Edexcel International A Level (IAL) Further Maths: Further Pure 1

Your notes

Transformations using Matrices

Contents

Your notes

Transforming Points using Matrices

How do I transform a point using a matrix?

- A point (x, y) in a 2D plane can be **transformed** (**mapped**) on to another point (x', y') by a 2 × 2 **matrix**, $\mathbf{M}$
  - This is called a **linear transformation**
    - (x, y) is the **object** and (x', y') is the **image**
- The **coordinates** of the image point can be found using **matrix multiplication**
- To transform (x, y) by the matrix
  - Write (x, y) as a **column vector,**
  - Use **matrix multiplication** to work out , which gives
  - **Write down** the image point **coordinates**, (x', y')
- If given image coordinates, (x', y'), and asked to find original coordinates
  - use **inverse** matrices
  - $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix}$

How do I transform a set of vertices?

- **Vertices** of a **shape** can be transformed **one-by-one** using the method above
  - This gives the vertices of the **new** shape
    - **Straight lines** between the vertices will still be straight lines in the new shape
  - Sometimes the order (**clockwise** or **anticlockwise**) of the vertices is **reversed**
    - This is also called changing the **sense** of the vertices
- An **alternative method** is to stack **column vectors** of **coordinates** into a **vertex matrix**
  - $(1, 2)$, $(3, 4)$, $(5, 6)$ and $(0, 10)$ become $\begin{pmatrix} 1 & 3 & 5 & 0 \\ 2 & 4 & 6 & 10 \end{pmatrix}$
  - Then **multiply** the **matrix** $\mathbf{M}$ by the **vertex matrix** above
  - The **new columns** represent the corresponding **image coordinates**

Worked Example

The matrix $\mathbf{M}$ is given by .
A triangle has coordinates $(1, 0)$, $(2, 3)$ and $(-1, 5)$.

Work out the coordinates of the triangle after the transformation represented by the matrix $\mathbf{M}$ has been applied.

Multiply the matrix $\mathbf{M}$ by each set **of coordinates, written as a column vectors**

$$\begin{pmatrix} 4 & 5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4\times 1 \ + \ 5\times 0 \\ 1\times 1 \ + \ -2\times 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4\times 2 \ + \ 5\times 3 \\ 1\times 2 \ + \ -2\times 3 \end{pmatrix} = \begin{pmatrix} 23 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 4\times(-1) \ + \ 5\times 5 \\ 1\times(-1) \ + \ -2\times 5 \end{pmatrix} = \begin{pmatrix} 21 \\ -11 \end{pmatrix}$$

Rewrite the answers as coordinates

$(4, 1)$, $(23, -4)$ and $(21, -11)$

Alternatively, you can multiply $\mathbf{M}$ by a vertex mat**rix:**

$$\begin{pmatrix} 4 & 5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 23 & 21 \\ 1 & -4 & -11 \end{pmatrix}$$

Determinants as Area Scale Factors

How are 2x2 determinants and areas related?

- When **transforming** vertices of an **object** using the **matrix** $\mathbf{M}$ to give vertices of the **image**
  - the **magnitude** of the **determinant** of $\mathbf{M}$ is the **area scale factor** of **enlargement** from the object to the image
  - $|\det(\mathbf{M})|$ is the area scale factor
- For example, if a shape is transformed by $\mathbf{M} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ then
  - $\det(\mathbf{M}) = 1\times 4 - 2\times 3 = -2$
  - so $|\det(\mathbf{M})| = |-2| = 2$
    - This transformation **doubles** the area of any shape
- A **negative determinant** means the **sense** (**clockwise** or **anticlockwise**) of the vertices is **reversed**

Examiner Tips and Tricks

Your notes

Remember the modulus signs around the determinant as this can often lead to two solutions in algebraic questions.

Your notes

Worked Example

A transformation, represented by the matrix $\mathbf{M} = \begin{pmatrix} 3 & 2 \\ 4 & k+1 \end{pmatrix}$, where k is a constant, maps a quadrilateral of area 45 square units to a quadrilateral of area 450 square units.

Find the possible values of k.

Find the scale factor of enlargement of the area from 45 to 450

Area scale factor is 10

The magnitude of the determinant is the area scale factor
Start by finding and simplifying the determinant of $\mathbf{M}$

$$\begin{aligned} \det(\mathbf{M}) &= 3(k+1) - 2 \times 4 \\ &= 3k + 3 - 8 \\ &= 3k - 5 \end{aligned}$$

Set the magnitude of the determinant equal to 10

$$|3k - 5| = 10$$

This means $3k - 5$ is either equal to 10 or -10
Solve each equation

$$\begin{aligned} 3k - 5 &= 10 \\ 3k &= 15 \\ k &= 5 \end{aligned}$$

or

$$\begin{aligned} 3k - 5 &= -10 \\ 3k &= -5 \\ k &= -\frac{5}{3} \end{aligned}$$

Write out the two answers

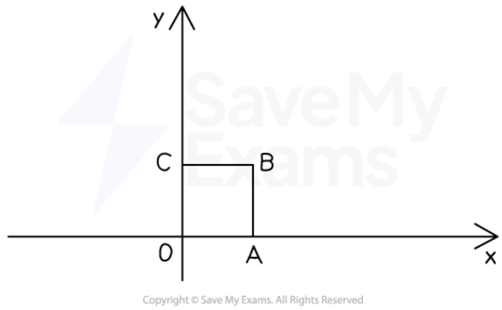
$$k = 5 \ \text{ or } \ k = -\frac{5}{3}$$

If the question had said "where k is an integer" then only $k = 5$ would be accepted

Reflection Matrices

How do I find reflection matrices?

- Imagine the **unit** square *OABC*
  - It has a side-length **1 unit**
  - *O* is the origin



- The **coordinates** of ***A*** **and C** as **column vectors** are
  - and
- Under a **reflection** about an **axis or** $y = \pm x$, *A* moves to *A*' and C moves to C'
  - The **matrix,** $\mathbf{M}$ representing this reflection is
  - ***A*' and C'** are column vectors of their **new positions**
    - The points **O and *B*** are **not needed**, as we can draw the reflected square using just *A*' and C' (*O* won't move)
- For example:
  - To find the matrix representing a **reflection about the** x**-axis**
    - *A* stays where it is, so
    - *C* goes to (on the negative y-axis)
    -
  - To find the matrix representing a **reflection in the line** $y = x$
    - *A* goes to (on the positive *y*-axis)
    - C goes to (on the positive *x*-axis)
    -
    - (This is not the same as the identity matrix, as the 1s are on the wrong diagonal)

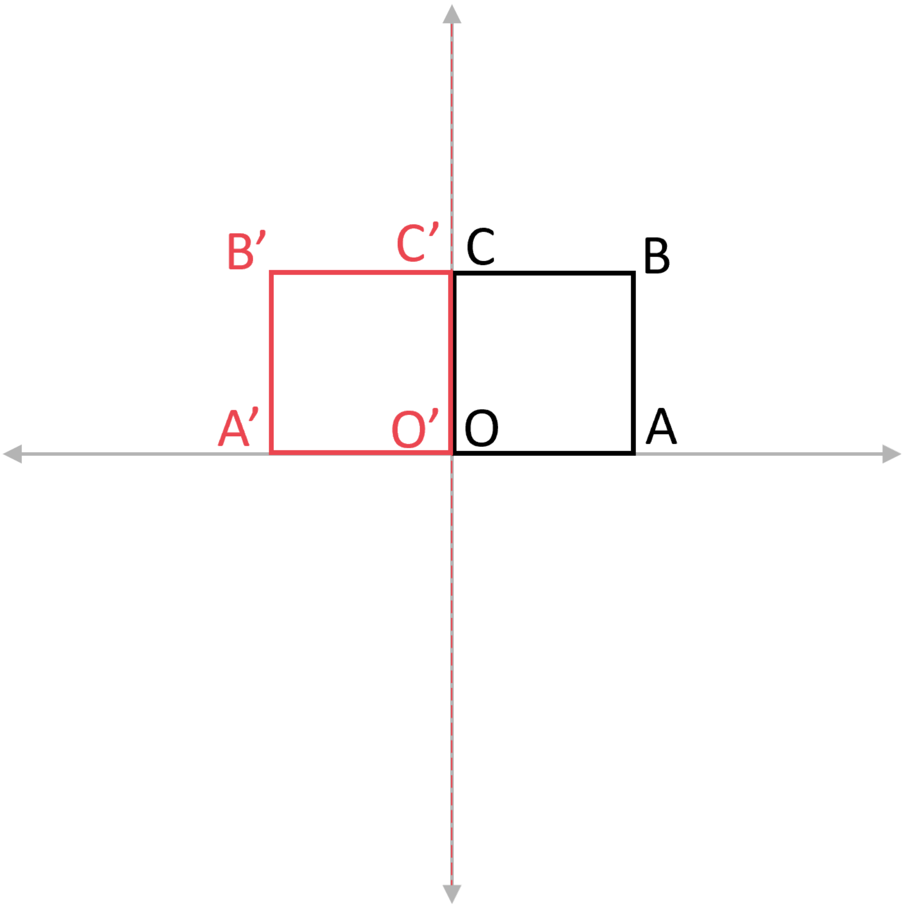
Your notes

Worked Example

(a) The matrix $\mathbf{M}$ represents a reflection in the y-axis.

Work out $\mathbf{M}$.

Consider how the points *A* and C on the unit square are transformed by a reflection in the y-axis



The point *A* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ moves to *A*' $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

The point C $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ remains in the same place

The transformation matrix is given by $\mathbf{M} = \left(A' \mid C' \right)$

$$\mathbf{M} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Your notes

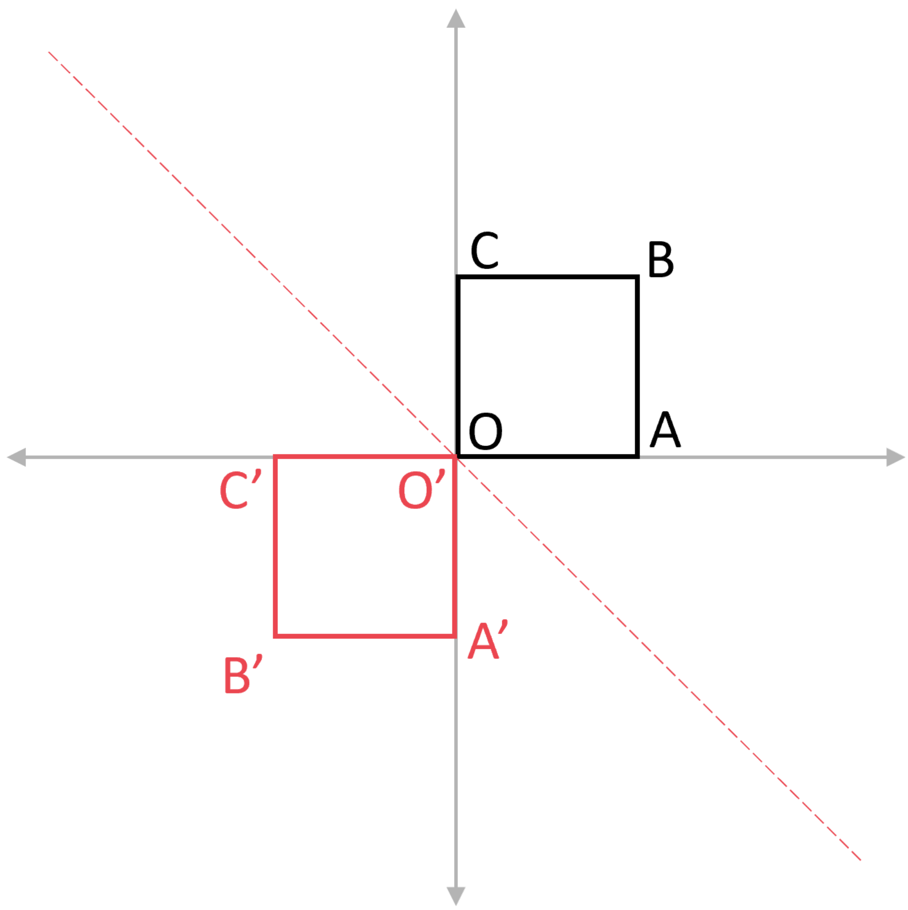
(b) Describe fully the transformation represented by the matrix $\mathbf{N} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

Consider how the points *A* and C on the unit square are transformed

The point *A* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ moves to *A'* $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

The point C $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ moves to C' $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

It helps to draw a picture of the unit square being transformed with vertices clearly labelled



This transformation could be a rotation of 180° about O or a reflection in $y = -x$
The vertices *A'* and *C'* are in the correct places for a reflection, but not a rotation

The matrix N represents a reflection in the line $y = -x$

Enlargement & Stretch Matrices

Your notes

Which matrix represents an enlargement?

- The matrix represents an **enlargement** of **scale factor** k about the **origin, O**
  - This is the same as
    - is the identity matrix

Which matrix represents a stretch?

- The matrix $\mathbf{M} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ represents a **stretch parallel** to the x**-axis** of **scale factor** a
  - The point (x, y) becomes (ax, y)
- The matrix $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ represents a **stretch parallel** to the y**-axis** of **scale factor** b
  - The point (x, y) becomes (x, by)
- The matrix $\mathbf{M} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ represents a **combined stretch** of **scale factor** a **parallel** to the x**-axis** and **scale factor** b **parallel** to the y**-axis**
  - If $a = b$, the combined stretch is an **enlargement**

Examiner Tips and Tricks

Use phrases like "parallel to the x-axis" or "parallel to the y-axis" to describe stretches (not "left" or "up"!)

Worked Example

A transformation is represented by the matrix $\mathbf{M} = \begin{pmatrix} 3+p & 0 \\ 0 & 3-p \end{pmatrix}$.

Describe fully the transformation in each of the following cases:

(a) $p = 0$

Substitute in $p = 0$

$$\mathbf{M} = \begin{pmatrix} 3+0 & 0 \\ 0 & 3-0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

This has the form $\mathbf{M} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ where $k = 3$

$\mathbf{M}$ represents an enlargement of scale factor 3 about the origin

You must give its scale factor and centre of enlargement

(b) $p = 2$

Substitute in $p = 2$

$$\mathbf{M} = \begin{pmatrix} 3+2 & 0 \\ 0 & 3-2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$$

This has the form $\mathbf{M} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ where $a = 5$

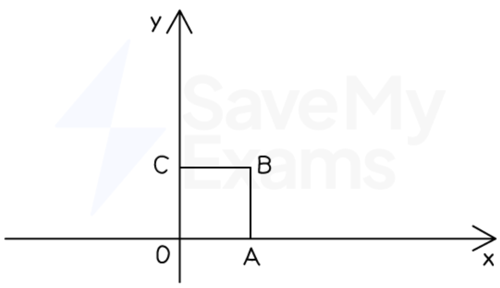
$\mathbf{M}$ represents a stretch of scale factor 5 parallel to the x-axis

You must give its scale factor and direction

Rotation Matrices

How do I find matrices for rotations by multiples of 90°?

- Imagine the **unit** square *OABC*
  - It has a side-length of **1 unit**
  - *O* is the origin



- The **coordinates** of *A* and *C* as **column vectors** are
  - and
- Under a **rotation about the origin**, *A* moves to *A*' and *C* moves to *C*'
  - The **matrix,** $\mathbf{M}$ representing this rotation is
  - *A*' and *C*' are **column vectors** of their new positions
    - The points *O* and *B* are not needed, as we can draw the rotated square using just *A*' and *C*' (*O* won't move)
- For example:
  - To find the matrix representing a **rotation of 90° anticlockwise** about the origin

- A goes to (on the positive y-axis)
- C goes to (on the negative x-axis)
-
- To find the matrix representing a **rotation of 180°** about the origin
    - A goes to (on the negative x-axis)
    - C goes to (on the negative y-axis)
    -
    - This is the same as where $\mathbf{I}$ is the identity matrix

Examiner Tips and Tricks

Students often confuse rotations of 180° with reflections in the lines $y = \pm x$.

How do I find matrices for rotations by any angle?

- A **rotation anticlockwise** by any **angle**, θ, about the **origin** is represented by the matrix:
    - $\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
        - θ can be in degrees or **radians**
- A **negative** value of θ represents a **clockwise** rotation
    - Remember that $\sin(-\theta) = -\sin\theta$ but that $\cos(-\theta) = \cos\theta$
- You can **substitute** in multiples of 90° to get the matrices above
- You may be required to recognise **common angles** from their ratios
    - For example, $\frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$

Examiner Tips and Tricks

You are given the rotation matrix in the Formulae Booklet.

Worked Example

Your notes

Your notes

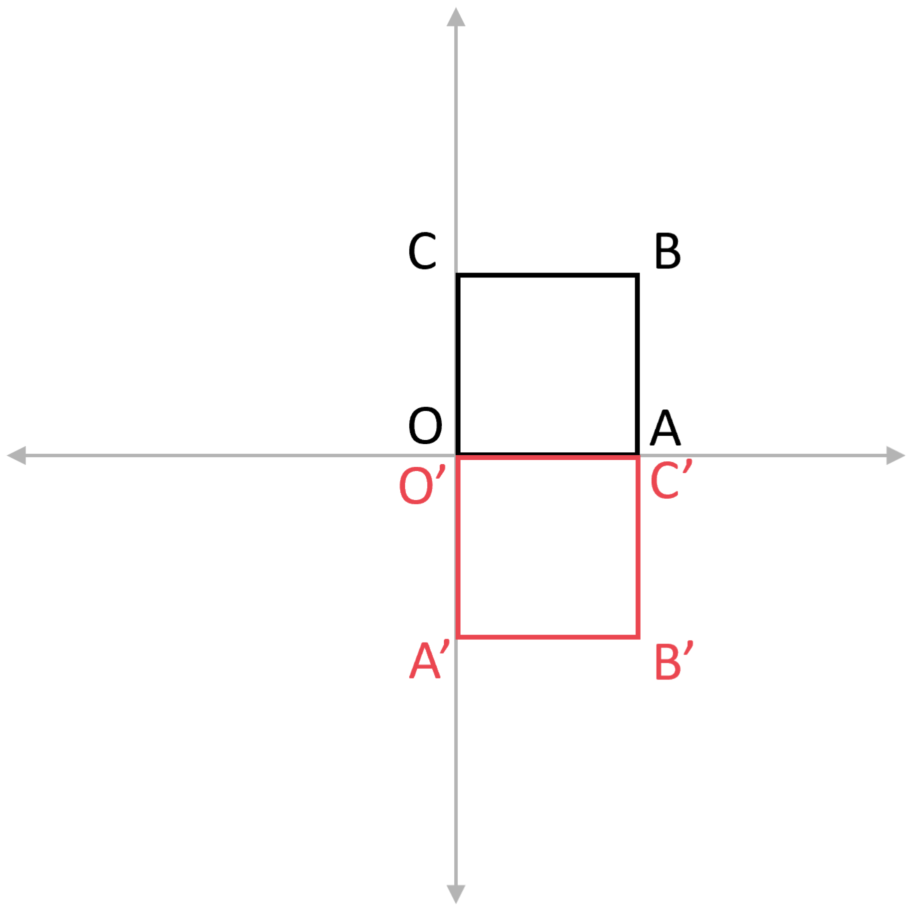
(a) Describe fully the transformation represented by the matrix $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Consider how the points A and C on the unit square are transformed

The point A $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ moves to A' $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

The point C $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ moves to C' $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

It helps to draw a picture of the unit square being transformed with vertices clearly labelled



This transformation could be a rotation of 90° **clockwise** about O or a reflection in the x-axis
The vertices A' and C' are not in the correct places for a reflection, but are for a rotation

The matrix P represents a rotation of 90° clockwise about the origin

You must give its angle, direction and centre of rotation
270° anticlockwise would also be accepted

(b) Find $\mathbf{Q}$, the matrix that represents a clockwise rotation of 120° about the origin, giving your answer in an exact form.

The matrix for an anticlockwise rotation by θ is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

θ is negative, as the rotation is clockwise

$$\theta = -120°$$

Substitute this value of θ into the matrix
Use that $\cos(-\theta) = \cos\theta$ and that $\sin(-\theta) = -\sin\theta$

$$\begin{pmatrix} \cos(-120°) & -\sin(-120°) \\ \sin(-120°) & \cos(-120°) \end{pmatrix} = \begin{pmatrix} \cos(120°) & \sin(120°) \\ -\sin(120°) & \cos(120°) \end{pmatrix}$$

Use a calculator to find these values (or common angles and symmetry)

$$\mathbf{Q} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Your notes

Your notes

Inverse Matrix Transformations

What are inverse matrix transformations?

- If the matrix $\mathbf{M}$ transforms the point *P* to *P'* then
  - the **inverse matrix** $\mathbf{M}^{-1}$ transforms *P'* **back** to *P*
- Inverse matrices represent the **reverse** of the transformation
- Applying a transformation then its inverse returns points to their **original positions**
  - $\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$

How do I use inverse matrix transformations?

- You can often interpret inverse matrix transformations **geometrically**
- For example, let $\mathbf{M}$ represent a rotation of 90° **clockwise**
  - Then $\mathbf{M}^{-1}$ must represent a rotation of 90° **anticlockwise**
  - $\mathbf{M}^{-1}$ is also the same as a rotation of 270° clockwise ($\mathbf{M}^3$)
    - So $\mathbf{M}^3 = \mathbf{M}^{-1}$ giving $\mathbf{M}^4 = \mathbf{I}$
    - This says four rotations of 90° clockwise returns to the original position
- If $\mathbf{M} = \mathbf{M}^{-1}$ then the **inverse** does the **same** thing as the **transformation**
  - For example. the reverse of reflecting in the *y*-axis is reflecting in the *y*-axis!
  - $\mathbf{M} = \mathbf{M}^{-1}$ also gives $\mathbf{M}^2 = \mathbf{I}$
    - This says reflecting twice about the *y*-axis returns to the original position

Worked Example

The matrix $\mathbf{Q}$ represents a rotation of 120° anticlockwise about the origin.

(a) Describe fully the single transformation represented by $\mathbf{Q}^{-1}$.

The inverse of $\mathbf{Q}$ reverses the transformation

$\mathbf{Q}^{-1}$ represents a rotation of 120° clockwise about the origin

(b) Use a geometrical argument to explain why $\mathbf{Q}^{-1} = \mathbf{Q}^2$.

$\mathbf{Q}^2$ means apply the transformation represented by $\mathbf{Q}$ twice

$\mathbf{Q}^2$ represents a rotation of 240° anticlockwise about the origin

Rotating 240° anticlockwise is the same as rotating 120° clockwise

A rotation of 120° clockwise about the origin, $\mathbf{Q}^{-1}$, is the same as doing a rotation of 240° anticlockwise about the origin, $\mathbf{Q}^2$

Your notes

Combinations of Matrix Transformations

How do I find a single matrix that represents a combination of transformations?

- A point (x, y) can be transformed **twice**
  - **Firstly** by the matrix , then **secondly** by the matrix
  - This is called a **combined** (or **composite** or **successive**) transformation
- A **single** matrix, , representing the combined transformation can be found using **matrix multiplication** as follows:
  -
    - The **order** matters: the **first** transformation is on the **right** in the multiplication
    - This order is the **reverse** of what you might expect!
  - Be careful: represents first, followed by second

How do I find the inverse of a combined transformation?

- The inverse of a **product** of matrices is the product of the inverses of the matrices in **reverse order**
  - $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- Let represent the transformation **first** by , then **second** by
  - That means from above
- Algebraically, $\mathbf{M}^{-1} = (\mathbf{QP})^{-1}$ which gives $\mathbf{M}^{-1} = \mathbf{P}^{-1}\mathbf{Q}^{-1}$
  - This shows that the **inverse**, $\mathbf{M}^{-1}$**,** first reverses , then reverses
    - That is the order we would expect

Worked Example

Three transformations in the - plane are represented by the matrices below.

represents a rotation of 180° about the origin
represents a reflection in the *y*-axis
represents a reflection in the *x*-axis

Your notes

(a) Use matrix multiplication to prove that a reflection in the y-axis followed by a reflection in the x-axis is equivalent to a rotation of 180° about the origin.

The question requires transformation $\mathbf{B}$ followed by transformation $\mathbf{C}$
This is the same as the matrix $\mathbf{CB}$ in that order (the first transformation appears on the right)

$$\mathbf{CB}$$

Use matrix multiplication to find $\mathbf{CB}$

$$\mathbf{CB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (1\times -1 + 0\times 0) & (1\times 0 + 0\times 1) \\ (0\times -1 + -1\times 0) & (0\times 0 + -1\times 1) \end{pmatrix}$$

The question claims that this is equivalent to transformation $\mathbf{A}$
Simplify the working above and show that it is the same as matrix $\mathbf{A}$

$$\mathbf{CB} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{A}$$

Therefore a reflection in the y-axis followed by a reflection in the x-axis is equivalent to a rotation of 180° about the origin

You would not get the marks for multiplying BC (it must be CB)

(b) A different transformation is represented by $\mathbf{CD}$ where $\mathbf{D}^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Find and simplify the matrix representing the inverse of the transformation.

You need to find the inverse of $\mathbf{CD}$
You need the rule that $(\mathbf{CD})^{-1} = \mathbf{D}^{-1}\mathbf{C}^{-1}$
You can substitute in $\mathbf{D}^{-1}$ from the question

$$\begin{aligned}(\mathbf{CD})^{-1} &= \mathbf{D}^{-1}\mathbf{C}^{-1} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{C}^{-1}\end{aligned}$$

You need to find $\mathbf{C}^{-1}$
Use that $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \mathbf{M}^{-1} = \dfrac{1}{\det \mathbf{M}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where $\det \mathbf{M} = ad - bc$

$$\begin{aligned}\mathbf{C}^{-1} &= \frac{1}{1\times(-1) - 0\times 0}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\end{aligned}$$

(Note that a reflection in the x-axis is its own inverse!)
Substitute this into the working above and multiply the matrices

Your notes

$$(\mathbf{CD})^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

There are other ways to do this question, for example finding $\mathbf{D}$ first
If you saw that $\mathbf{C}^{-1} = \mathbf{C}$ (as it is a reflection in the x-axis), explain why clearly