Edexcel International A Level (IAL) Further Maths: Further Pure 1

Your notes

Solving Equations with Complex Roots

Contents

Your notes

Quadratics with Complex Roots

How do I solve a quadratic with complex roots?

- **Quadratic equations** with **complex roots** can still be solved using the **quadratic formula** or by **completing the square**
  - They have the **form** $az^2 + bz + c = 0$ where a, b and c are **real** numbers
  - For **complex roots**, the **discriminant** is **negative**
    - $b^2 - 4ac < 0$
  - You must therefore rewrite the **square root** using i
    - $\sqrt{-16} = 4\mathrm{i}$, $\sqrt{-2} = \mathrm{i}\sqrt{2}$, ... and so on
- The **two roots**, z_1 and z_2 are **complex conjugates** of each other
  - (provided a, b and c from the equation are all **real** numbers)
  - So $z_1 = x + \mathrm{i}y$ and $z_2 = x - \mathrm{i}y$
    - $z_2 = z_1{}^*$

Examiner Tips and Tricks

Complex conjugate roots is a key concept that is used in many exam questions.

How do I solve a quadratic with one known complex root?

- If you **know** one complex root, the **other** will be its **complex conjugate**
  - If $z_1 = 4 + 5\mathrm{i}$ is a root to $z^2 - 8z + 41 = 0$
    - then $z_2 = 4 - 5\mathrm{i}$ must be the **other** root
    - No solving is required!

How do I find a quadratic from its complex root?

- If you are **given** one **complex** root, z_1, the **other** root is its **complex conjugate**, $z_1{}^*$
- You can then write its **quadratic equation** in **factorised form**
  - $(z - z_1)(z - z_1{}^*) = 0$

- **Expand** this to give the equation in **expanded form** $az^2 + bz + c = 0$
  - Any **multiple** of this equation also works
- A good **trick** when expanding is to **group** the **first two terms** in each bracket
  - So $(z - (2 + 3i))(z - (2 - 3i))$ becomes $((z - 2) - 3i)((z - 2) + 3i)$
  - Then use the **difference of two squares**
    - $(z - 2)^2 - (3i)^2 \equiv (z - 2)^2 + 9 \equiv z^2 - 4z + 4 + 9$
  - The equation is therefore $z^2 - 4z + 13 = 0$

Your notes

Examiner Tips and Tricks

Always check how the question wants the answer, for example asking for integer coefficients.

Worked Example

(a) Solve $z^2 - 2z + 37 = 0$.

Use the quadratic formula with $a = 1$, $b = -2$ and $c = 37$
Find the discriminant $b^2 - 4ac$

$$\begin{aligned} b^2 - 4ac &= (-2)^2 - 4 \times 1 \times 37 \\ &= 4 - 148 \\ &= -144 \end{aligned}$$

Substitute these values into the quadratic formula $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\begin{aligned} z &= \frac{-(-2) \pm \sqrt{-144}}{2 \times 1} \\ &= \frac{2 \pm \sqrt{-144}}{2} \end{aligned}$$

Use that $\sqrt{-144} = \sqrt{144i^2} = 12i$

$$z = \frac{2 \pm 12i}{2}$$

Simplify the answers
Check that they are complex conjugates of each other

$$z = 1 \pm 6i$$

(b) If $-\frac{1}{2}+\frac{3\mathrm{i}}{2}$ is one root of the equation $az^2+bz+c=0$ where a, b and c are positive integers, find a, b and c .

Write down the other root (the complex conjugate)

$$-\frac{1}{2}-\frac{3\mathrm{i}}{2}$$

Write down the quadratic equation in factorised form, $(z-z_1)(z-z_1{}^*)=0$

$$\left(z-\left(-\frac{1}{2}+\frac{3\mathrm{i}}{2}\right)\right)\left(z-\left(-\frac{1}{2}-\frac{3\mathrm{i}}{2}\right)\right)=0$$

Expand inside each bracket

$$\left(z+\frac{1}{2}-\frac{3\mathrm{i}}{2}\right)\left(z+\frac{1}{2}+\frac{3\mathrm{i}}{2}\right)=0$$

Group the first two terms in each bracket

$$\left(\left(z+\frac{1}{2}\right)-\frac{3\mathrm{i}}{2}\right)\left(\left(z+\frac{1}{2}\right)+\frac{3\mathrm{i}}{2}\right)=0$$

Use the difference of two squares to expand

$$\left(z+\frac{1}{2}\right)^2-\left(\frac{3\mathrm{i}}{2}\right)^2=0$$

Expand and collect, using $\mathrm{i}^2=-1$

$$z^2+z+\frac{1}{4}+\frac{9}{4}=0$$

$$z^2+z+\frac{5}{2}=0$$

The question wants the final answer to have positive integer coefficients
Multiply both sides of the equation by 2

$$2z^2+2z+5=0$$

Write down the values of a, b and c

$a=2$, $b=2$ and $c=5$

Positive integer multiples of these answers are also accepted

Cubics & Quartics with Complex Roots

How do I solve a cubic with complex roots?

- **Cubic equations** have either
  - **3 real** roots
  - or **1 real** root and one **complex conjugate pair** of roots

Your notes

- Solve $az^3 + bz^2 + cz + d = 0$, given that $m + n\mathrm{i}$ is a root
  - Another root is
    - Its complex conjugate
  - So and are **factors**
  - **Multiply** the factors together to get a **quadratic factor**
    -
  - You need to find the remaining **linear** factor $(Pz + Q)$
  - Either use **equating coefficients** to find
    - $(Az^2 + Bz + C)(Pz + Q) \equiv ax^3 + bx^2 + cx + d$
    - Expand and simplify the left-hand side
    - Match each coefficient with the right-hand side
  - Or use **polynomial division** to find $(Pz + Q)$
    - $(az^3 + bz^2 + cz + d) \div (Az^2 + Bz + C)$
  - Write out your **three roots** clearly

How do I solve a quartic with complex roots?

- **Quartic equations** have either
  - **4 real** roots
  - or **2 real** roots and **1 pair** of **complex conjugate** roots
  - or **2 pairs** of **complex conjugate** roots
- Solve $az^4 + bz^3 + cz^2 + dz + e = 0$, given that $m + n\mathrm{i}$ is a root
  - Another root is
    - Its complex conjugate
  - So and are **factors**
  - **Multiply** the factors together to get a **quadratic factor**
    -
  - You need to find the remaining **quadratic** factor $(Pz^2 + Qz + R)$
  - Either use **equating coefficients** to find P, Q and R
    - $(Az^2 + Bz + C)(Pz^2 + Qz + R) \equiv az^4 + bz^3 + cz^2 + dz + e$
    - Expand and simplify the left-hand side
    - Match each coefficient with the right-hand side
  - Or use **polynomial division** to find $(Pz^2 + Qz + R)$

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- $(az^4 + bz^3 + cz^2 + dz + e) \div (Az^2 + Bz + C)$
- Solve $Pz^2 + Qz + R = 0$
- Write out your **four roots** clearly

How do I find unknown coefficients of equations?

- Find as many **factors** as possible from **given** roots
  - **Real** roots give **linear** factors
    - $z = \alpha$ gives $(z - \alpha)$
  - **Complex** roots give **quadratic** factors (from **conjugate** pairs)
  - $z = m + ni$ gives $(z - (m + ni))(z - (m - ni)) \equiv Az^2 + Bz + C$
- Then write out the rest **algebraically** and use **equating coefficients**
  - Start by matching coefficients of the **highest powers** and of the **constants**
    - These can often been seen **by inspection**

Examiner Tips and Tricks

Whilst polynomial division can be used instead of equating coefficients, it is not recommended when exam questions have algebraic coefficients!

Worked Example

(a) Given that $-4 + 2\mathrm{i}$ is a root of the equation $z^3 + 5z^2 - 4z - 60 = 0$, find the other two roots.

Write down the other complex root by finding the complex conjugate

$$-4 - 2\mathrm{i}$$

Write down a factorised quadratic factor of the cubic
Use the form $(z - (-4 + 2\mathrm{i}))(z - (-4 - 2\mathrm{i}))$

$$(z - (-4 + 2\mathrm{i}))(z - (-4 - 2\mathrm{i}))$$

Expand inside each bracket

$$= (z + 4 - 2\mathrm{i})(z + 4 + 2\mathrm{i})$$

Group the first pair of terms in each bracket
Then expand using the difference between two squares
Simplify the quadratic factor

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$$= ((z+4) - 2i)((z+4) + 2i)$$
$$= (z+4)^2 - (2i)^2$$
$$= z^2 + 8z + 16 + 4$$
$$= z^2 + 8z + 20$$

Write the cubic as the quadratic factor multiplied by a linear factor $Pz + Q$

$$z^3 + 5z^2 - 4z - 60 \equiv (z^2 + 8z + 20)(Pz + Q)$$

Either expand the right-hand side and equate coefficients
Or see that $P = 1$ to get z^3 on the left, and $Q = -3$ to get -60 on the left

$$z^3 + 5z^2 - 4z - 60 \equiv (z^2 + 8z + 20)(z - 3)$$

$z^3 + 5z^2 - 4z - 60 = 0$ so set the factorised cubic equal to zero

$$(z^2 + 8z + 20)(z - 3) = 0$$

The solutions from the quadratic factor are the complex conjugates
The solution from $(z - 3) = 0$ is $z = 3$
Write down all three solutions

$z = -4 + 2i$, $-4 - 2i$ or 3

(b) It is known that the equation $z^4 - 8z^3 + Az^2 + Bz + 90$ has a complex root $1 + 3i$ and a repeated real root. Find A and B.

Write down the other complex root by finding the complex conjugate

$$1 - 3i$$

Write down a factorised quadratic factor of the quartic
Use the form $(z - (1 + 3i))(z - (1 - 3i))$

$$(z - (1 + 3i))(z - (1 - 3i))$$

Expand inside each bracket

$$= (z - 1 - 3i)(z - 1 + 3i)$$

Group the first pair of terms in each bracket
Then expand using the difference between two squares
Simplify the quadratic factor

$$= ((z-1) - 3i)((z-1) + 3i)$$
$$= (z-1)^2 - (3i)^2$$
$$= z^2 - 2z + 1 + 9$$
$$= z^2 - 2z + 10$$

The other two roots are repeated and real
Write the quartic as the quadratic factor multiplied by the square of a linear factor $(Pz + Q)^2$

$$z^4 - 8z^3 + Az^2 + Bz + 90 \equiv (z^2 - 2z + 10)(Pz + Q)^2$$

Imagine expanding the right-hand side
To get z^4 on the left, you can let P equal 1

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$$z^4 - 8z^3 + Az^2 + Bz + 90 \equiv (z^2 - 2z + 10)(z + Q)^2$$
$$\equiv (z^2 - 2z + 10)(z^2 + 2Qz + Q^2)$$

To get 90 on the left, let $Q^2 = 9$ so $Q = \pm 3$
To find out which value Q takes, expand and collect the right-hand side further

$$z^4 - 8z^3 + Az^2 + Bz + 90 \equiv z^4 + 2Qz^3 + 9z^2 - 2z^3 - 4Qz^2 - 18z + 10z^2 + 20Qz + 90$$
$$\equiv z^4 + (2Q - 2)z^3 + (9 - 4Q + 10)z^2 + (-18 + 20Q)z + 90$$

The z^4 and $+90$ are correct
Equate the z^3 coefficients to get an equation in Q and solve

$$-8 = 2Q - 2$$
$$-6 = 2Q$$
$$-3 = Q$$

Substitute $Q = -3$ into the coefficient of z^2 to get A

$$A = 9 - 4(-3) + 10$$
$$= 31$$

Substitute $Q = -3$ into the coefficient of z to get B

$$B = -18 + 20(-3)$$
$$= -78$$

Write out the answers clearly

$A = 31$ and $B = -78$

Polynomial division is not recommended as A and B are algebraic
$P = -1$ and $Q = 3$ also work, as the bracket $(Pz + Q)^2$ is squared

Your notes