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Edexcel International A Level (IAL) Further Maths: Further Pure 1

Numerical Solutions of Equations

Contents

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Roots in Intervals

What is a root of an equation?

- A **root** of the **equation** $\mathrm{f}(x) = 0$ is a **solution**
  - If $x = \alpha$ is a root of $\mathrm{f}(x) = 0$ then $\mathrm{f}(\alpha) = 0$
    - The Greek letter **alpha** is often used
- To find α **exactly**, you need to solve the equation **algebraically** (analytically)
- Some equations **cannot** be solved algebraically
  - In which case you can **approximate** a root to a given **accuracy**
    - For example, to 3 decimal places
  - This is called solving equations **numerically**

What is an interval?

- If you don't know the root but know that it lies between $x = a$ and $x = b$, where $a < b$, then
  - $a < x < b$ is an **interval** containing the root
- Intervals can be written using **bracket notation**
  - $(a,\ b)$ is $a < x < b$
  - $[a,\ b]$ is $a \le x \le b$

How do I show that an interval contains a root?

- Use the **sign-change and continuity test** to show that the **interval** $a < x < b$ contains a **root** to the equation $\mathrm{f}(x) = 0$
- **Show** that $\mathrm{f}(a)$ and $\mathrm{f}(b)$ have **different signs** and **check** that $\mathrm{f}(x)$ is **continuous** across $a < x < b$
  - Then the interval $a < x < b$ **must** contains a **root**
  - **Continuous** means **no jumps or asymptotes** in the interval
    - Check the equation does not **divide by zero** within the interval
    - (Jumps or asymptotes outside of the interval are fine)
  - You must write a **conclusion** to the test in words, for example:
    - "$\mathrm{f}(x)$ has a sign change in the interval $a < x < b$ and $\mathrm{f}(x)$ is continuous in the interval, so a root must lie in the interval $a < x < b$"

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Examiner Tips and Tricks

When writing your conclusion in the exam, don't forget to mention $f(x)$ being continuous in the interval!

How do I show that an equation has a root to a given accuracy?

- If asked to show that $x = 1.39$ is the root of an equation to **2 decimal places**
  - write an **interval** using the **lower** and **upper bound** of the root
    - $1.385 < x < 1.395$
  - use the **sign-change and continuity test** on this interval

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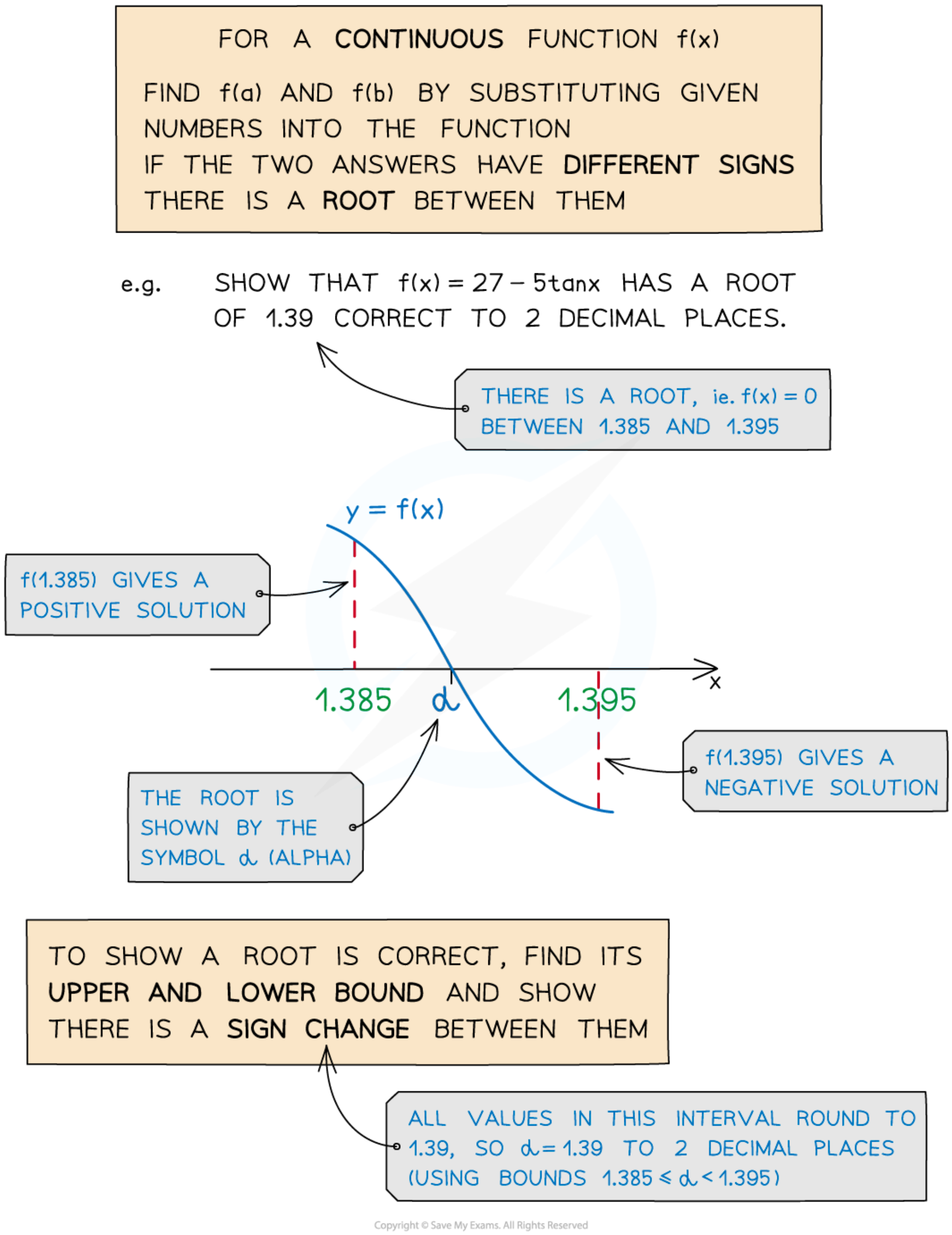
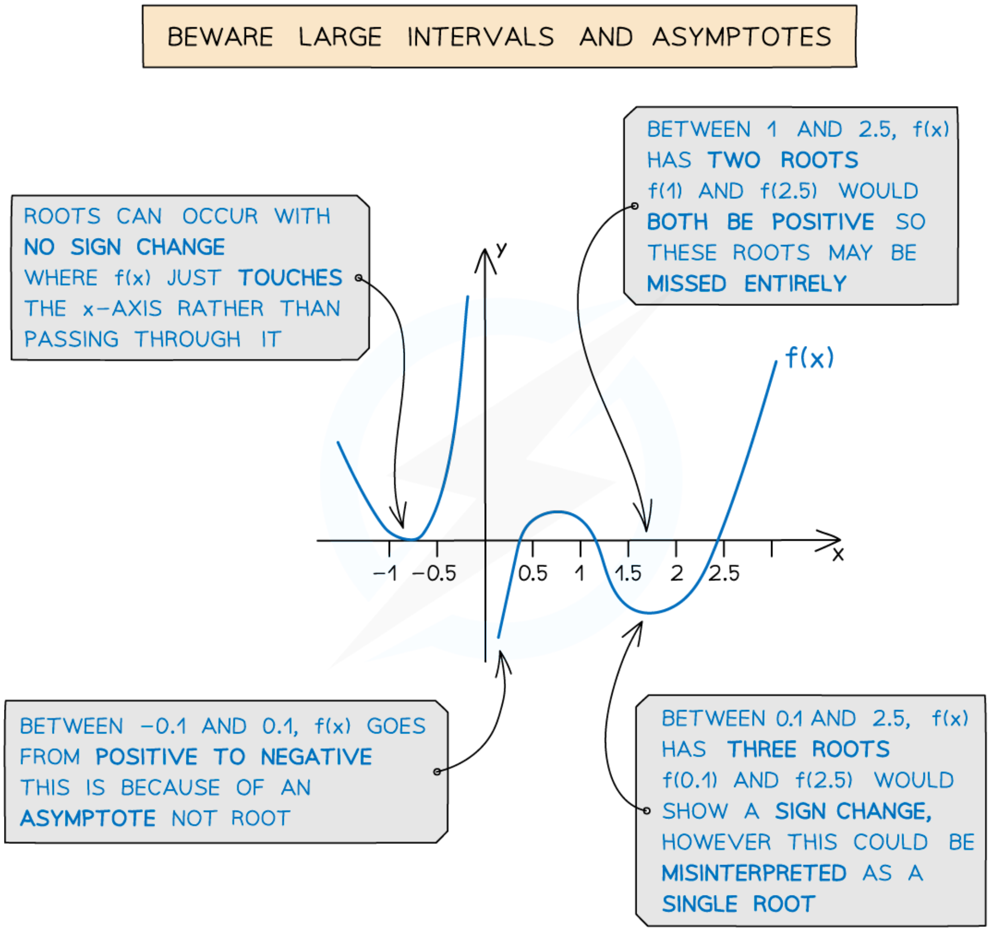


- A suitable conclusion would be
  - $\mathrm{f}(x)$ has a **sign change** in the interval $1.385 < x < 1.395$
  - and $\mathrm{f}(x)$ is **continuous** in the interval
  - so a root **must** lie in the interval $1.385 < x < 1.395$
  - All values in the interval **round** to 1.39
  - so the root is 1.39 to **2 decimal places**

How do I know when the sign-change test fails?

- You need to know the **three** cases when the sign-change and continuity test **fails** to work properly:
  - If the curve $y = \mathrm{f}(x)$ only **touches** the x-axis at the root, you will **not** see a sign change (even though there **is** a root)
  - If the curve $y = \mathrm{f}(x)$ has an **asymptote** in the interval, then you may see a sign change (but there is **no** root)
    - The asymptote means $\mathrm{f}(x)$ is **not** continuous
  - If the interval is **too large**, there may be **more than one** root in it
    - You may see a sign change (an **odd** number of roots)
    - Or you may **not** see a sign change (an **even** number of roots)



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Worked Example

The equation $f(x)=0$ where $f(x)=x-x^4+\frac{1}{2x-1}$ has exactly one positive root.

(a) Show that the intervals $[0,\ 1]$ and $[1,\ 2]$ both have a change of sign in $f(x)$.

Substitute $x=0$, $x=1$ and $x=2$ into $f(x)$

$$f(0)=0-0^4+\frac{1}{2\times 0-1}=-1$$
$$f(1)=1-1^4+\frac{1}{2\times 1-1}=1$$
$$f(2)=2-2^4+\frac{1}{2\times 2-1}=-\frac{41}{3}$$

Comment that the signs are different for each interval

$f(0)=-1<0$ and $f(1)=1>0$ so there is a sign change in $[0,\ 1]$

$f(1)=1>0$ and $f(2)=-\frac{41}{3}<0$ so there is a sign change in $[1,\ 2]$

(b) Determine, with a reason, which of the intervals contains the root.

An interval contains a root if $f(x)$ has a sign change and is continuous in that interval
Check to see if $f(x)$ is continuous in $[0,\ 1]$
You need to look for any asymptotes

$\frac{1}{2x-1}$ is undefined when $x=\frac{1}{2}$

The asymptote $x=\frac{1}{2}$ lies in the interval $[0,\ 1]$
There are no other asymptotes and only one root

The interval $[1,\ 2]$ contains the root as there is a sign change and $f(x)$ is continuous

The interval $[0,\ 1]$ contains the asymptote $x=\frac{1}{2}$ so $f(x)$ is not continuous

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Interval Bisection

What is interval bisection?

- **Interval Bisection** means **splitting** an **interval** containing a root into two **halves** and using a **sign change test** to find out which half contains the **root**
- For example, if $f(1) < 0$ and $f(2) > 0$ and $f(x)$ is continuous
  - Then a root must lie in $[1, 2]$
  - Find the **midpoint** of the interval
    - $x = 1.5$
  - Check the **sign** at the midpoint
    - For example, $f(1.5) < 0$
  - Write down the **half-interval** that has the **sign change**
    - $f(1) < 0$, $f(1.5) < 0$ and $f(2) > 0$
    - So $[1.5, 2]$ has the sign change
- The process can be **repeated**
  - For example, if $f(1.75) > 0$ then $[1.5, 1.75]$ contains the sign change

How do I use interval bisection to estimate a root?

- If a root lies in the interval $[1, 2]$, then the **first approximation** of the root, x_1, is simply the **midpoint**
  - $x_1 = 1.5$
- The **second approximation**, x_2, requires using Interval Bisection once
  - For example, giving $[1.5, 2]$
  - Then find the **midpoint**
    - $x_2 = 1.75$
  - And so on

Examiner Tips and Tricks

Read the question carefully to see if an approximation to the root is required, or the interval containing the root is required.

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Worked Example

The equation $x^3 + x - 16 = 0$ has a root in the interval $[2.1, 2.9]$.

Use interval bisection to find an interval of width 0.2 that contains the root.

Find the width of the interval given in the question

$$2.9 - 2.1 = 0.8$$

Interval bisection halves the width
To get a width of 0.2, interval bisection is needed twice
Find the signs of the interval end-points

$$\mathrm{f}(2.1) = -4.639 < 0$$
$$\mathrm{f}(2.9) = 11.289 > 0$$

Find the midpoint of the interval

$$x = \frac{2.1 + 2.9}{2} = 2.5$$

Find the sign of $\mathrm{f}(x)$ at $x = 2.5$

$$\mathrm{f}(2.5) = 2.125 > 0$$

Find the new halved interval containing the sign change
Check the signs: $\mathrm{f}(2.1) < 0$, $\mathrm{f}(2.5) > 0$ and $\mathrm{f}(2.9) > 0$

$$[2.1, 2.5]$$

Repeat the process
Find the midpoint of the interval

$$x = \frac{2.1 + 2.5}{2} = 2.3$$

Find the sign of $\mathrm{f}(x)$ at $x = 2.3$

$$\mathrm{f}(2.3) = -1.533 < 0$$

Find the new halved interval containing the sign change
Check the signs: $\mathrm{f}(2.1) < 0$, $\mathrm{f}(2.3) < 0$ and $\mathrm{f}(2.5) > 0$

$$[2.3, 2.5]$$

Linear Interpolation

What is linear interpolation?

- **Linear interpolation** draws a **straight line** between the points *A* and *B* with coordinates $(a,\ \mathrm{f}(a))$ and $(b,\ \mathrm{f}(b))$, where $a < x < b$ is an interval containing a root to the equation $\mathrm{f}(x) = 0$
  - As $\mathrm{f}(a)$ and $\mathrm{f}(b)$ have **different signs**, *A* and *B* are on **either side** of the *x*-axis
  - The **approximation** to the root is where this straight line **cuts the *x*-axis**
    - This is at the point *X* with coordinates $(x,\ 0)$
    - Points *A*, *X* and *B* lie on the same straight line
- It helps to use the **gradient formula**
  - The gradient between $(x_1,\ y_1)$ and $(x_2,\ y_2)$ is $\dfrac{y_2 - y_1}{x_2 - x_1}$

How do I apply linear interpolation using gradients?

- This is best explained with an example
- If the root of $\mathrm{f}(x) = 0$ lies in the interval $[1,\ 2]$ and $\mathrm{f}(1) = -5$ and $\mathrm{f}(2) = 10$, then:
  - *A* is the point $(1,\ -5)$
  - *B* is the point $(2,\ 10)$
  - *X* is the point $(x,\ 0)$
    - This is the *x*-intercept of the line *AB*
    - *A*, *X* and *B* lie on the same **straight line**
  - The **gradient** of ***AX*** equals the **gradient** of ***XB***
    - The **order** matters (not *BX*)
  - Use the **gradient formula** to set these equal
    - $\dfrac{0 - (-5)}{x - 1} = \dfrac{10 - 0}{2 - x}$ giving $\dfrac{5}{x - 1} = \dfrac{10}{2 - x}$
  - **Solve** the equation to find x
    - $5(2 - x) = 10(x - 1)$
    - expand and solve to get $x = \dfrac{4}{3}$
- This process can be **repeated** to get the next approximation

- This gradient method does **not** require a **sketch**

What other methods can I use to do linear interpolation?

- You can **sketch** the points *A*, *B* and *X* on a diagram and use **similar triangles** either side of the *x*-axis to find an **equation** in x
  - This leads to the same equation as the gradients above
- Or you can use **coordinate geometry** to find the **equation** of the **line *AB***
  - Use $y - y_1 = m(x - x_1)$
  - Then find its **x-intercept**
    - Substitute in $y = 0$ and solve for x

Examiner Tips and Tricks

If the calculations involve lengthy decimals, you can use $\mathrm{f}(a)$ and $\mathrm{f}(b)$ to stand for their numbers in your working!

Worked Example

The equation $\mathrm{f}(x) = 0$ where $\mathrm{f}(x) = x^3 - \sqrt{x} - 2$ has a root in the interval $[1.4,\ 1.5]$.

Use linear interpolation once to find an approximation to the root, giving your answer to 3 decimal places.

Find $\mathrm{f}(1.4)$ and $\mathrm{f}(1.5)$

$$\mathrm{f}(1.4) = 1.4^3 - \sqrt{1.4} - 2 = -0.439215...$$
$$\mathrm{f}(1.5) = 1.5^3 - \sqrt{1.5} - 2 = 0.150255...$$

It is easier to write them as $\mathrm{f}(1.4)$ and $\mathrm{f}(1.5)$ for now
Let *A*, *B* and *X* be the points $(1.4,\ \mathrm{f}(1.4))$, $(1.5,\ \mathrm{f}(1.5))$ and $(x,\ 0)$

Use the gradient formula $\dfrac{y_2 - y_1}{x_2 - x_1}$ to find the gradient of *AX*

gradient of *AX* is $\dfrac{0 - \mathrm{f}(1.4)}{x - 1.4}$

Use the gradient formula $\dfrac{y_2 - y_1}{x_2 - x_1}$ to find the gradient of *XB*

gradient of *XB* is $\dfrac{\mathrm{f}(1.5) - 0}{1.5 - x}$

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Set the two gradients equal to each other

$$\frac{0-\mathrm{f}(1.4)}{x-1.4}=\frac{\mathrm{f}(1.5)-0}{1.5-x}$$

Simplify and cross multiply

$$\frac{-\mathrm{f}(1.4)}{x-1.4}=\frac{\mathrm{f}(1.5)}{1.5-x}$$
$$-\mathrm{f}(1.4)(1.5-x)=\mathrm{f}(1.5)(x-1.4)$$

Expand and collect the x terms on the right-hand side

$$-1.5\mathrm{f}(1.4)+\mathrm{f}(1.4)x=\mathrm{f}(1.5)x-1.4\mathrm{f}(1.5)$$
$$1.4\mathrm{f}(1.5)-1.5\mathrm{f}(1.4)=\mathrm{f}(1.5)x-\mathrm{f}(1.4)x$$

Factorise out the x, then make x the subject

$$1.4\mathrm{f}(1.5)-1.5\mathrm{f}(1.4)=x[\mathrm{f}(1.5)-\mathrm{f}(1.4)]$$
$$\frac{1.4\mathrm{f}(1.5)-1.5\mathrm{f}(1.4)}{\mathrm{f}(1.5)-\mathrm{f}(1.4)}=x$$

Substitute in $\mathrm{f}(1.4)$ and $\mathrm{f}(1.5)$ from above

$$\frac{1.4\times 0.150255...-1.5\times(-0.439215...)}{0.150255...-(-0.439215...)}=x$$
$$1.47451...=x$$

Give your final answer to 3 decimal places

1.475 to 3 decimal places

If you did the working without $\mathrm{f}(1.4)$ and $\mathrm{f}(1.5)$, keep lots of decimal places

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Newton-Raphson Method

How do I apply the Newton-Raphson method?

- The **Newton-Raphson method** is a process for finding **roots** of equations
  - Equations must be **rearranged** into the form $\mathrm{f}(x) = 0$
- If $x = \alpha$ is a **root** of the equation $\mathrm{f}(x) = 0$, then you need to **choose** x_0, the **initial** (first) **approximation**
  - This is usual **given** in the question (and is a value **near** to the root)
- You can then find successive **approximations** $x_1, x_2, x_3, \ldots$ using the **Newton-Raphson formula**:
  - $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$
  - This is an iterative formula
  - $\mathrm{f}'(x)$ is the derivative of $\mathrm{f}(x)$
    - Work this out beforehand
- **Increasing** the **number** of approximations improves the **accuracy**
  - The Newton-Raphson method usually converges quickly to the root compared to other methods

Examiner Tips and Tricks

The formula for the Newton-Raphson method is given in the Formulae Booklet.

How do I apply the Newton-Raphson method on my calculator?

- If your calculator has a **recursion** function
  - Input the **formula**, **start value** and **number of steps**
  - The output is a **table** showing all the approximations
- Alternatively:
  - Type in the value of x_0 and press =
    - This **stores** it in the "Ans" (answer) button

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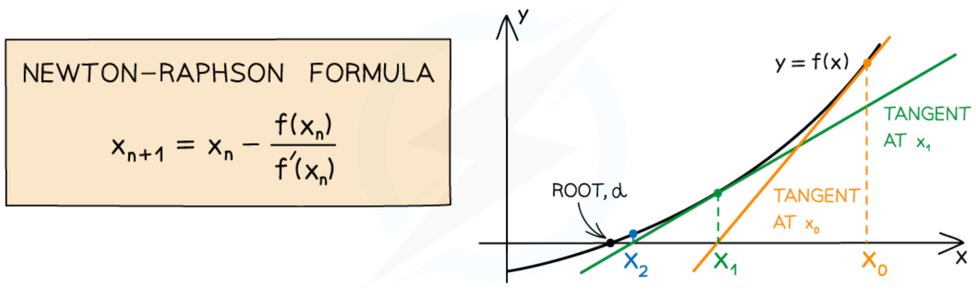
- Type $\text{Ans} - \dfrac{\text{f}(\text{Ans})}{\text{f}'(\text{Ans})}$ into your calculator and press =
  - This finds x_1
- Without pressing any other button, press = again
  - This finds x_2
- Repeatedly pressing = gives further approximations
  - The more you press it, the closer to the root it becomes

How does the Newton-Raphson method work geometrically?

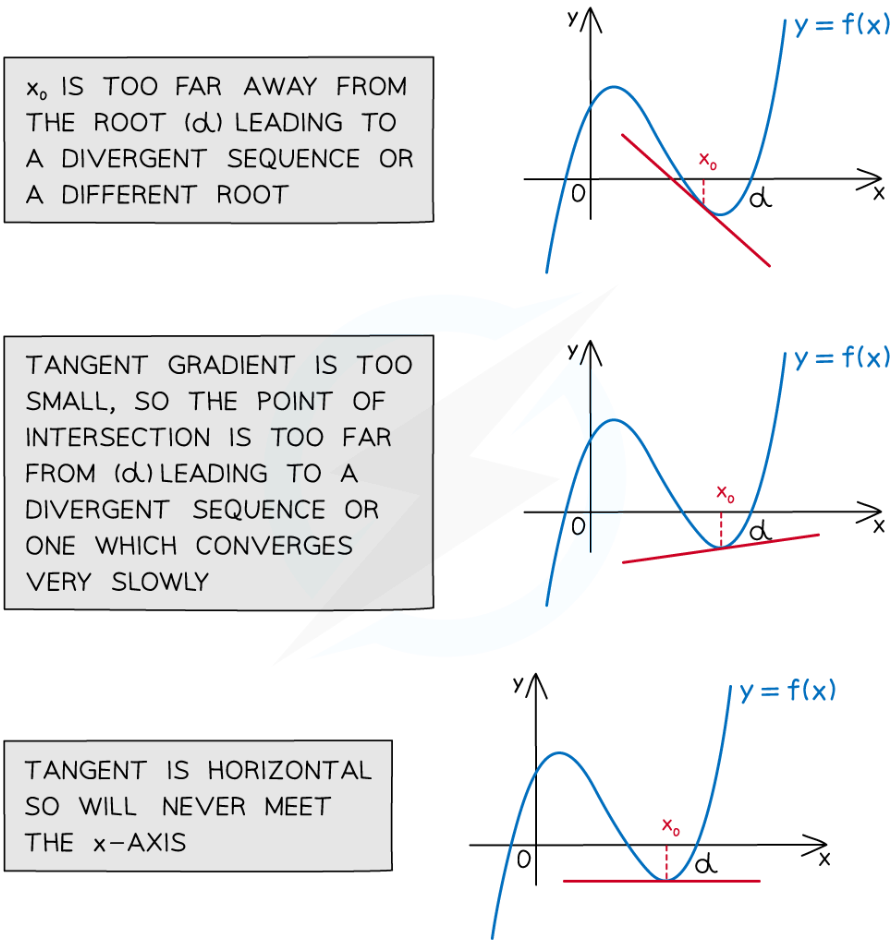
- The method works by drawing a **tangent** to the curve $y = \text{f}(x)$ at $x = x_0$
  - then finding where the tangent **cuts the *x*-axis**
  - This *x*-intercept is the **new** approximation, x_1
- This process is then **repeated**
  - Draw the tangent to $y = \text{f}(x)$ at $x = x_1$
  - Find its *x*-intercept and call it x_2
- The approximations get closer and closer to the **root**



How do I know if the Newton-Raphson method will fail?

- The Newton-Raphson method **fails** if the **initial value**, x_0, satisfies $\text{f}'(x_0) = 0$
  - Algebraically, this is because the formula $x_{n+1} = x_n - \dfrac{\text{f}(x_n)}{\text{f}'(x_n)}$ cannot **divide by zero**
  - Geometrically, this is because x_0 is at a stationary point on the curve $y = \text{f}(x)$
    - A **tangent** drawn at a **stationary point** is **horizontal**

- This will **never intersect** the x-axis
- The Newton-Raphson method also fails if the sequence $x_1, x_2, x_3, ...$ diverges
  - This can happen if x_0 is chosen:
    - **too far away** from the root
    - or at a point where the **gradient** is **very small**
- The Newton-Raphson method is sometimes avoided when $f(x)$ is too tricky to differentiate



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Worked Example

The equation $2x^4 - 4x^3 = -1$ has a solution in the interval $[1, 2]$.

(a) Using $x_0 = 2$ as the first approximation to the solution, apply the Newton-Raphson method to find a second approximation.

Rearrange the formula into the form $f(x) = 0$

$$2x^4 - 4x^3 + 1 = 0 \text{ so } f(x) = 2x^4 - 4x^3 + 1$$

(You could also use $-1 + 4x^3 - 2x^4$)
Find $f'(x)$ using differentiation

$$f'(x) = 8x^3 - 12x^2$$

Substitute $f(x)$ and $f'(x)$ into the Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \frac{(2x_n^{\ 4} - 4x_n^{\ 3} + 1)}{(8x_n^{\ 3} - 12x_n^{\ 2})}$$

Substitute $x_0 = 2$ into the formula to get x_1

$$x_1 = 2 - \frac{(2 \times 2^4 - 4 \times 2^3 + 1)}{(8 \times 2^3 - 12 \times 2^2)}$$

$$= 2 - \frac{1}{16}$$

$$= 1.9375$$

Check this solution lies in the interval $[1, 2]$

$$x_1 = 1.9375$$

$\frac{31}{16}$ is also accepted

(b) Explain why the midpoint of the interval $(1, 2)$ cannot be used as the first approximation when applying the Newton-Raphson method.

The Newton-Raphson method fails if x_0 satisfies $f'(x_0) = 0$
Find the midpoint of the interval

$$x_0 = \frac{1+2}{2} = 1.5$$

Check if $f'(1.5) = 0$

$$f'(1.5) = 8 \times 1.5^3 - 12 \times 1.5^2 = 0$$

Write a conclusion either about dividing by zero, or about having a horizontal tangent

$f'(1.5) = 0$, but you cannot divide by zero in the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
So 1.5 cannot be used as the first approximation

You could also say that $x = 1.5$ is at a stationary point where the tangent is horizontal, meaning it cannot intersect the x-axis to make x_1

Your notes