Edexcel International A Level (IAL) Further Maths: Further Pure 1

Your notes

Series

Contents

Sums of Natural Numbers

What is sigma notation?

- **Sigma notation** represents **sums** as follows: $\sum_{r=1}^{5} r = 1+2+3+4+5$
    - r counts in **integers** from the **lower limit** to the **upper limit**
- It works for **functions** of r
    - $\sum_{r=1}^{5}(3r-1) = (3\times1-1)+(3\times2-1)+(3\times3-1)+\ldots+(3\times5-1)$
- The sum of the **first n terms** has an upper limit of n
    - $\sum_{r=1}^{n} r = 1+2+3+\ldots+n$

How do I write a series as the difference of two sums?

- When the **lower limit** is **not 1**, such as $\sum_{r=5}^{10} r = 5+6+7+\ldots+10$, you can write it as the **difference between two sums**:
    - $\sum_{r=5}^{10} r = \sum_{r=1}^{10} r - \sum_{r=1}^{4} r$
    - This is because $5+6+7+8+9+10 = (1+2+3+4+\ldots+10)-(1+2+3+4)$
        - The 1, 2, 3 and 4 **cancel out**, leaving 5, 6, ..., 10
    - Be careful: the **upper limit** of the second sum is **one less** than the **lower limit** of the original sum!

What properties of sigma notation do I need to know?

- You need to know the following rules:
    - $\sum_{r=1}^{n} k = k+k+\ldots+k = kn$
        - The **sum** of a **constant** is the constant $\times\ n$
    - $\sum_{r=1}^{n}(ar+b) = a\sum_{r=1}^{n} r + bn$
        - **Coefficients** of r can come **out** of the sum

- **Constants** are **multiplied** by n

What is the formula for the sum of the natural numbers?

- $\sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots + n$ is the sum of the **natural numbers**
  - Natural numbers are **positive integers**
- It has the **standard formula**:
  - $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$
  - You can work the formula out using **arithmetic series**
    - first term 1, common difference 1
- You may use the formula in calculations **without proof**

Examiner Tips and Tricks

The formula for the sum of natural numbers is **not given** in the Formulae Booklet!

Worked Example

A sum of n terms is given by $\sum_{r=1}^{n}(2r+1)$.

(a) Using any standard summation formulae, show that $\sum_{r=1}^{n}(2r+1) = n(n+A)$ where A is a positive integer to be found.

Use the rule that $\sum_{r=1}^{n}(ar+b) = a\sum_{r=1}^{n} r + bn$

$$\sum_{r=1}^{n}(2r+1) = 2\sum_{r=1}^{n} r + n$$

Substitute in the formula $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

$$\sum_{r=1}^{n}(2r+1) = 2\left(\frac{1}{2}n(n+1)\right) + n$$

Expand and simplify

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$$\sum_{r=1}^{n}(2r+1)=n(n+1)+n$$
$$=n^2+2n$$

Factorise the right-hand side

$$\sum_{r=1}^{n}(2r+1)=n(n+2)$$

Check it has the right form, $n(n+A)$

$$\sum_{r=1}^{n}(2r+1)=n(n+2) \text{ where } A=2$$

(b) Hence find the sum of the odd numbers from 3 to 81.

It often helps to write out the first few terms and the last term of the series

$$\sum_{r=1}^{n}(2r+1)=(2\times1+1)+(2\times2+1)+(2\times3+1)+\ldots+(2\times n+1)$$

Simplify these terms

$$\sum_{r=1}^{n}(2r+1)=3+5+7+\ldots+(2n+1)$$

This is the sum of the odd numbers from 3 to $(2n+1)$
The question wants the sum of odd numbers from 3 to 81
Use the last term to find n

$$2n+1=81$$
$$2n=80$$
$$n=40$$

The sum of the first 40 terms gives the sum of odd numbers from 3 to 81
Substitute $n=40$ into the formula in part (a)

$$40\times(40+2)$$

1680

Sums of Squares

What is the formula for the sum of the squares of the natural numbers?

- $\sum_{r=1}^{n} r^2=1^2+2^2+3^2+\ldots+n^2=1+4+9+\ldots+n^2$ is the sum of the **squares** of the **natural numbers**
- It has it the **standard formula**:

- $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$
  - You may use the formula in calculations **without proof**
- Note that $\sum_{r=1}^{n}(ar^2+b) = a\sum_{r=1}^{n} r^2 + bn$
  - **Coefficients** of r^2 can come **out** of the sum
  - **Constants** are **multiplied** by n

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Examiner Tips and Tricks

This formula is given in the Formulae Booklet.

Worked Example

(a) Evaluate $\sum_{r=1}^{30} r^2$

Evaluate means "find the value of"
This is the sum of the first 30 square numbers

Substitute $n = 30$ into the formula $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$$\sum_{r=1}^{30} r^2 = \frac{1}{6}\times 30\times(30+1)(2\times 30+1)$$

$$= \frac{1}{6}\times 30\times 31\times 61$$

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(b) Show that $\sum_{r=m+1}^{2m} r^2 = \frac{1}{6}m(Pm+1)(Qm+1)$ where $P < Q$ and where P and Q are prime numbers to be found.

The lower limit is not 1, so write it as the difference between two sums

It can help to imagine simpler numbers, $\sum_{r=5}^{10} r^2 = \sum_{r=1}^{10} r^2 - \sum_{r=1}^{4} r^2$

$$\sum_{r=m+1}^{2m} r^2 = \sum_{r=1}^{2m} r^2 - \sum_{r=1}^{m} r^2$$

Now use the formula $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ for each sum on the right

$$\sum_{r=m+1}^{2m} r^2 = \frac{1}{6}(2m)(2m+1)(4m+1) - \frac{1}{6}m(m+1)(2m+1)$$

Factorise out $\frac{1}{6}$, m and $(2m+1)$ then simplify

$$\sum_{r=m+1}^{2m} r^2 = \frac{1}{6}m(2m+1)[2(4m+1)-(m+1)]$$
$$= \frac{1}{6}m(2m+1)(7m+1)$$

Check this is in the right form
2 and 7 are prime numbers and 2 < 7

$$\sum_{r=m+1}^{2m} r^2 = \frac{1}{6}m(2m+1)(7m+1) \text{ where } P=2 \text{ and } Q=7$$

Sums of Cubes

What is the formula for the sum of the cubes of the natural numbers?

- $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = 1 + 8 + 27 + \ldots + n^3$ is the sum of the **cubes** of the **natural numbers**
- It has it the **standard formula**:
  - $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$
  - You may use the formula in calculations **without proof**
  - It is the **square** of the **formula** for the **sum** of the **natural numbers**
    - $\left(\sum_{r=1}^{n} r\right)^2 = \left(\frac{1}{2}n(n+1)\right)^2 = \frac{1}{4}n^2(n+1)^2$
- Note that $\sum_{r=1}^{n}(ar^3+b) = a\sum_{r=1}^{n} r^3 + bn$
  - **Coefficients** of r^3 can come **out** of the sum
  - **Constants** are **multiplied** by n

Examiner Tips and Tricks

This formula is given in the Formulae Booklet.

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Worked Example

(a) Find $1^3 + 2^3 + 3^3 + \ldots + 24^3$

This is the sum of the first 24 cube numbers

Substitute $n = 24$ into the formula $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

$$\sum_{r=1}^{24} r^3 = \frac{1}{4} \times 24^2 \times (24+1)^2$$
$$= \frac{1}{4} \times 24^2 \times 25^2$$

90000

(b) Find and simplify a formula for the sum of the first n even cube numbers.

The sum of even cube numbers is $2^3 + 4^3 + 6^3 + \ldots$
The first 5 would end on 10^3
The first n would end on $(2n)^3$

$$2^3 + 4^3 + 6^3 + \ldots + (2n)^3$$

The standard formula is for $\sum_{r=1}^{n} r^3$
To get just the even numbers, replace r with $2r$

$$\sum_{r=1}^{n} (2r)^3$$

Cube the expression and use $\sum_{r=1}^{n} ar^3 = a\sum_{r=1}^{n} r^3$

$$\sum_{r=1}^{n} (2r)^3 = \sum_{r=1}^{n} 8r^3$$
$$= 8\sum_{r=1}^{n} r^3$$

Now substitute in the standard formula $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ and simplify

$$\sum_{r=1}^{n} (2r)^3 = 8\left(\frac{1}{4}n^2(n+1)^2\right)$$
$$= 2n^2(n+1)^2$$

$2n^2(n+1)^2$

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Combinations of Series

How do I find combinations of series?

- Use the **three formulae**:
  - $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$
  - $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$
  - $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$
- **Coefficients** of r, r^2 and r^3 can come out of the sum
  - $\sum_{r=1}^{n}(ar^3 + br^2 + cr + d) = a\sum_{r=1}^{n} r^3 + b\sum_{r=1}^{n} r^2 + c\sum_{r=1}^{n} r + dn$
    - **Constants** are **multiplied** by n
  - You may have to **expand** expressions in r
    - $\sum_{r=1}^{n}(r+3)(r+4) = \sum_{r=1}^{n}(r^2 + 7r + 12)$
- It is often possible to **factorise final answers**, in particular taking out:
  - fractional coefficients
  - n
  - $(n+1)$
- You may be required to write series as the **difference of two sums**
  - $\sum_{r=5}^{10} r = \sum_{r=1}^{10} r - \sum_{r=1}^{4} r$

Examiner Tips and Tricks

Exam questions often refer to these three formulae as the "standard summation formulae".

Worked Example

(a) Using standard summation formulae, show that

$$\sum_{r=1}^{n}(r-1)r(r+1)=\frac{1}{4}(n-1)n(n+1)(n+2)$$

Expand and simplify the left-hand side

$$\begin{aligned}(r-1)r(r+1)&=r(r^2-1)\\&=r^3-r\end{aligned}$$

Substitute in the formulae $\sum_{r=1}^{n}r^3=\frac{1}{4}n^2(n+1)^2$ and $\sum_{r=1}^{n}r=\frac{1}{2}n(n+1)$

$$\begin{aligned}\sum_{r=1}^{n}(r^3-r)&=\sum_{r=1}^{n}r^3-\sum_{r=1}^{n}r\\&=\frac{1}{4}n^2(n+1)^2-\frac{1}{2}n(n+1)\end{aligned}$$

Factorise out the quarter, n and $(n+1)$ then simplify

$$\begin{aligned}\sum_{r=1}^{n}(r^3-r)&=\frac{1}{4}n(n+1)[n(n+1)-2]\\&=\frac{1}{4}n(n+1)(n^2+n-2)\end{aligned}$$

Factorise the quadratic on the right-hand side

$$\sum_{r=1}^{n}(r^3-r)==\frac{1}{4}n(n+1)(n+2)(n-1)$$

Rearrange into the form asked for by the question

$$\sum_{r=1}^{n}(r-1)r(r+1)=\frac{1}{4}(n-1)n(n+1)(n+2)$$

(b) Hence find the value of

$$0\times1\times2\ +\ 1\times2\times3\ +\ 2\times3\times4\ +\ldots+\ 20\times21\times22$$

It helps to write out the first few terms in the series $\sum_{r=1}^{n}(r-1)r(r+1)$

$$\begin{aligned}&(1-1)\times1\times(1+1)\ +\ (2-1)\times2\times(2+1)\ +\ (3-1)\times3\times(3+1)\ +\ldots\\&=0\times1\times2\ +\ 1\times2\times3\ +\ 2\times3\times4\ +\ldots\end{aligned}$$

The question stops at 20 x 21 x 22

This is actually the 21st term (not the 20th term)

To find the sum of the first 21 terms, substitute $n=21$ into part (a)

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$$\sum_{r=1}^{21}(r-1)r(r+1)=\frac{1}{4}(21-1)\times 21\times(21+1)(21+2)$$
$$=\frac{1}{4}\times 20\times 21\times 22\times 23$$

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(c) If $\sum_{r=1}^{n}(r-1)r(r+1)=10\sum_{r=1}^{n}r^2$, find n.

Substitute the answer from part (a) into the left-hand side

Substitute the standard formula $\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1)$ into the right-hand side

$$\frac{1}{4}(n-1)n(n+1)(n+2)=10\times\frac{1}{6}n(n+1)(2n+1)$$

Cancel the terms n and $(n+1)$ from both sides
It also helps to multiply both sides by 12 to remove fractions

$$3(n-1)(n+2)=20(2n+1)$$

Expand, then form and solve a quadratic in

$$3(n^2+n-2)=40n+20$$
$$3n^2-37n-26=0$$
$$(3n+2)(n-13)=0$$
$$n=-\frac{2}{3}\quad\text{or}\quad n=13$$

n must be a positive integer (as it is used in sigma notation, $\sum_{r=1}^{n}$)

$n=13$