Edexcel International A Level (IAL) Further Maths: Further Pure 1

Your notes

Roots of Quadratic Equations

Contents

Your notes

Sums & Products of Roots

How do I find the sum and product of the roots of a quadratic?

- The **quadratic equation** $ax^2 + bx + c = 0$ can be written as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
  - by **dividing** by a (assuming $a \neq 0$)
- If α and β are **roots** of the quadratic equation
  - then the equation is also $(x - \alpha)(x - \beta) = 0$ in **factorised** form
    - This works for real and complex roots
- Both left-hand sides of the equations are **identical**
  - $(x - \alpha)(x - \beta) \equiv x^2 + \frac{b}{a}x + \frac{c}{a}$
  - **Expand** and **collect** the left-hand side
  - $x^2 - (\alpha + \beta)x + \alpha\beta \equiv x^2 + \frac{b}{a}x + \frac{c}{a}$
  - **Equating coefficients** gives
    - the **sum** of the roots is $\alpha + \beta = -\frac{b}{a}$
    - the **product** of the roots is $\alpha\beta = \frac{c}{a}$
- Be careful: the **sum** of the roots has a **negative** sign
  - The product does not

Examiner Tips and Tricks

These two relationships are not given in the Formulae Booklet, so must be learnt!

Worked Example

The quadratic equation $5x^2 - 30x - 3 = 0$ has roots α and β.

Without solving the equation, find the value of:

(a) $\alpha+\beta$

Either use $\alpha+\beta=-\frac{b}{a}=-\frac{(-30)}{5}$

Or first divide both sides of the equation by 5

$$x^2-6x-\frac{3}{5}=0$$

Then use $x^2-(\text{sum of roots})x+\text{product of roots}=0$

The sum of the roots is the negative of the middle number

$$\alpha+\beta=-(-6)$$

$$\alpha+\beta=6$$

(b) $\alpha\beta$

Either use $\alpha\beta=\frac{c}{a}=\frac{(-3)}{5}$

Or first divide both sides of the equation by 5

$$x^2-6x-\frac{3}{5}=0$$

Then use $x^2-(\text{sum of roots})x+\text{product of roots}=0$

The product of the roots is the constant term on the end

No change of sign is needed

$$\alpha\beta=-\frac{3}{5}$$

Your notes

Expressions with Roots

How do I simplify expressions involving roots of a quadratic?

- You can use **algebra** to write expressions **in terms of** $(\alpha+\beta)$ and $\alpha\beta$
- Then use the fact that the equation $ax^2+bx+c=0$ has roots α and β where
  - $\alpha+\beta=-\frac{b}{a}$
  - $\alpha\beta=\frac{c}{a}$
    - **Substitute** these into your expression to find its value

Which identities do I need to know?

- You should know **identities** involving **powers** of **products** of roots
  - $\alpha^2\beta^2\equiv(\alpha\beta)^2$
  - $\alpha^3\beta^3\equiv(\alpha\beta)^3$

- and so on

- You should know the **identity** for the **sum** of the **squares** of the roots
  - $\alpha^2+\beta^2 \equiv (\alpha+\beta)^2 - 2\alpha\beta$
    - which comes from **expanding** $(\alpha+\beta)^2 \equiv \alpha^2+\beta^2+2\alpha\beta$
- You should know the **identity** for the **sum** of the **cubes** of the roots
  - $\alpha^3+\beta^3 \equiv (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$
    - which comes from the **binomial expansion** of $(\alpha+\beta)^3$
    - $(\alpha+\beta)^3 \equiv \alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3 \equiv \alpha^3+\beta^3+3\alpha\beta(\alpha+\beta)$
    - then rearranging
- You should know **how** to use **algebraic fractions** (but do not need to learn these identities)
  - $\frac{1}{\alpha}\times\frac{1}{\beta}\equiv\frac{1}{\alpha\beta}$
  - $\frac{1}{\alpha}+\frac{1}{\beta}\equiv\frac{\alpha+\beta}{\alpha\beta}$
  - $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\equiv\frac{\alpha^2+\beta^2}{\alpha\beta}$
  - and so on

Examiner Tips and Tricks

None of the identities are given in the Formulae Booklet, so you either need to learn them, or learn how to find them.

Worked Example

The quadratic equation $2x^2+10x+9=0$ has roots α and β.

Without solving the equation, find the exact value of

(a) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

Use $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$ to find the sum and product of the roots

$$\alpha+\beta=-\frac{10}{2}=-5 \text{ and } \alpha\beta=\frac{9}{2}$$

Add the algebraic fractions $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

Use a lowest common denominator of $\alpha\beta$

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2}{\alpha\beta}+\frac{\beta^2}{\alpha\beta}$$

$$=\frac{\alpha^2+\beta^2}{\alpha\beta}$$

Now use the identity for the sum of the squares of the roots $\alpha^2+\beta^2\equiv(\alpha+\beta)^2-2\alpha\beta$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

Substitute in $\alpha+\beta=-5$ and $\alpha\beta=\frac{9}{2}$

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{(-5)^2-2\left(\frac{9}{2}\right)}{\left(\frac{9}{2}\right)}$$

$$=\frac{25-9}{\frac{9}{2}}$$

$$=16\div\frac{9}{2}$$

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{32}{9}$$

(b) $\alpha^3+\beta^3$

Either use $\alpha^3+\beta^3\equiv(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$ if learnt

Or start by expanding the binomial $(\alpha+\beta)^3$

$$(\alpha+\beta)^3=\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3$$

Factorise fully the middle two terms

$$(\alpha+\beta)^3=\alpha^3+3\alpha\beta(\alpha+\beta)+\beta^3$$

Make $\alpha^3+\beta^3$ the subject

$$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$$

Substitute in $\alpha+\beta=-5$ and $\alpha\beta=\frac{9}{2}$

$$\alpha^3+\beta^3=(-5)^3-3\left(\frac{9}{2}\right)(-5)$$

$$=-125+\frac{135}{2}$$

$$\alpha^3+\beta^3=-\frac{115}{2}$$

Your notes

-57.5 is also accepted

Your notes

Forming Quadratics with New Roots

How do I form quadratics with new roots?

- The quadratic equation $ax^2 + bx + c = 0$ has roots α and β where
  - $\alpha + \beta = -\frac{b}{a}$
  - $\alpha\beta = \frac{c}{a}$
- Any **new quadratic equations** with **new roots** will have the form
  - $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$
- For example, to find the quadratic equation with roots α^2 and β^2
  - The new equation is $x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0$
  - You can then work out these **coefficients** using the **identities** from before
    - $\alpha^2 + \beta^2 \equiv (\alpha + \beta)^2 - 2\alpha\beta$ and $\alpha^2\beta^2 \equiv (\alpha\beta)^2$
    - **Substitute** in $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Which identities do I need to know?

- You should know **identities** involving **powers** of **products** of roots
  - $\alpha^2\beta^2 \equiv (\alpha\beta)^2$
  - $\alpha^3\beta^3 \equiv (\alpha\beta)^3$
  - and so on
- You should know the **identity** for the **sum** of the **squares** of the roots
  - $\alpha^2 + \beta^2 \equiv (\alpha + \beta)^2 - 2\alpha\beta$
    - which comes from **expanding** $(\alpha + \beta)^2 \equiv \alpha^2 + \beta^2 + 2\alpha\beta$
- You should know the **identity** for the **sum** of the **cubes** of the roots
  - $\alpha^3 + \beta^3 \equiv (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
    - which comes from the **binomial expansion** of $(\alpha + \beta)^3$
    - $(\alpha + \beta)^3 \equiv \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 \equiv \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta)$
    - then rearranging
- You should know **how** to use **algebraic fractions** (but do not need to learn these identities)

- $\frac{1}{\alpha} \times \frac{1}{\beta} \equiv \frac{1}{\alpha\beta}$
- $\frac{1}{\alpha} + \frac{1}{\beta} \equiv \frac{\alpha+\beta}{\alpha\beta}$
- $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \equiv \frac{\alpha^2+\beta^2}{\alpha\beta}$
- and so on

Your notes

Examiner Tips and Tricks

- Don't forget to add $=0$ to your quadratic equations!
- Read the question carefully to see how to give your final answer
  - For example, if asked for integer coefficients

Worked Example

The quadratic equation $3x^2 + 12x - 1 = 0$ has roots α and β.

Without solving the equation, find a quadratic equation with roots

$$\frac{\beta}{\alpha} - \alpha \text{ and } \frac{\alpha}{\beta} - \beta$$

giving your answer in the form $Ax^2 + Bx + C = 0$ where A, B and C are integers to be determined.

Start with the equation $3x^2 + 12x - 1 = 0$

Use $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ to find the sum and product of the roots

$$\alpha + \beta = -\frac{12}{3} = -4 \text{ and } \alpha\beta = -\frac{1}{3}$$

Form the new equation using $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$$x^2 - \left(\frac{\beta}{\alpha} - \alpha + \frac{\alpha}{\beta} - \beta\right)x + \left(\frac{\beta}{\alpha} - \alpha\right)\left(\frac{\alpha}{\beta} - \beta\right) = 0$$

Simplify the coefficient of x

Add the algebraic fractions

$$\begin{aligned}\frac{\beta}{\alpha} - \alpha + \frac{\alpha}{\beta} - \beta &= \frac{\beta}{\alpha} + \frac{\alpha}{\beta} - (\alpha + \beta) \\ &= \frac{\beta^2 + \alpha^2}{\alpha\beta} - (\alpha + \beta)\end{aligned}$$

Your notes

Use (or derive) the identity for the sum of the squares of the roots

$\alpha^2 + \beta^2 \equiv (\alpha + \beta)^2 - 2\alpha\beta$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} - (\alpha + \beta)$$

Substitute in $\alpha + \beta = -4$ and $\alpha\beta = -\frac{1}{3}$

$$= \frac{(-4)^2 - 2\left(-\frac{1}{3}\right)}{\left(-\frac{1}{3}\right)} - (-4)$$

$$= -46$$

Now simplify the constant term

Expand the brackets and add the algebraic fractions

$$\left(\frac{\beta}{\alpha} - \alpha\right)\left(\frac{\alpha}{\beta} - \beta\right) = \frac{\beta\alpha}{\alpha\beta} - \frac{\alpha^2}{\beta} - \frac{\beta^2}{\alpha} + \alpha\beta$$

$$= 1 - \left(\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}\right) + \alpha\beta$$

$$= 1 - \frac{(\alpha^3 + \beta^3)}{\alpha\beta} + \alpha\beta$$

Use (or derive) the identity for the sum the cubes of the roots

$\alpha^3 + \beta^3 \equiv (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$$= 1 - \frac{[(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)]}{\alpha\beta} + \alpha\beta$$

Substitute in $\alpha + \beta = -4$ and $\alpha\beta = -\frac{1}{3}$

$$= 1 - \frac{\left[(-4)^3 - 3\left(-\frac{1}{3}\right)(-4)\right]}{\left(-\frac{1}{3}\right)} + \left(-\frac{1}{3}\right)$$

$$= 1 - 204 - \frac{1}{3}$$

$$= -\frac{610}{3}$$

Write out the new quadratic equation

Simplify the coefficients

$$x^2 - (-46)x + \left(-\frac{610}{3}\right) = 0$$

$$x^2 + 46x - \frac{610}{3} = 0$$

The question asks for the equation to have integer coefficients

Multiply both sides by 3

$$3x^2 + 138x - 610 = 0$$

Don't forget to write $= 0$ to get full marks

Any multiple of the answer (e.g. $6x^2 + 276x - 1220 = 0$) would also be accepted

Your notes