies that $z^7+\frac{1}{z^7}=2\cos 7\theta$ A1: Correct expression for $2\cos 7\theta$ in terms of z	M1A1
	$2\cos 7\theta=128\cos^7\theta-7\left(z^5+\frac{1}{z^5}\right)-21\left(z^3+\frac{1}{z^3}\right)-35\left(z+\frac{1}{z}\right)$ Starts the process of replacing $\left(z+\frac{1}{z}\right)^n$ with $(2\cos\theta)^n$	M1
	$=128\cos^7\theta-7(2\cos\theta)^5+14\left(z^3+\frac{1}{z^3}\right)+35\left(z+\frac{1}{z}\right)$	
	$=128\cos^7\theta-7(2\cos\theta)^5+14(2\cos\theta)^3-7\left(z+\frac{1}{z}\right)$	
	$=128\cos^7\theta-7(2\cos\theta)^5+14(2\cos\theta)^3-14\cos\theta$ Reaches an expression in terms of cos only	M1
	$\cos 7\theta=64\cos^7\theta-112\cos^5\theta+56\cos^3\theta-7\cos\theta$	A1

(b)	$\cos 7\theta+1=0\Rightarrow\cos 7\theta=-1$	$\cos 7\theta=-1$ ($\cos 7x=-1$ is B0)	B1
	$7\theta=\pm180, \pm540, \pm900, \pm1260, \ldots$ or $7\theta=\pm\pi, \pm3\pi, \pm5\pi, \pm7\pi, \ldots$	At least one correct value for 7θ. Condone the use of $7x$ here.	M1
	$\theta=\pm\frac{180}{7}, \pm\frac{540}{7}, \pm\frac{900}{7}, \pm\frac{1260}{7}, \ldots\Rightarrow\cos\theta=\ldots$ or $\theta=\pm\frac{\pi}{7}, \pm\frac{3\pi}{7}, \pm\frac{5\pi}{7}, \pm\frac{7\pi}{7}, \ldots\Rightarrow\cos\theta=\ldots$	Divides by 7 and attempts at least one value for $\cos\theta$. Condone the use of x for θ here.	M1
	$x=\cos\theta=0.901, 0.223, -1, -0.623$	A1: Awrt 2 correct values for x A1: Awrt all 4 x values correct and no extras	A1A1
			(5)
			Total 11

Q3.

Question Number	Scheme	Notes	Marks
	$-8-8\mathrm{i}\sqrt{3}$		
(a)	$r=\left(\sqrt{(8)^2+\left(8\sqrt{3}\right)^2}\right)=16$	16	B1
	$\theta=-\pi+\tan^{-1}\left(\frac{8\sqrt{3}}{8}\right)$ or e.g. $\theta=-\frac{\pi}{2}-\tan^{-1}\left(\frac{8}{8\sqrt{3}}\right)$	Correct strategy for the argument (may see correct value only from calculator) or can be implied by a correct argument not in range e.g. $\frac{4\pi}{3}$	M1
	$16\left(\cos\left(-\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{2\pi}{3}\right)\right)$	Correct form and correct values. Condone careless use of brackets as long as the intention is clear.	A1
			(3)
(b)	**Note that in (b) the candidate may legitimately work with e.g.** $16\left(\cos\left(\frac{2\pi}{3}\right)-\mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right)$		
	$z^4=16\left(\cos\left(2k\pi-\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(2k\pi-\frac{2\pi}{3}\right)\right)$ Correct use of $2k\pi$ seen or implied. This may be implied by the above expression seen or used with any non-zero integer value (positive or negative) for k		M1
	$z=16^{\frac{1}{4}}\left(\cos\left(\frac{2k\pi}{4}-\frac{2\pi}{12}\right)+\mathrm{i}\sin\left(\frac{2k\pi}{4}-\frac{2\pi}{12}\right)\right)$ Divide their angle by 4 after $\pm 2k\pi$ and find 4th root of their 16 **Dependent on the previous mark.**		dM1
	$z=2\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$, $2\left(\cos\left(\frac{\pi}{6}\right)-\mathrm{i}\sin\left(\frac{\pi}{6}\right)\right)$, $2\mathrm{e}^{-\frac{\pi}{6}\mathrm{i}}$, $-\mathrm{i}+\sqrt{3}$		
	$z=2\left(\cos\left(\frac{\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$, $2\left(\cos\left(\frac{5\pi}{3}\right)-\mathrm{i}\sin\left(\frac{5\pi}{3}\right)\right)$, $2\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}$, $1+\mathrm{i}\sqrt{3}$		
	$z=2\left(\cos\left(\frac{5\pi}{6}\right)+\mathrm{i}\sin\left(\frac{5\pi}{6}\right)\right)$, $2\left(\cos\left(\frac{7\pi}{6}\right)-\mathrm{i}\sin\left(\frac{7\pi}{6}\right)\right)$, $2\mathrm{e}^{\frac{5\pi}{6}\mathrm{i}}$, $\mathrm{i}-\sqrt{3}$		
	$z=2\left(\cos\left(-\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{2\pi}{3}\right)\right)$, $2\left(\cos\left(\frac{2\pi}{3}\right)-\mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right)$, $2\mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}}$, $-1-\mathrm{i}\sqrt{3}$		
	1 correct root in any form		A1
	All 4 roots correct in any form (allow equivalent arguments)		A1
	All 4 roots in correct surd form or exact equivalent in required form.		A1
	The A marks must follow correct work and should not be awarded if any correct roots are obtained fortuitously. So must follow a correct answer in part (a) although the argument may not be in the required range.		
			(5)
			Total 8

Special Case in (b):

Candidates who do not consider $\pm 2k\pi$ **at any stage** and apply De Moivre correctly to get

$$z^4=16\left(\cos\left(-\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{2\pi}{3}\right)\right)\Rightarrow z=2\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)\left(=-\mathrm{i}+\sqrt{3}\right)$$

Can score a B1 special case and this should be awarded as the first A mark on ePEN

Q4.

Question Number	Scheme	Marks
(a)	$\lvert 18\sqrt{3}-18\mathrm{i}\rvert = 18\sqrt{(3+1)} = 36$	B1
	$\tan\theta = \dfrac{-18}{18\sqrt{3}} \quad \theta = -\dfrac{\pi}{6}, \quad 18\sqrt{3}-18\mathrm{i} = 36\left(\cos\left(-\dfrac{\pi}{6}\right)+\mathrm{i}\sin\left(-\dfrac{\pi}{6}\right)\right)$	M1,A1cao (3)
(b)	$z^4 = 36\left(\cos-\dfrac{\pi}{6}+\mathrm{i}\sin-\dfrac{\pi}{6}\right) = 36\left(\cos\left(2k\pi-\dfrac{\pi}{6}\right)+\mathrm{i}\sin\left(2k\pi-\dfrac{\pi}{6}\right)\right)$	M1
	$z = \sqrt{6}\left(\cos\left(\dfrac{12k\pi-\pi}{24}\right)+\mathrm{i}\sin\left(\dfrac{12k\pi-\pi}{24}\right)\right)$	M1
	$k=0 \quad z_0 = \sqrt{6}\left(\cos\left(\dfrac{-\pi}{24}\right)+\mathrm{i}\sin\left(\dfrac{-\pi}{24}\right)\right) = \sqrt{6}\,\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{24}\right)}$	B1
	$k=1 \quad z_1 = \sqrt{6}\left(\cos\left(\dfrac{11\pi}{24}\right)+\mathrm{i}\sin\left(\dfrac{11\pi}{24}\right)\right) = \sqrt{6}\mathrm{e}^{\mathrm{i}\frac{11\pi}{24}}$	A1ft
	$k=2 \quad z_2 = \sqrt{6}\left(\cos\left(\dfrac{23\pi}{24}\right)+\mathrm{i}\sin\left(\dfrac{23\pi}{24}\right)\right) = \sqrt{6}\mathrm{e}^{\mathrm{i}\frac{23\pi}{24}}$	
	$k=-1 \quad z_3 = \sqrt{6}\left(\cos\left(-\dfrac{13\pi}{24}\right)+\mathrm{i}\sin\left(-\dfrac{13\pi}{24}\right)\right) = \sqrt{6}\mathrm{e}^{\mathrm{i}\left(-\frac{13\pi}{24}\right)}$	A1ft (5)
		[8]

(a)	
B1	Correct modulus
M1	Attempt argument using $\tan\theta = \dfrac{\pm 18}{18\sqrt{3}}$ or other valid method. Can be implied by $\theta = \pm\dfrac{\pi}{6}$
A1cao	Correct answer in the required form.
(b)	
M1	Valid method for generating at least 2 roots, rotation through $\dfrac{\pi}{2}$ accepted
M1	Apply de Moivre or use the rotation method
B1	Any one correct root
A1ft	Second root in required form
A1ft	All 4 roots in the required form Follow through their $\sqrt[4]{36}$ but 36 not acceptable.
NB	Argument in degrees – M1M1B1A0A0 (ie treat as mis-read) Incorrect argument: B0A1ftA1ft available Answers in $r(\cos\theta+\mathrm{i}\sin\theta)$ form – deduct final A marks

Q5.

<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td rowspan="5"></td><td>$2(\cos 0 + \mathrm{i}\sin 0)$ or 2</td><td>$(z =)\ 2$ or $(z =)\ 2(\cos 0 + \mathrm{i}\sin 0)$
or $2\cos 0 + \mathrm{i}\sin 0$ or $2 + 0\mathrm{i}$
Allow $2(\cos 0\pi + \mathrm{i}\sin 0\pi)$</td><td>B1</td></tr>
<tr><td>$2\left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)$</td><td>This answer in this form.
Do not allow e.g. $2\mathrm{e}^{\frac{2\pi}{5}\mathrm{i}}$ but allow $2\cos\frac{2\pi}{5} + 2\mathrm{i}\sin\frac{2\pi}{5}$</td><td>B1</td></tr>
<tr><td>$2\left(\cos\frac{2k\pi}{5} + \mathrm{i}\sin\frac{2k\pi}{5}\right), (k = 2, 3, 4)$</td><td>Attempts at least 2 more solutions whose arguments differ by $\frac{2\pi}{5}$. Allow this mark if the arguments are out of range. May be implied by their answers.</td><td>M1</td></tr>
<tr><td colspan="2">Note that this answer in general solution form can score full marks if correct i.e. the A marks below can be implied.
E.g. $z = 2\left(\cos\frac{2k\pi}{5} + \mathrm{i}\sin\frac{2k\pi}{5}\right), (k = 0, 1, 2, 3, 4)$ scores full marks</td><td></td></tr>
<tr><td>$2\left(\cos\frac{4\pi}{5} + \mathrm{i}\sin\frac{4\pi}{5}\right)$
$2\left(\cos\frac{6\pi}{5} + \mathrm{i}\sin\frac{6\pi}{5}\right)$
$2\left(\cos\frac{8\pi}{5} + \mathrm{i}\sin\frac{8\pi}{5}\right)$</td><td>A1: One further correct answer, allow the brackets to be expanded.

A1: All correct, allow the brackets to be expanded.</td><td>A1 A1</td></tr>
<tr><td colspan="4">Do not allow $2\left(\cos\frac{4\pi}{5} - \mathrm{i}\sin\frac{4\pi}{5}\right)$ or $2\left(\cos\left(-\frac{4\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{5}\right)\right)$ for $2\left(\cos\frac{6\pi}{5} + \mathrm{i}\sin\frac{6\pi}{5}\right)$
Do not allow $2\left(\cos\frac{2\pi}{5} - \mathrm{i}\sin\frac{2\pi}{5}\right)$ or $2\left(\cos\left(-\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{5}\right)\right)$ for $2\left(\cos\frac{8\pi}{5} + \mathrm{i}\sin\frac{8\pi}{5}\right)$</td></tr>
<tr><td colspan="4">Ignore answers outside the range.
For a fully correct solution that has extra solutions in range, deduct the final A mark.</td></tr>
<tr><td colspan="4">Answers in degrees: Penalise once the first time it occurs.
Answers in degrees are: 0, 72, 144, 216, 288</td></tr>
<tr><td></td><td></td><td></td><td>(5)</td></tr>
<tr><td></td><td></td><td></td><td>Total 5</td></tr>
</table>

Q6.

Question Number	Scheme	Notes	Marks
(a)(i)	$\cos 5\theta + \mathrm{i}\sin 5\theta = (c+\mathrm{i}s)^5 = c^5 + 5c^4\mathrm{i}s + 10c^3\mathrm{i}^2s^2 + 10c^2\mathrm{i}^3s^3 + 5c\mathrm{i}^4s^4 + \mathrm{i}^5s^5$ Attempts to expand $(c+\mathrm{i}s)^5$ including binomial coefficients (NB may only see real terms here)		M1
	$\cos 5\theta = \mathrm{Re}(c+\mathrm{i}s)^5 = c^5 + 10c^3\mathrm{i}^2s^2 + 5c\mathrm{i}^4s^4 = c^5 - 10c^3s^2 + 5cs^4$ Extracts real terms and uses $\mathrm{i}^2 = -1$ to eliminate i.		M1
	$\cos 5\theta \equiv \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ * Achieves the printed result with no errors seen.		A1*
	Alternative: $\left(z = \cos\theta + \mathrm{i}\sin\theta,\ z^{-1} = \cos\theta - \mathrm{i}\sin\theta,\ z^n = \cos n\theta + \mathrm{i}\sin n\theta\right)$ $\left(z+\frac{1}{z}\right)^5 = z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) + 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) + \frac{1}{z^5}$ $(2\cos\theta)^5 = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$ **M1**: Expands $\left(z+\frac{1}{z}\right)^5$ including binomial coefficients and uses $z^n + \frac{1}{z^n} = 2\cos n\theta$ at least once to obtain an equation in cos $\cos 5\theta = 16\cos^5\theta - 5\cos 3\theta - 10\cos\theta$ $\cos 3\theta = 4\cos^3\theta - 3\cos\theta \Rightarrow \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 15\cos\theta - 10\cos\theta$ **M1**: Uses **correct** identity for cos3θ to obtain $\cos 5\theta$ in terms of single angles $= \cos^5\theta + 15\cos\theta\left(1-\sin^2\theta\right)^2 - 20\cos^3\theta + 5\cos\theta$ $= \cos^5\theta - 10\cos\theta\sin^2\theta + 15\cos\theta\sin^4\theta$ $= \cos^5\theta + 5\cos\theta\sin^4\theta + 10\cos\theta\sin^2\theta\left(\sin^2\theta - 1\right)$ $\cos 5\theta \equiv \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ * **A1**: Achieves the printed result with no errors seen (may need careful checking)		
(ii)	$\sin 5\theta \equiv 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ This expression (or equivalent) with no i's seen. **Note that some candidates may re-start and expand here as above.**		B1
			(4)

(b)	$\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta} = \dfrac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$ Uses $\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta}$ and substitutes the results from part (a)	M1
	$= \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta} = \dfrac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1}$ * Achieves the printed result with no errors seen.	A1*
	Note that a minimum could be: $\tan 5\theta = \dfrac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta} = \dfrac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1}$ * **Note that some candidates may work backwards which is acceptable:** E.g. $\dfrac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1} = \dfrac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{5\tan^4\theta - 10\tan^2\theta + 1}$ $= \dfrac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta} = \tan 5\theta$	
		(2)

(c)	$\tan 5\theta = 0$ or $\dfrac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{5\tan^4\theta - 10\tan^2\theta + 1} = 0$ or $\dfrac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1} = 0$	Considers $\tan 5\theta = 0$. This may be implied by $\tan^5\theta - 10\tan^3\theta + 5\tan\theta = 0$ or $t^5 - 10t^3 + 5t = 0$ or $\tan^4\theta - 10\tan^2\theta + 5 = 0$ or $t^4 - 10t^2 + 5 = 0$	M1
	$\tan^5\theta - 10\tan^3\theta + 5\tan\theta = 0$ or $t^5 - 10t^3 + 5t = 0$	Equate numerator to 0 This may be implied by $\tan^4\theta - 10\tan^2\theta + 5 = 0$ or $t^4 - 10t^2 + 5 = 0$	M1
	$\tan^4\theta - 10\tan^2\theta + 5 = 0$ or $t^4 - 10t^2 + 5 = 0$	Correct quartic	A1
	$x^2 - 10x + 5 = 0$	$x^2 - 10x + 5 = 0$ or equivalent	A1
			(4)

(d)	Product of roots: $\tan^2\dfrac{\pi}{5}\tan^2\dfrac{2\pi}{5} = 5$ Or solves "$x^2 - 10x + 5 = 0$" and attempts to multiply roots together e.g. $x = \dfrac{10 \pm \sqrt{100 - 20}}{2} = 5 \pm 2\sqrt{5}$ and $\left(5 + 2\sqrt{5}\right)\left(5 - 2\sqrt{5}\right) = \ldots$	Must clearly state product of roots or e.g. $\alpha\beta = 5$ or $x_1x_2 = 5$ and uses their constant in (c) or solves their quadratic and attempts product of roots.	M1
	$\tan^2\dfrac{\pi}{5}\tan^2\dfrac{2\pi}{5} = 5 \Rightarrow \tan\dfrac{\pi}{5}\tan\dfrac{2\pi}{5} = \sqrt{5}$ *	Shows the given result with no errors.	A1
			(2)
			Total 12