Edexcel International A Level (IAL) Further Maths: Further Pure 1

Your notes

Proof by Induction

Contents

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Introduction to Proof by Induction

What is proof by induction?

- **Proof by induction** is a way of proving a **result is true for a set of integers** by showing that if it is **true for one integer then it is true for the next integer**
- It can be thought of as falling dominoes:
  - Assume one domino falls
    - The **assumption** step
  - Show that if this domino falls, the next domino falls
    - The **inductive** step
- If you want all dominoes to fall (from the beginning) then
  - Show also that the **first** domino falls
    - The **basic** step

What are the steps for proof by induction?

- **STEP 1**

  The basic step: Show the result is **true** for the **base case**
  - This is normally $\boldsymbol{n = 1}$ **or** $\boldsymbol{n = 0}$
- **STEP 2**
  The assumption step: Assume the result is true for $\boldsymbol{n = k}$ where k is some integer
  - There is nothing to do for this step apart from writing down the assumption
- **STEP 3**
  The inductive step: Use the **assumption** to show the result is true for $n = k + 1$
  - This involves investigating $n = k + 1$ and bringing in the $n = k$ assumption
- **STEP 4**
  The conclusion step: Explain in **words** how the above steps make the result **true** for **all** integers, using the following **two** sentences:
  - "**If** it is true for $n = k$, **then** it is true for $n = k + 1$."
  - "**As** it is true for $\boldsymbol{n = 1}$, the statement is true **for all** $n \in \mathbb{Z}^+$."

Examiner Tips and Tricks

Learn the exact wording from the conclusion above! You can lose the final mark if your conclusion does not make sense.

Your notes

Proof by Induction with Series

What are the steps for proof by induction with series?

- For example: prove that
  - It has a **left-hand** side, $LHS = \sum_{r=1}^{n} r^2$
  - and a **right-hand** side, $RHS = \frac{1}{6}n(n+1)(2n+1)$
- STEP 1

  The basic step: Show result is true for $\boldsymbol{n = 1}$
  - **Substitute** $n = 1$ into both sides **individually**

    $LHS = \sum_{r=1}^{1} r^2 = 1^2 = 1$

    $RHS = \frac{1}{6} \times 1 \times (1+1)(2 \times 1 + 1) = \frac{1}{6} \times 2 \times 3 = 1$
- STEP 2

  The assumption step: Assume the *LHS* and *RHS* are equal for some $\boldsymbol{n = k}$ where k is some integer
  - Replace n with k in the statement
  - Write: "Assume is true"
- STEP 3

  The inductive step: You need to **prove** that the *LHS* and *RHS* are equal for $n = k + 1$
  - **Start** with the ***LHS*** for $n = k + 1$
    - Take the **last** term out: $\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2$
    - **Substitute** in the **assumption** (STEP 2) :
    - $\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$

- **Factorise**:

$$\frac{1}{6}(k+1)[k(2k+1)+6(k+1)]$$

$$=\frac{1}{6}(k+1)[2k^2+7k+6]$$

$$=\frac{1}{6}(k+1)(k+2)(2k+3)$$

- **Check** if it is the **same** as the ***RHS*** for $n=k+1$

$$RHS=\frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$$

$$=\frac{1}{6}(k+1)(k+2)(2k+3)$$

- So ***LHS*** **=** ***RHS*** for $n=k+1$, as long as the **assumption** is true

- STEP 4

The conclusion step: Explain in **words** how the above steps make the result **true** for **all** integers, using the following **two** sentences:

- "**If** it is true for $n=k$, **then** it is true for $n=k+1$."
- "**As** it is true for $\mathbf{n=1}$, the statement is true **for all** $n\in\mathbb{Z}^+$."

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Worked Example

Prove by induction that for .

STE P 1: The basic step
Find the LHS when $n=1$

$$\sum_{r=1}^{1} r(r-3)=1(1-3)=-2$$

Find the RHS when $n=1$

$$\frac{1}{3}\times 1\times(1-4)(1+1)=\frac{1}{3}\times(-6)=-2$$

The LHS and RHS are equal, which shows the statement is true for $n=1$
State this

LHS = RHS, so it is true for $n=1$

STEP 2: The assumption step
Assume the statement is true for some $n=k$ where k is a positive integer

Assume that $\sum_{r=1}^{k} r(r-3)=\frac{1}{3}k(k-4)(k+1)$

STEP 3: The inductive step

Find the RHS when $n = k + 1$

$$\frac{1}{3}(k+1)(k+1-4)(k+1+1) = \frac{1}{3}(k+1)(k-3)(k+2)$$

Find the LHS when $n = k + 1$

$$\sum_{r=1}^{k+1} r(r-3)$$

Prove that the LHS when $n = k + 1$ equals the RHS when $n = k + 1$

Start by pulling out the final term in the sum

$$\sum_{r=1}^{k+1} r(r-3) = \sum_{r=1}^{k} r(r-3) + (k+1)(k+1-3)$$
$$= \sum_{r=1}^{k} r(r-3) + (k+1)(k-2)$$

Now use the assumption in STEP 2 to replace the sum from 1 to k with its answer

$$\sum_{r=1}^{k+1} r(r-3) = \frac{1}{3}k(k-4)(k+1) + (k+1)(k-2)$$

Now factorise the right-hand side

$$\sum_{r=1}^{k+1} r(r-3) = (k+1)\left[\frac{1}{3}k(k-4) + (k-2)\right]$$
$$= \frac{1}{3}(k+1)[k(k-4) + 3(k-2)]$$

Expand inside the square brackets then continue to factorise

$$\sum_{r=1}^{k+1} r(r-3) = \frac{1}{3}(k+1)[k^2 - 4k + 3k - 6]$$
$$= \frac{1}{3}(k+1)[k^2 - k - 6]$$
$$= \frac{1}{3}(k+1)(k-3)(k+2)$$

This is the same as the RHS when $n = k + 1$

The LHS and RHS are equal, which shows that the statement is true for $n = k + 1$

State this

LHS = RHS, so it is true for $n = k + 1$

STEP 4: The conclusion

We have just proved that if the case when $n = k$ is true, then the case when $n = k + 1$ is true

We have also shown that the case when $n = 1$ is true

Those two parts come together to mean that $n = 1$ is true, so $n = 2$ is true, so $n = 3$ is true, ... and so on

Your notes

If it is true for $n = k$, then it is true for $n = k + 1$.
As it is true for $n = 1$, the statement is true for all $n \in \mathbb{Z}^+$.

Your notes

Proof by Induction with Divisibility

What are the steps for proof by induction with divisibility?

- For example: prove that $f(n) = 4^n - 1$ is divisible by 3 for all positive integers *n*
  - Being **divisible** by 3 is the same as being a **multiple** of 3
- STEP 1

  The basic step: Show result is true for $\boldsymbol{n = 1}$
  - **Substitute** $n = 1$ into the function
  - $f(1) = 4^1 - 1 = 3$ which is divisible by 3
- STEP 2

  The assumption step: Assume $f(k) = 4^k - 1$ is divisible by 3 for some $\boldsymbol{n = k}$ where k is some integer
  - Replace n with k in the statement
  - Write "divisible by 3" as "$= 3m$" where m is a positive integer
  - So $f(k) = 4^k - 1 = 3m$ where $m \in \mathbb{Z}^+$
  - It helps to make 4^k the **subject**: $4^k = 1 + 3m$
- STEP 3

  The inductive step (method 1): You need to show that the $n = k + 1$ case is divisible by 3
  - Substitute $n = k + 1$ into the function
    - $f(k + 1) = 4^{k+1} - 1$
  - Use **index laws** to substitute in the **assumption** (STEP 2) $4^k = 1 + 3m$
    - $f(k + 1) = 4(4^k) - 1 = 4(1 + 3m) - 1 = 3(4m + 1)$ which is a multiple of 3
  - So $f(k + 1)$ is divisible by 3 (as long as the **assumption** is true)
- STEP 3

  The inductive step (method 2): An **alternative method** is to show that the **difference** $f(k + 1) - f(k)$ is a multiple of 3
  - i.e. $f(k + 1) - f(k) = 3 \times ...$
  - Making $f(k + 1)$ the subject gives $f(k + 1) = f(k) + 3 \times ...$
  - So $f(k + 1)$ is divisible by 3 (as long as the **assumption** is true, i.e. that $f(k)$ is divisible by 3)

- This method does not always work
  - When it does, a hint will be given in the question
- STEP 4

  The conclusion step: Explain in **words** how the above steps make the result **true** for **all** integers, using the following **two** sentences:

  - "**If** it is true for $n=k$, **then** it is true for $n=k+1$."
  - "**As** it is true for $\boldsymbol{n=1}$, the statement is true **for all** $n \in \mathbb{Z}^{+}$."

Your notes

Worked Example

Prove by induction that $\mathrm{f}(n)=6^{n}+13^{n+1}$ is divisible by 7 for all integers satisfying .

STE P 1: The basic step
Prove that it is true for $n=0$

$$\begin{aligned}\mathrm{f}(0)&=6^{0}+13^{0+1}\\&=1+13\\&=14\\&=7\times 2\end{aligned}$$

State that it is true for $n=0$

14 is divisible by 7, so it is true for $n=0$

STEP 2: The assumption step
Assume the statement is true for some $n=k$ where k is a positive integer

Assume that $\mathrm{f}(k)=6^{k}+13^{k+1}$ is divisible by 7

Replace "being divisible by 7" with being equal to $7m$, where m is a positive integer

$$6^{k}+13^{k+1}=7m$$

STEP 3: The inductive step
Substitute $n=k+1$ into the function

$$\begin{aligned}\mathrm{f}(k+1)&=6^{k+1}+13^{(k+1)+1}\\&=6^{k+1}+13^{k+2}\end{aligned}$$

Now make either 6^{k} or 13^{k+1} the subject of STEP 2 and substitute it in
It helps to use index laws first

$$\begin{aligned}\mathrm{f}(k+1)&=6^{k}\times 6+13^{k+2}\\&=\left(7m-13^{k+1}\right)\times 6+13^{k+2}\end{aligned}$$

Simplify the terms, using index laws to group the 13^{k+1} terms

$$\begin{aligned} f(k+1) &= 42m - 6 \times 13^{k+1} + 13^{k+2} \\ &= 42m - 6 \times 13^{k+1} + 13 \times 13^{k+1} \\ &= 42m + (-6+13) \times 13^{k+1} \\ &= 42m + 7 \times 13^{k+1} \end{aligned}$$

The goal is to show that this result is divisible by 7
Factorise out a 7 to show this

$$f(k+1) = 7 \times (6m + 13^{k+1})$$

State that it is true for $n = k + 1$

$f(k+1)$ is divisible by 7, so it is true for $n = k + 1$

STEP 4: The conclusion
We have just proved that if the case when $n = k$ is true, then the case when $n = k + 1$ is true
We have also shown that the case when $n = 0$ is true
Those two parts come together to mean that $n = 0$ is true, so $n = 1$ is true, so $n = 2$ is true, ... and so on

If it is true for $n = k$, then it is true for $n = k + 1$.
As it is true for $n = 0$, the statement is true for all integers $n \geq 0$.

Proof by Induction with Sequences

What are the steps for proof by induction with sequences?

- For example: prove that the sequence given **recursively** by $u_{n+1} = 3u_n + 4$ where $u_1 = 1$ has the **nth term formula** for $n \geq 1$
- **STEP 1**

  The basic step: Show result is true for $\boldsymbol{n = 1}$

  - **Substitute** $n = 1$ into the formulae **separately**
  - $u_1 = 1$ from the recursive formula
  - $u_1 = 3^1 - 2 = 1$ from the n^{th} term formula
- **STEP 2**

  The assumption step: Assume that the term u_k satisfies $u_k = 3^k - 2$ for some $\boldsymbol{n = k}$ where k is some integer

  - Replace n with k in the statement
  - The assumption is on the n^{th} term formula
    - (not on the recursive formula)
- **STEP 3**

The inductive step: Show that the n^{th} term formula is true for $n = k + 1$

- Use the recursive formula to write
- Substitute in the **assumption** (STEP 2) and simplify
  - $u_{k+1} = 3(3^k - 2) + 4 = 3^{k+1} - 2$
- **Check** this is the same as the *n*th term formula when $n = k + 1$
  - $u_{k+1} = 3^{k+1} - 2$
  - It is the same

- STEP 4

  The conclusion step: Explain in **words** how the above steps make the result **true** for **all** integers, using the following **two** sentences:

  - "**If** it is true for $n = k$, **then** it is true for $n = k + 1$."
  - "**As** it is true for $\boldsymbol{n = 1}$, the statement is true **for all** $n \in \mathbb{Z}^+$."

Your notes

Examiner Tips and Tricks

- If the recursive formula involves the previous **two** terms, then the basic step must be done for both $n = 1$ and $n = 2$.

Worked Example

A sequence is defined by $u_{n+1} = 5u_n - 4$ where $u_1 = 2$ and .

Prove by mathematical induction that , where .

STE P 1: The basic step
Find u_1 from the recursive sequence

$$u_1 = 2$$

Find u_1 from the n^{th} term formula

$$\begin{aligned} u_1 &= 5^{1-1} + 1 \\ &= 5^0 + 1 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

These two terms are equal
State that it is true for $n = 1$

It is true for $n = 1$

STEP 2: The assumption step
Assume the formula is true for some $n = k$ where k is a positive integer

$$\text{Assume that } u_k = 5^{k-1} + 1$$

STEP 3: The inductive step
Use the recursive sequence to write u_{k+1} in terms of u_k

$$u_{k+1} = 5u_k - 4$$

Replace u_k with the assumption in STEP 2 and simplify

$$\begin{aligned} u_{k+1} &= 5(5^{k-1} + 1) - 4 \\ &= 5^k + 5 - 4 \\ &= 5^k + 1 \end{aligned}$$

Check that this is the same as substituting $n = k + 1$ into the n^{th} term formula

$$\begin{aligned} u_{k+1} &= 5^{(k+1)-1} + 1 \\ &= 5^k + 1 \end{aligned}$$

State that the result is true for $n = k + 1$

It is true for $n = k + 1$

STEP 4: The conclusion
We have just proved that if the case when $n = k$ is true, then the case when $n = k + 1$ is true
We have also shown that the case when $n = 1$ is true
Those two parts come together to mean that $n = 1$ is true, so $n = 2$ is true, so $n = 3$ is true, ... and so on

If it is true for $n = k$, then it is true for $n = k + 1$.
As it is true for $n = 1$, the statement is true for all $n \in \mathbb{Z}^+$.

Proof by Induction with Matrices

What are the steps for proof by induction with matrices?

- For example: prove that $\mathbf{M}^n = \begin{pmatrix} 2^n & 0 \\ 1 - 2^n & 1 \end{pmatrix}$ where $\mathbf{M} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$ for all integers .
- **STEP 1**

  The basic step: Show result is true for $n = 1$

  - **Substitute** $n = 1$ into formula
  - $\mathbf{M}^1 = \begin{pmatrix} 2^1 & 0 \\ 1 - 2^1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = \mathbf{M}$
- **STEP 2**

Your notes

The assumption step: Assume $\mathbf{M}^k = \begin{pmatrix} 2^k & 0 \\ 1-2^k & 1 \end{pmatrix}$ is true for some $\boldsymbol{n = k}$ where k is some integer

- Replace n with k in the statement

- STEP 3

  The inductive step: You need to show that the $n = k + 1$ case is true

  - Use **index laws** to write $\mathbf{M}^{k+1} = \mathbf{M}\mathbf{M}^k$
  - Substitute in the **assumption**: $\mathbf{M}^{k+1} = \mathbf{M}\begin{pmatrix} 2^k & 0 \\ 1-2^k & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2^k & 0 \\ 1-2^k & 1 \end{pmatrix}$
  - **Multiply** the matrices
    - $\mathbf{M}^{k+1} = \begin{pmatrix} 2^{k+1} & 0 \\ 1-2\times 2^k & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 0 \\ 1-2^{k+1} & 1 \end{pmatrix}$
  - **Check** if it is the same as the formula with $n = k + 1$
    - $\mathbf{M}^{k+1} = \begin{pmatrix} 2^{k+1} & 0 \\ 1-2^{k+1} & 1 \end{pmatrix}$
    - It is the same

- STEP 4

  The conclusion step: Explain in **words** how the above steps make the result **true** for **all** integers, using the following **two** sentences:

  - "**If** it is true for $n = k$, **then** it is true for $n = k + 1$."
  - "**As** it is true for $\boldsymbol{n = 1}$, the statement is true **for all** $n \in \mathbb{Z}^+$."

Worked Example

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$.

Prove, using mathematical induction, that $\mathbf{M}^n = \begin{pmatrix} 2^n & 2(2^n - 1) \\ 0 & 1 \end{pmatrix}$ for all integers satisfying $n \geq 1$.

STE P 1: The basic step
Prove that it is true for $n = 1$

Your notes

$$\mathbf{M}^1 = \begin{pmatrix} 2^1 & 2(2^1 - 1) \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$
$$= \mathbf{M}$$

State that it is true for $n = 1$

It is true for $n = 1$

STEP 2: The assumption step
Assume the formula is true for some $n = k$ where k is a positive integer

Assume that $\mathbf{M}^k = \begin{pmatrix} 2^k & 2(2^k - 1) \\ 0 & 1 \end{pmatrix}$

STEP 3: The inductive step
Look at the power of $n = k + 1$

$$\mathbf{M}^{k+1}$$

Use index laws, then substitute in the assumption from STEP 2

$$\mathbf{M}^{k+1} = \mathbf{M}^k \mathbf{M}^1$$
$$= \begin{pmatrix} 2^k & 2(2^k - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

Use matrix multiplication to work out the right-hand side
Simplify the terms

$$\mathbf{M}^{k+1} = \begin{pmatrix} 2^k \times 2 + 0 & 2^k \times 2 + 2(2^{k+1} - 1) \times 1 \\ 0 + 0 & 0 + 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2^{k+1} & 2 \times 2^{k+1} - 2 \\ 0 & 1 \end{pmatrix}$$

Check that this is the same as substituting $n = k + 1$ into the formula for $\mathbf{M}^n$

$$\mathbf{M}^{k+1} = \begin{pmatrix} 2^{k+1} & 2(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2^{k+1} & 2 \times 2^{k+1} - 2 \\ 0 & 1 \end{pmatrix}$$

State that the result is true for $n = k + 1$

It is true for $n = k + 1$

STEP 4: The conclusion
We have just proved that if the case when $n = k$ is true, then the case when $n = k + 1$ is true
We have also shown that the case when $n = 1$ is true
Those two parts come together to mean that $n = 1$ is true, so $n = 2$ is true, so $n = 3$ is true, ... and so on

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If it is true for $n = k$, then it is true for $n = k + 1$.
As it is true for $n = 1$, the statement is true for all $n \in \mathbb{Z}^+$.

Your notes