Edexcel International A Level (IAL) Further Maths: Further Pure 1

Your notes

Coordinate Systems

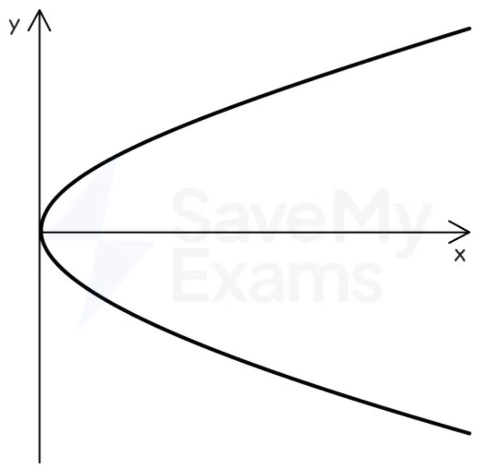
Contents

The Equation of a Parabola

What is a parabola?

- A **parabola** is a **U-shaped quadratic curve** with a **line of symmetry**
  - $y = x^2$ is a parabola
    - The line of symmetry is the y-axis
  - $x = y^2$ is a parabola
    - The line of symmetry is the x-axis
- A parabola is part of a family of curves called the **conics** (or **conic sections**)
  - Conics are parabolae, hyperbolae and ellipses

What is the general equation of a parabola?



The general parabola

- The **general equation** for a **parabola** is
  - $y^2 = 4ax$ in **Cartesian form**
  - $x = at^2, \ \ y = 2at$ in **Parametric form**
    - where a is a positive constant
- The x-axis is the **line of symmetry**, as shown
- The general equation can be **rearranged**
  - $y = \pm\sqrt{4ax}$

- $y = \sqrt{4ax}$ is the **branch above** the x-axis
- $y = -\sqrt{4ax}$ is the **branch below** the x-axis

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Examiner Tips and Tricks

You are given the Cartesian and parametric equations of a parabola in the Formulae Booklet.

Worked Example

A parabola has the parametric equations $x = at^2$ and $y = 2at$ where $a > 0$.

Show that its Cartesian equation is $y^2 = 4ax$.

To find the Cartesian equation, eliminate t from the parametric equations
It is easier to make t the subject of $y = 2at$

$$t = \frac{y}{2a}$$

Substitute this into the equation for x and simplify

$$x = a\left(\frac{y}{2a}\right)^2$$
$$x = \frac{ay^2}{4a^2}$$
$$x = \frac{y^2}{4a}$$

Make y^2 the subject

$$y^2 = 4ax$$

The Focus & Directrix of a Parabola

What are the focus and directrix of a parabola?

- The **focus** of the parabola $y^2 = 4ax$ is the **point** $(a, 0)$ on the x-axis
- The **directrix** is the **vertical line** $x = -a$
- For example, the parabola $y^2 = 12x$ where $a = 3$ has
  - a focus at $(3, 0)$
  - the directrix $x = -3$

What is the focus-directrix property?

- The **focus-directrix property** says that:
  - "Points on a parabola are the **same distance** from the focus as they are horizontally from the directrix"
- In other words:
  - If S is the **focus** of a parabola
  - and P is **any point** on the parabola
  - and X is the point on the **directrix** horizontally from P
  - Then the distance PS equals PX
    - $PS = PX$
- This property is an example of a **locus** of points

Examiner Tips and Tricks

You are given the focus and directrix of a parabola in the Formulae Booklet, but not the focus-directrix property.

Worked Example

The diagram shows the point $S(a,0)$, the vertical line $x = -a$ and a general point $P(x,y)$ that can move in the plane.

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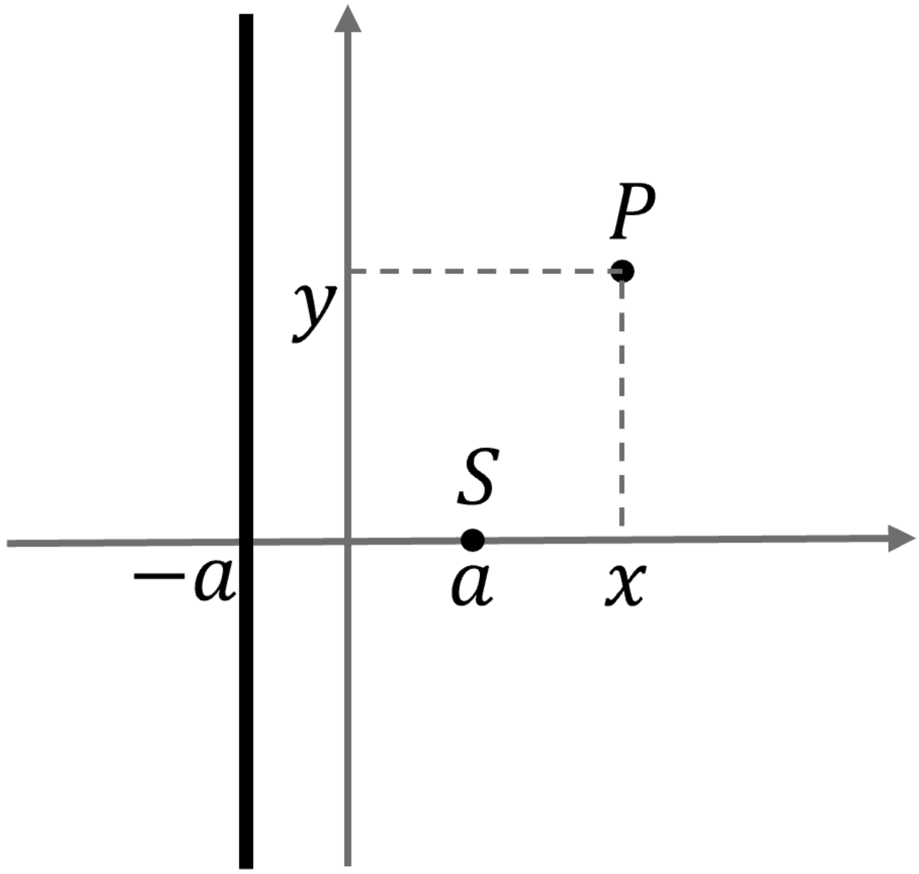
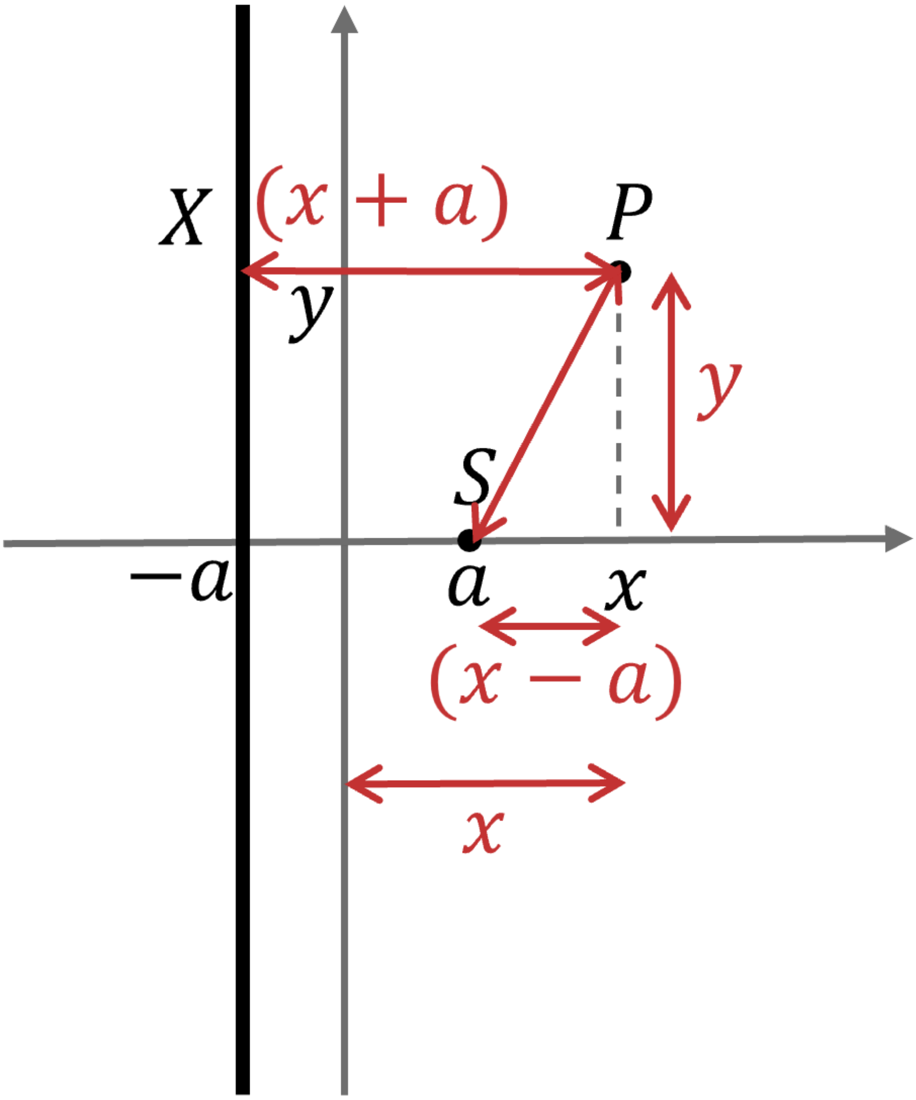


If P is restricted to always be the same distance from S as it is horizontally from the vertical line, use coordinate geometry to prove that it must lie on the parabola $y^2 = 4ax$.

It helps to add the point X on to the diagram, on the line $x = -a$ horizontally from P
The question says that $PS = PX$
The distance PS can be found using Pythagoras' theorem



$$PS^2 = (x - a)^2 + y^2$$

The distance PX can be found from the diagram

$$PX = x + a$$

Square $PS = PX$ and substitute in the results above

$$PS^2 = PX^2$$
$$(x - a)^2 + y^2 = (x + a)^2$$

Expand and simplify both sides

$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$
$$-2ax + y^2 = 2ax$$
$$y^2 = 4ax$$

$P(x, y)$ must lie on this curve

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$y^2 = 4ax$

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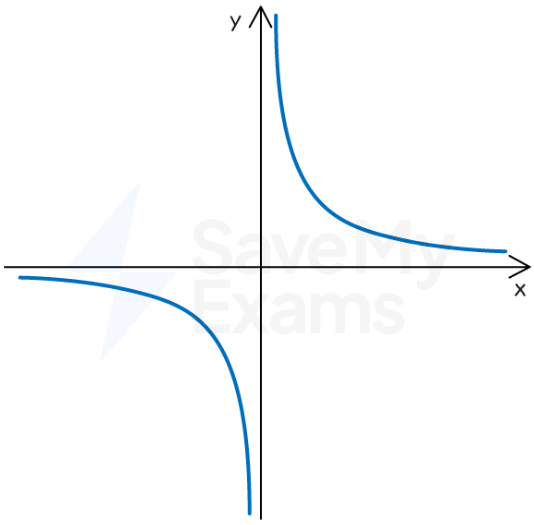
The Equation of a Rectangular Hyperbola

What is a rectangular hyperbola?

- A **rectangular hyperbola** is a **reciprocal curve** with two **L-shaped** branches and **asymptotes** along the axes
    - $y = \frac{1}{x}$ is a rectangular hyperbola
- A rectangular hyperbola is part of a family of curves called the **conics** (or **conic sections**)
    - Conics are parabolae, hyperbolae and ellipses

What is the general equation of a rectangular hyperbola?



The rectangular hyperbola

- The **general equation** for a **rectangular hyperbola** is
    - $xy = c^2$ in **Cartesian form**
    - $x = ct,\ \ y = \frac{c}{t}$ in **Parametric form**
        - $t \neq 0$
    - where c is a positive constant
- The **asymptotes** are the lines $y = 0$ and $x = 0$
    - These are **rectangular** (horizontal and vertical)

- Non-rectangular hyperbola have asymptotes at angles

- The general equation can be **rearranged**
  - $y = \frac{c^2}{x}$
    - This is a more familiar reciprocal form

Your notes

Examiner Tips and Tricks

You are given the Cartesian and parametric equations of a rectangular hyperbola in the Formulae Booklet.

Worked Example

A rectangular hyperbola has the parametric equations $x = ct$ and $y = \frac{c}{t}$ where $t \neq 0$ and $c > 0$.

Show that its Cartesian equation is $xy = c^2$.

To find the Cartesian equation, eliminate t from the parametric equations
A quick way here is to first make t the subject of both

$$t = \frac{x}{c} \text{ and } t = \frac{c}{y}$$

Then set them equal to each other and cross-multiply

$$\frac{x}{c} = \frac{c}{y}$$
$$xy = c^2$$

$$xy = c^2$$

Other correct ways to eliminate t are also accepted

Your notes

Tangents & Normals to a Parabola

What is a general point on a parabola?

- $(3, 6)$ is a **fixed point** on the parabola $y^2 = 12x$
- $(3t^2, 6t)$ is a **general point** on the parabola $y^2 = 12x$
  - It is **algebraic**
  - but it still satisfies the **equation**
  - and it can **move** along the curve (depending on t)
- General points are formed from **parametric equations**
  - A general point on $y^2 = 4ax$ is $(at^2, 2at)$
- You can have two or more **distinct** general points
  - $(3p^2, 6p)$ and $(3q^2, 6q)$ are general points P and Q on $y^2 = 12x$

How do I find the tangent or normal to a parabola at a general point?

- To find the **tangent** to the parabola $y^2 = 12x$ at the point $(3t^2, 6t)$
  - use $y - y_1 = m(x - x_1)$
  - substitute in $x_1 = 3t^2$ and $y_1 = 6t$
  - find m using **calculus** from $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $(3t^2, 6t)$
    - Make y the subject of $y^2 = 12x$
    - $y = \sqrt{12x} = 2\sqrt{3}\, x^{\frac{1}{2}}$
    - Ignore ±√ for now
    - $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{3} \times \frac{1}{2} x^{-\frac{1}{2}} = \sqrt{\frac{3}{x}}$
    - At $x_1 = 3t^2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{3}{3t^2}} = \frac{1}{t}$
  - The tangent is $y - 6t = \frac{1}{t}(x - 3t^2)$
- The **normal** has a perpendicular gradient $-\frac{1}{m}$

- $y - 6t = -t(x - 3t^2)$

How do I use tangents and normals that pass through general points?

- All methods in **coordinate geometry** still work
  - However points of intersection will be **algebraic** in t
- Questions can involve **forming** and **solving** equations in t

Examiner Tips and Tricks

If you know **parametric** and **implicit** differentiation, they are also acceptable methods to find $\frac{dy}{dx}$.

Worked Example

The point P with coordinates $(4t^2, 8t)$ lies on the parabola $y^2 = 16x$.

(a) Use calculus to show that the equation of the normal at P is $y + tx = 4t(t^2 + 2)$

The normal at P has the form $y - y_1 = m(x - x_1)$
Substitute in $x_1 = 4t^2$ and $y_1 = 8t$

$$y - 8t = m(x - 4t^2)$$

The gradient of the normal is the negative reciprocal of the gradient of the tangent
Find the gradient of the tangent
For example first make y the subject of the curve
(Use the positive square root)

$$y = \sqrt{16x} = 4x^{\frac{1}{2}}$$

Then differentiate

$$\frac{dy}{dx} = 4 \times \frac{1}{2}x^{-\frac{1}{2}}$$
$$= \frac{2}{\sqrt{x}}$$

Then substitute in $x_1 = 4t^2$

Your notes

$$\frac{dy}{dx} = \frac{2}{\sqrt{4t^2}}$$
$$= \frac{1}{t}$$

The negative reciprocal is $-t$
Therefore the equation of the normal is

$$y - 8t = -t(x - 4t^2)$$

Expand and rearrange

$$\begin{aligned} y - 8t &= -tx + 4t^3 \\ y &= -tx + 4t^3 + 8t \\ y + tx &= 4t^3 + 8t \end{aligned}$$

Then factorise the right-hand side

$$y + tx = 4t(t^2 + 2)$$

(b) Hence find the coordinates of the three points on the parabola at which the normal passes through the point $(9, 0)$.

The equation of the normal above must pass through the point $(9, 0)$
Substitute in $x = 9$ and $y = 0$

$$0 + 9t = 4t(t^2 + 2)$$

This forms an equation in t
Rearrange and solve

$$\begin{aligned} 9t &= 4t^3 + 8t \\ 0 &= 4t^3 - t \\ 0 &= t(4t^2 - 1) \\ 0 &= t(2t - 1)(2t + 1) \end{aligned}$$

$$t = 0, \quad t = \frac{1}{2}, \quad t = -\frac{1}{2}$$

The question asks for the coordinates of the points on the curve, P, $(4t^2, 8t)$
Substitute in the three values of t

$$(0, 0), \ (1, 4), \ (1, -4)$$

It helps to imagine the situation visually (below) to check the answers make sense

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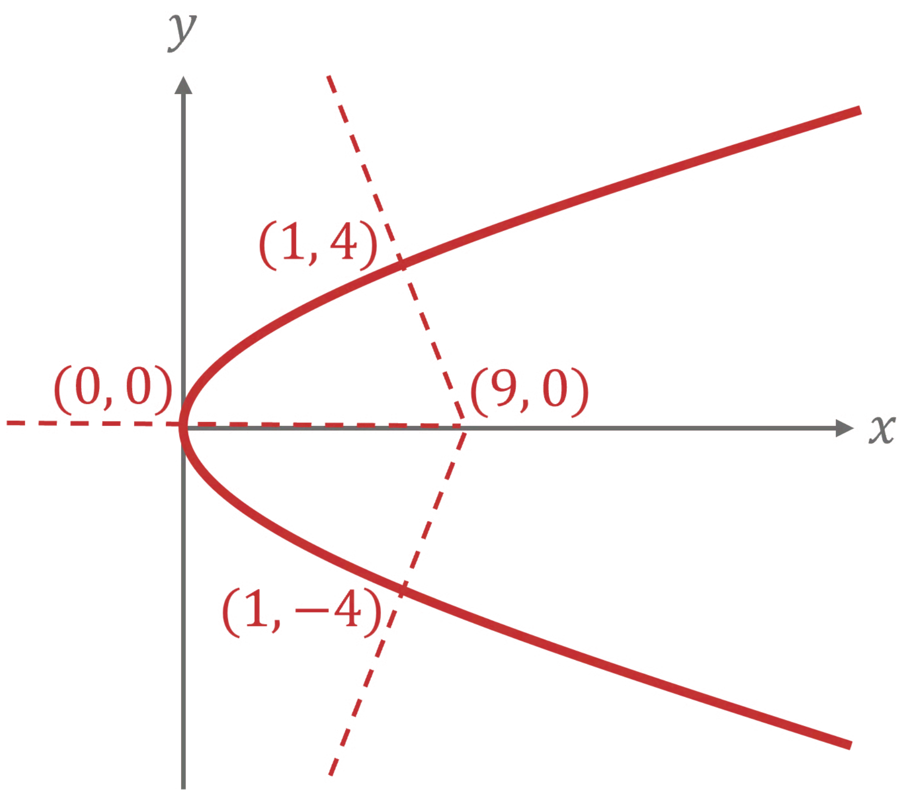


Your notes

Tangents & Normals to a Rectangular Hyperbola

What is a general point on a rectangular hyperbola?

- $(5, 5)$ is a **fixed point** on the rectangular hyperbola $xy = 25$
- $\left(5t, \frac{5}{t}\right)$ is a **general point** on the rectangular hyperbola $xy = 25$
  - It is **algebraic**
  - but it still satisfies the **equation**
  - and it can **move** along the curve (depending on t)
    - $t \neq 0$
- General points are formed from **parametric equations**
  - A general point on $xy = c^2$ is $\left(ct, \frac{c}{t}\right)$
- You can have two or more **distinct** general points
  - $\left(5p, \frac{5}{p}\right)$ and $\left(5q, \frac{5}{q}\right)$ are general points P and Q on $xy = 25$

Your notes

How do I find the tangent or normal to a rectangular hyperbola at a general point?

- To find the **tangent** to the rectangular hyperbola $xy = 25$ at the point $\left(5t, \frac{5}{t}\right)$
  - use $y - y_1 = m(x - x_1)$
  - substitute in $x_1 = 5t$ and $y_1 = \frac{5}{t}$
  - find m using **calculus** from $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $\left(5t, \frac{5}{t}\right)$
    - Make y the subject of $xy = 25$
    - $y = \frac{25}{x} = 25x^{-1}$
    - $\frac{\mathrm{d}y}{\mathrm{d}x} = 25 \times (-1)x^{-2} = -\frac{25}{x^2}$
    - At $x_1 = 5t$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{25}{(5t)^2} = -\frac{1}{t^2}$
  - The tangent is $y - \frac{5}{t} = -\frac{1}{t^2}(x - 5t)$
- The **normal** has a perpendicular gradient $-\frac{1}{m}$
  - $y - \frac{5}{t} = t^2(x - 5t)$

How do I use tangents and normals that pass through general points?

- All methods in **coordinate geometry** still work
  - However points of intersection will be **algebraic** in t
- Questions can involve **forming** and **solving** equations in t

Examiner Tips and Tricks

If you know **parametric** and **implicit** differentiation, they are also acceptable methods to find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Worked Example

The point P with coordinates $\left(2p, \frac{2}{p}\right)$ and the point Q with coordinates $\left(2q, \frac{2}{q}\right)$ lie on the rectangular hyperbola $xy = 4$, where $p \neq \pm q$.

(a) Use calculus to show that the equation of the tangent to the curve at P is $p^2 y = -x + 4p$.

The tangent at P has the form $y - y_1 = m(x - x_1)$

Substitute in $x_1 = 2p$ and $y_1 = \frac{2}{p}$

$$y - \frac{2}{p} = m(x - 2p)$$

Find the gradient of the tangent, for example, first make y the subject of the curve

$$y = \frac{4}{x} = 4x^{-1}$$

Then differentiate

$$\frac{dy}{dx} = -4x^{-2}$$
$$= -\frac{4}{x^2}$$

Then substitute in $x_1 = 2p$ and simplify

$$\frac{dy}{dx} = -\frac{4}{(2p)^2}$$
$$= -\frac{4}{4p^2}$$
$$= -\frac{1}{p^2}$$

Therefore the equation of the tangent is

$$y - \frac{2}{p} = -\frac{1}{p^2}(x - 2p)$$

Multiply both sides by p^2 then rearrange

$$p^2 y - 2p = -(x - 2p)$$
$$p^2 y - 2p = -x + 2p$$

Make $p^2 y$ the subject

$$p^2 y = -x + 4p$$

(b) Hence find the x-coordinate of the point of intersection between the tangent at P and the tangent at Q, giving your answer in fully simplified form.

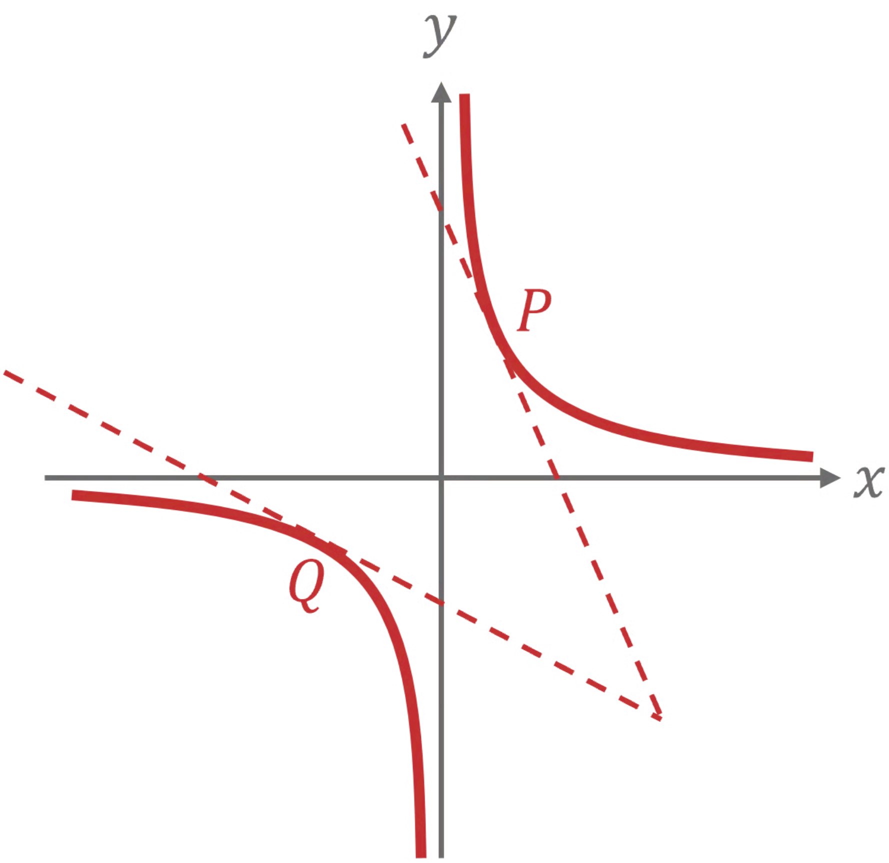
Finding the tangent at Q requires the exact same steps as above, but with q instead of p

Your notes

You can therefore write down the tangent at Q with no working

$$q^2y = -x + 4q$$

The tangent at P intersects the tangent at Q



Make y the subject of each tangent and set them equal to each other

$$\frac{-x+4p}{p^2} = \frac{-x+4q}{q^2}$$

Cross-multiply and expand

$$q^2(-x+4p) = p^2(-x+4q)$$
$$-q^2x + 4pq^2 = -p^2x + 4p^2q$$

Bring the x terms to one side and factorise

$$p^2x - q^2x = 4p^2q - 4pq^2$$
$$(p^2 - q^2)x = 4pq(p-q)$$

Make x the subject and use the difference of two squares

$$x = \frac{4pq(p-q)}{p^2-q^2}$$
$$x = \frac{4pq(p-q)}{(p-q)(p+q)}$$

Your notes

As $p \neq \pm q$, the brackets will never be zero

This means the $(p - q)$ bracket can be cancelled, giving the final answer

$$x = \frac{4pq}{p + q}$$

Note: if $p = q$ then point P is the same as point Q

Also, if $p = -q$, then the tangents at P and Q are parallel

Your notes