Edexcel International A Level (IAL) Further Maths: Further Pure 1

Your notes

Operations with Complex Numbers

Contents

Your notes

Cartesian Form of Complex Numbers

What is an imaginary number?

- Equations like $x^2 = -9$ have **no real** solutions
  - **Squaring** real numbers always gives a **positive** value
    - No real number squared could give -9
  - $x = \pm 3$ are real numbers, but **neither** work
    - They give +9
- To get round this, mathematicians introduce the **imaginary number**, , as follows:
  -
    - This can be thought of as $\mathrm{i} = \sqrt{-1}$
- Rules for **surds** and **indices** can be used
  - $x^2 = -9$ means $x^2 = 9 \times (-1) = 9\mathrm{i}^2$
    - The **imaginary solutions** are $x = \pm 3\mathrm{i}$

What is a complex number?

- **Complex numbers** have both a **real part** and an **imaginary part**
  - For example,
    - The real part is 3
    - The imaginary part is 4
- This is called **Cartesian form**
- In general, Cartesian form is written using the **notation**:
  - $z = x + y\mathrm{i}$
  - $\mathrm{Re}(z) = x$
  - $\mathrm{Im}(z) = y$

How are real numbers and complex numbers related?

- The letter stands for all complex numbers
  - $z \in \mathbb{C}$
- Complex numbers with **no imaginary parts** are **real numbers**
  - The **real** numbers, $\mathbb{R}$, are a **subset** of the **complex** numbers,

- $\mathbb{R} \subset \mathbb{C}$

- In Cartesian form $z = x + y\text{i} \in \mathbb{C}$
  - $x \in \mathbb{R}$
  - $y \in \mathbb{R}$ as y itself takes real values
    - Multiplying it by makes it imaginary: $y\text{i}$
- Complex numbers with **no real parts** are called **imaginary numbers**

How do I add or subtract complex numbers?

- **Add** or **subtract** their **real** parts and **imaginary** parts **separately**
  - $(3 + 4\text{i}) + (2 + 8\text{i}) = (3 + 2) + (4 + 8)\text{i} = 5 + 12\text{i}$
  - $(3 + 4\text{i}) - (2 + 8\text{i}) = (3 - 2) + (4 - 8)\text{i} = 1 - 4\text{i}$

How do I multiply or divide complex numbers by real numbers?

- **Multiply** or **divide** their **real** parts and **imaginary** parts **separately**
  - $10(3 + 4\text{i}) = 30 + 40\text{i}$
  - $(3 + 4\text{i}) \div 10 = 0.3 + 0.4\text{i}$
    - This can also be written $\frac{1}{10}(3 + 4\text{i}) = \frac{3}{10} + \frac{4}{10}\text{i}$

Examiner Tips and Tricks

- Avoid these handwriting misinterpretations when writing in the exam:
  - $2\sqrt{5}\,\text{i}$ can look like $2\sqrt{5\text{i}}$
    - Alternatives are $(2\sqrt{5})\text{i}$ or $2\text{i}\sqrt{5}$
  - $\frac{3}{2}\text{i}$ can look like $\frac{3}{2\text{i}}$
    - An alternative is $\frac{3\text{i}}{2}$

Worked Example

Two complex numbers are given by $z_1 = p + 2\text{i}$ and $z_2 = -7 + q\text{i}$, where p and q are real.

Given that $z_1 + 2z_2 = 4 - 8\text{i}$, find p and q.

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Substitute the complex numbers into the left-hand side

$$(p+2\mathrm{i})+2(-7+q\mathrm{i})$$

Expand the brackets and collect real and imaginary terms

$$\begin{aligned}&= p+2\mathrm{i}-14+2q\mathrm{i}\\&=(\mathrm{p}-14)+(2+2q)\mathrm{i}\end{aligned}$$

Compare this to the right-hand side
Set the real parts equal to each other and solve

$$\begin{aligned}p-14&=4\\p&=18\end{aligned}$$

Set the imaginary parts equal to each other and solve

$$\begin{aligned}2+2q&=-8\\2q&=-10\\q&=-5\end{aligned}$$

$p=18$ and $q=-5$

Your notes

Multiplying Complex Numbers

How do I multiply complex numbers?

- All rules of **expanding brackets** still work
  - You need to remember that
- For example, $(a+b\mathrm{i})(c+d\mathrm{i})=ac+ad\mathrm{i}+bc\mathrm{i}+bd\mathrm{i}^2$
  - Use $\mathrm{i}^2=-1$ in the last term
    - $ac+ad\mathrm{i}+bc\mathrm{i}-bd$
  - Then group **real** and **imaginary** parts
    - **Factorise** out the
    - $ac-bd+(ad+bc)\mathrm{i}$
- Note that the **difference between two squares** becomes
  - $(a+b\mathrm{i})(a-b\mathrm{i})=a^2-b^2\mathrm{i}^2=a^2+b^2$

How do I find powers of i?

- Use the fact that
  - Below are the first few powers of :
    - $\mathrm{i}^0=1$
    - $\mathrm{i}^1=\mathrm{i}$
    - $\mathrm{i}^2=-1$

- $i^3 = -i$ from $i^2 \times i = (-1) \times i$
- $i^4 = 1$ from $(i^2)^2 = (-1)^2 = 1$
- $i^5 = i$ from $i^5 = (i^2)^2 \times i = i$
- The **pattern** above continues
  - $i^6 = -1$
  - $i^7 = -i$
- Find **higher powers** of using a **base** of i^2
  - Remember that **-1** to an **even** power is 1 (or to an **odd** power is -1)
    - $i^{23} = (i^2)^{11} \times i = (-1)^{11} \times i = -i$

Examiner Tips and Tricks

Questions that say "show your working clearly" won't accept answers written down from a calculator.

Worked Example

Showing your working clearly, find and simplify:

(a) $(4 + i)(2 + 9i)$

Expand the brackets

$$\begin{aligned} &= 4 \times 2 + 4 \times 9i + i \times 2 + i \times 9i \\ &= 8 + 36i + 2i + 9i^2 \end{aligned}$$

Collect the imaginary parts
Use that then collect the real parts

$$\begin{aligned} &= 8 + 38i + 9 \times (-1) \\ &= 8 + 38i - 9 \end{aligned}$$

$$-1 + 38i$$

(b) $(3 - 4i)^2$

Write using double brackets then expand

$$\begin{aligned} &= (3 - 4i)(3 - 4i) \\ &= 9 - 12i - 12i + 16i^2 \end{aligned}$$

Collect the imaginary parts
Use that then collect the real parts

Your notes

$$= 9 - 24i + 16 \times (-1)$$
$$= 9 - 24i - 16$$

$$-7 - 24i$$

(c) $(-2i)^{11}$

Use index laws to move the power onto the individual terms

$$(-2)^{11} \times i^{11}$$

Work out $(-2)^{11}$

$$(-2)^{11} = -2048$$

Work out i^{11}
It helps to write it in terms of i^{10} then i^2
Use

$$\begin{aligned} i^{11} &= i^{10} \times i \\ &= (i^2)^5 \times i \\ &= (-1)^5 \times i \\ &= -1 \times i \\ &= -i \end{aligned}$$

Multiply both parts together
Two minus signs make a plus

$$-2048 \times (-i)$$

$$2048i$$

Complex Conjugates & Division

What is a complex conjugate?

- If $z = x + yi$ then the **complex conjugate** of z is $z^* = x - yi$
  - The **sign** of the **imaginary part** changes
- Note that
  - $z + z^*$ is always **real**
    - since $x + yi + x - yi = 2x$
  - $z - z^*$ is always **imaginary**
    - since $x + yi - (x - yi) = 2yi$
  - zz^* is always **real** (and non-negative)
    - since $(x + yi)(x - yi) = x^2 - y^2i^2 = x^2 + y^2$
    - $zz^* = z^*z$

How do I divide complex numbers?

Your notes

- To **divide** z_1 by z_2, **multiply top and bottom** of $\frac{z_1}{z_2}$ by $z_2{}^*$
  - $z_2{}^*$ is the **complex conjugate** of the **denominator**
- This makes the denominator a **real** number
  - which allows you to write the final answer in Cartesian form, $x + y\mathrm{i}$
- The process is called **realising the denominator**
  - It is a very similar to rationalising the denominator with surds
- For example, to work out $\frac{50+75\mathrm{i}}{3+4\mathrm{i}}$
  - calculate $\frac{(50+75\mathrm{i})}{(3+4\mathrm{i})} \times \frac{(3-4\mathrm{i})}{(3-4\mathrm{i})}$
    - It helps to write the **brackets** in
  - then **expand** and **simplify**

Your notes

Examiner Tips and Tricks

To check your answer in an exam, multiply it by the denominator and see if you get the numerator.

Worked Example

Let $z_1 = 1 + 7\mathrm{i}$ and $z_2 = 3 - \mathrm{i}$.

Find and simplify $\frac{z_1}{z_2}$, giving your answer in the form $x + y\mathrm{i}$ where x and y are real numbers.

Find the complex conjugate of the denominator

$$z_2{}^* = 3 + \mathrm{i}$$

Multiply the top and bottom of $\frac{z_1}{z_2}$ by $z_2{}^*$

$$\frac{(1+7\mathrm{i})}{(3-\mathrm{i})} \times \frac{(3+\mathrm{i})}{(3+\mathrm{i})}$$

Write as one single fraction then expand top and bottom separately

$$\frac{(1+7i)(3+i)}{(3-i)(3+i)} = \frac{3+i+21i+7i^2}{9-i^2}$$

Use that then collect real and imaginary parts

$$= \frac{3+i+21i-7}{9-(-1)}$$
$$= \frac{-4+22i}{10}$$

To give your answer in the form $x+yi$, split the fraction then simplify

$$-\frac{4}{10}+\frac{22}{10}i$$

$$-\frac{2}{5}+\frac{11}{5}i$$

$-0.4+2.2i$ is also accepted

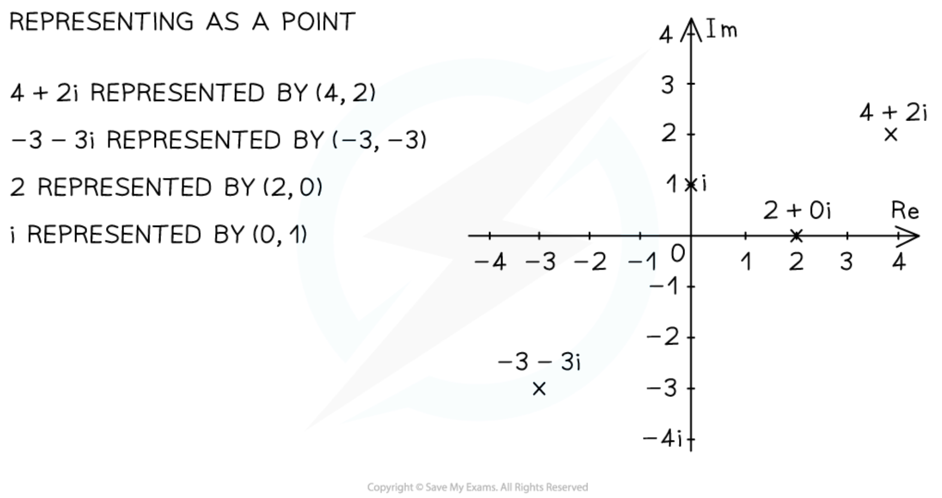
Your notes

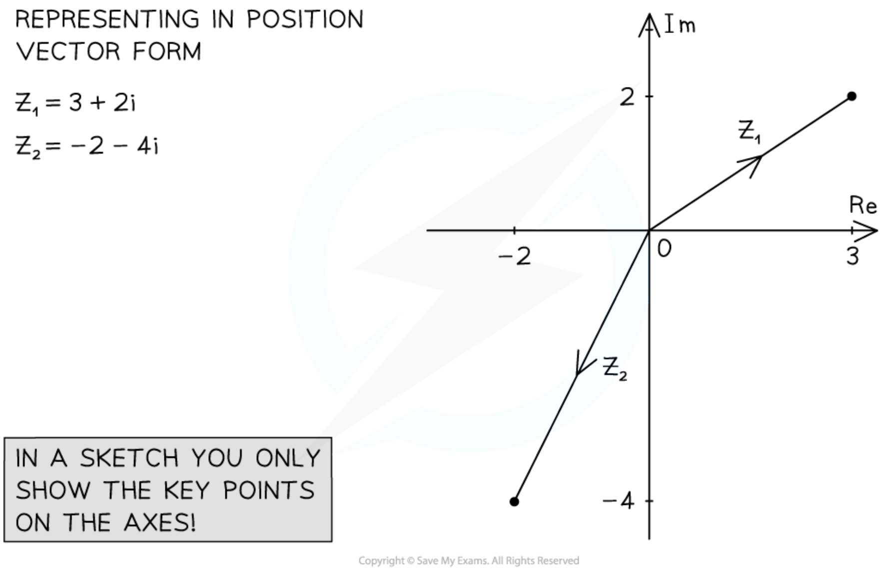
Your notes

Argand Diagrams

What is an Argand diagram?

- An **Argand diagram** is a **2D Cartesian grid** used to **visualise** complex numbers
- The complex number is represented by the **point** with **coordinates**
  - The **real** part is measured along the ***x*** **-axis**
    - called the **real axis**, written "**Re**"
  - The **imaginary** part is measured along the ***y*** **-axis**
    - called the **imaginary axis**, written "**Im**"
- Complex numbers can be thought of as **points**, or as **vectors** from the origin



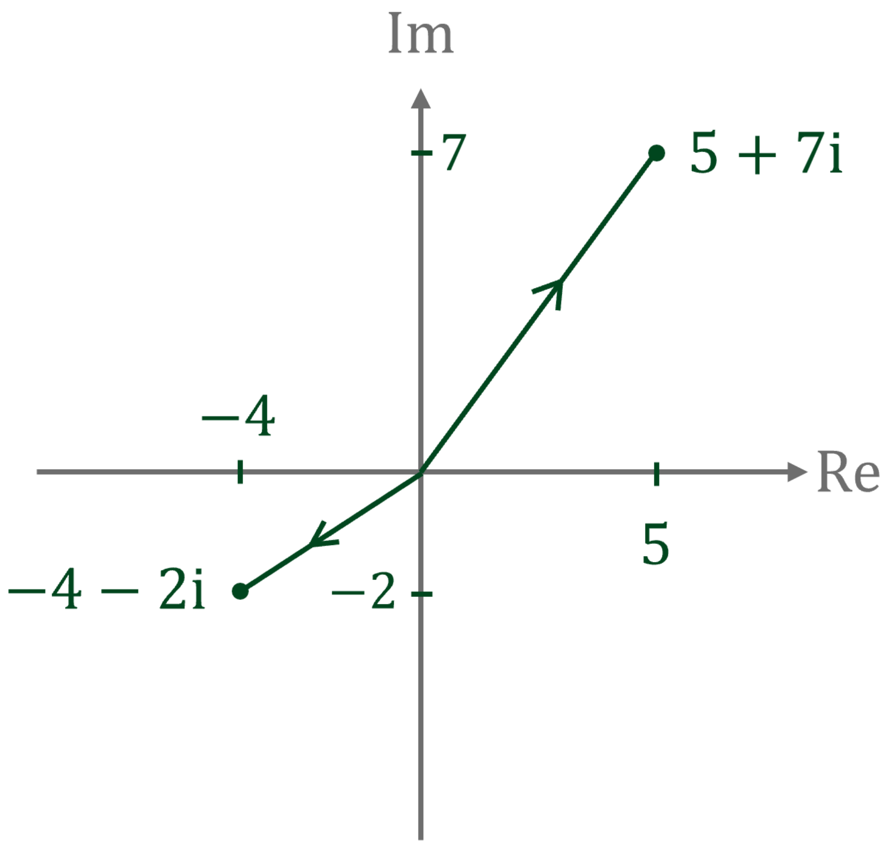


Examiner Tips and Tricks

If asked to sketch an Argand diagram, it does not need to be to scale (plotted), but should roughly show the correct positions.

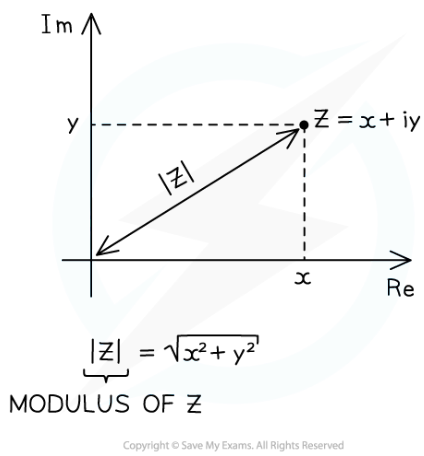
Worked Example

Sketch, on the same Argand diagram, the complex numbers $5 + 7\text{i}$ and $-4 - 2\text{i}$.



Modulus & Argument

How do I find the modulus of a complex number?



- The **modulus** of a complex number is its **distance** from the **origin** on an **Argand diagram**
  - It is written
  - If , then by **Pythagoras' theorem**
    -

Your notes

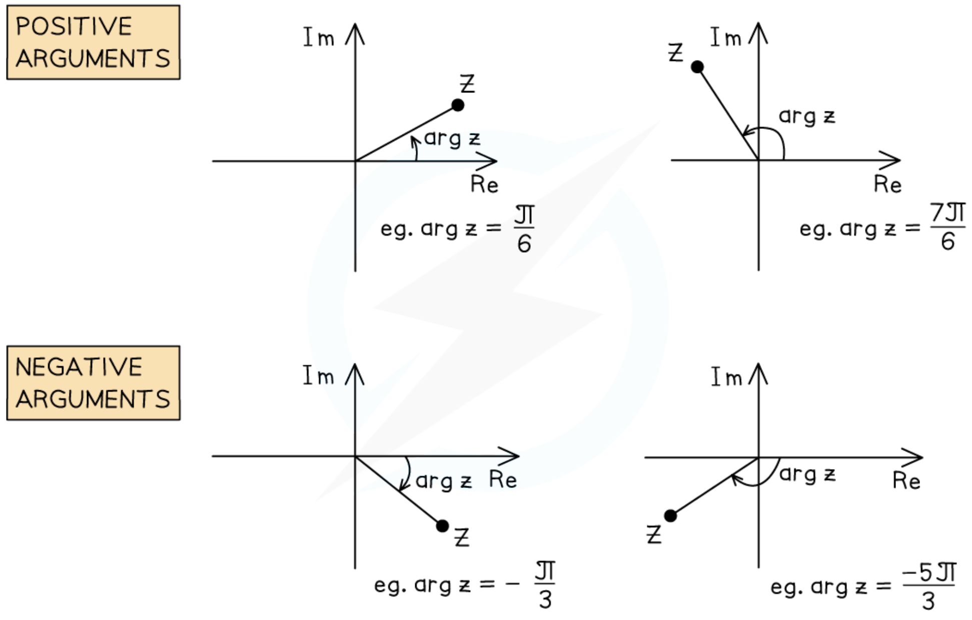
- A modulus is always **positive** or **zero**
  - It cannot be negative
- For example
  - $|3+4i| = \sqrt{3^2+4^2} = 5$
  - $|1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$
  - $|-5| = 5$
  - $|8i| = 8$
  - $|0+0i| = 0$

What rules does the modulus follow?

- Helpful **modulus rules** are:
  - $|z_1 z_2| = |z_1||z_2|$
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
  - $|z| = |z^*|$
  - $zz^* = z^* z = |z|^2$
    - Proved using $z = x + yi$ and $z^* = x - yi$
- Be careful:
  - For example, and
    - $|z_1| = \sqrt{3^2+4^2} = 5$ and $|z_2| = \sqrt{(-3)^2+4^2} = 5$
    - so $|z_1| + |z_2| = 10$
    - but $z_1 + z_2 = 8i$ so $|z_1 + z_2| = 8$

How do I find the argument of a complex number?

Your notes



Your notes

- The **argument** of a complex number is the **angle** in **radians** that it makes to the **positive real** axis
  - It is written as $\arg z$
  - The positive direction is **anticlockwise**
- The range normally used is
  - This is called the **principal range**
  - The sign of the angle depends on the **quadrant**:
    - The **1st** quadrant is **positive acute**
    - The **2nd** quadrant is **positive obtuse**
    - The **3rd** quadrant is **negative obtuse**
    - The **4th** quadrant is **negative acute**
- Arguments are found by
  - drawing a **sketch**
  - forming a **right-angled** triangle
  - using **trigonometry**
- The argument of the **origin**, $\arg\left(0+0\mathrm{i}\right)$, is **undefined**
  - No angle can be drawn

Your notes

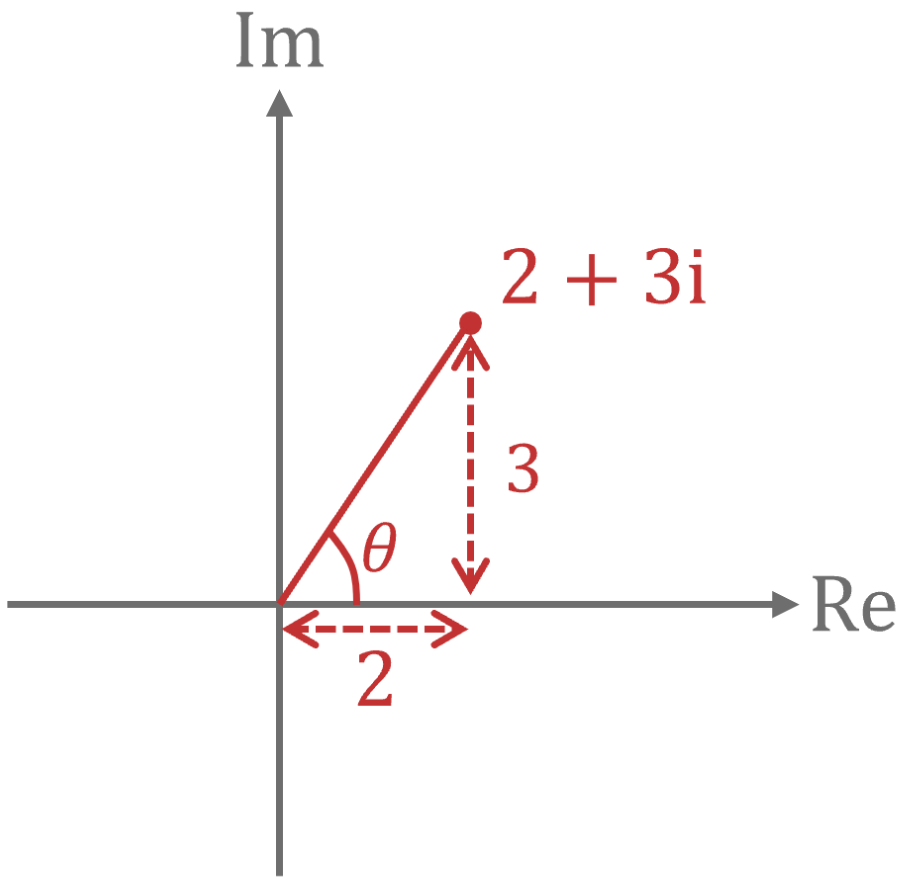
Examiner Tips and Tricks

Always draw a sketch to see which quadrant the complex number is in.

Worked Example

(a) Find the modulus and argument of , giving your answers correct to 3 significant figures.

Sketch this on an Argand diagram
Form a right-angled triangle



Use (or Pythagoras) to find the modulus

$$\begin{aligned}|z| &= \sqrt{2^2+3^2}\\ &= \sqrt{13}\\ &= 3.60555127...\end{aligned}$$

z is in the first quadrant so the argument is positive and acute
Use trigonometry to find the argument θ in radians

$$\tan\theta = \frac{3}{2}$$

$$\theta = \tan^{-1}\left(\frac{3}{2}\right)$$

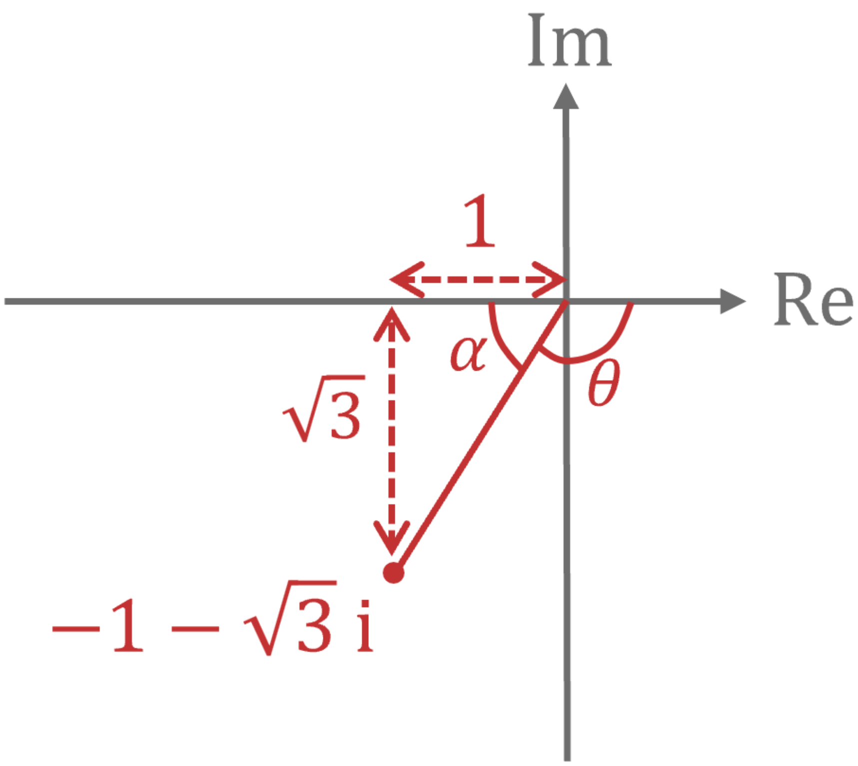
$$\theta = 0.98279372...$$

Round the answers to 3 significant figures

$|z| = 3.61$ and $\arg z = 0.983$ to 3 significant figures

(b) Find the modulus and argument of , leaving your answers as exact values.

Sketch this on an Argand diagram
Form a right-angled triangle



Use (or Pythagoras) to find the modulus

$$|z| = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2}$$
$$= \sqrt{4}$$
$$= 2$$

z is in the third quadrant so the argument is negative and obtuse
Use trigonometry to first find α in radians

$$\tan\alpha = \frac{\sqrt{3}}{1}$$

$$\alpha = \tan^{-1}\left(\sqrt{3}\right)$$

$$\alpha = \frac{\pi}{3}$$

Then find θ by subtracting α from 180° (π radians)

$$\theta = \pi - \alpha$$

$$= \pi - \frac{\pi}{3}$$

$$= \frac{2\pi}{3}$$

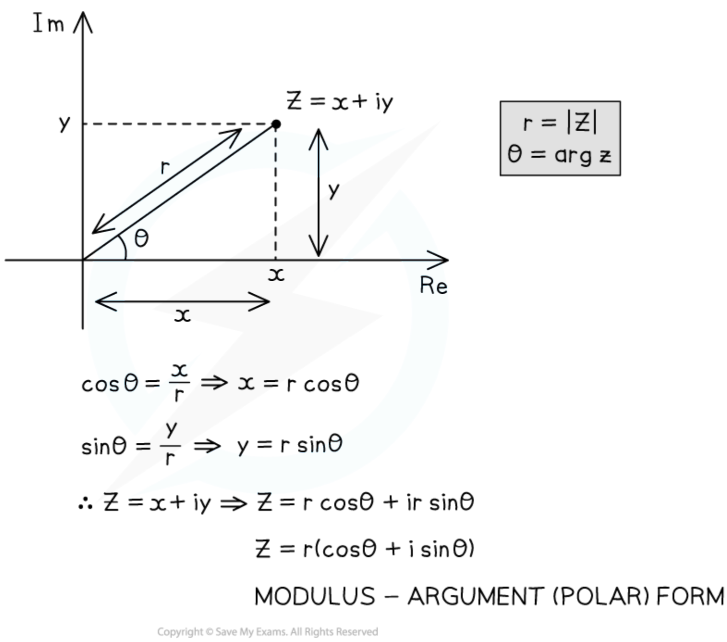
Remember that the argument here must be a negative angle

$|z| = 2$ and $\arg z = -\frac{2\pi}{3}$

These answers must be exact

Modulus-Argument Form

What is modulus-argument form?



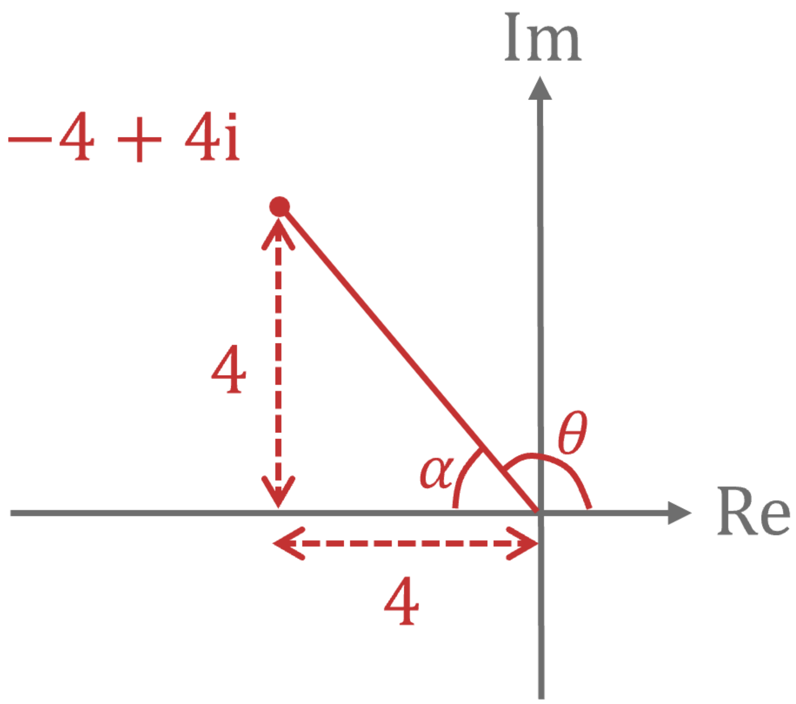
- All complex numbers can be written in the form:
  -
  - where and
    - This is called the **modulus-argument** (or polar) form

- **Negative arguments** should be shown clearly **without simplifying**
  -
    - Simplifying them gives the **Cartesian form**
    -
- Be careful: is **not** in modulus-argument form (due to the **minus sign**)
  - Rewrite it as
    - This uses the **symmetries** $\cos(-\theta) \equiv \cos(\theta)$ and $\sin(-\theta) \equiv -\sin\,\theta$
  - The argument is

Worked Example

Write in the exact form $r(\cos\,\theta\,+\,\mathrm{i}\,\sin\,\theta)$ where $r > 0$ and $-\pi < \theta \leq \pi$.

Draw a sketch to find the modulus, r, and argument, θ, of z
Form a right-angled triangle



Use (or Pythagoras) to find the modulus

$$\begin{aligned} |z| &= \sqrt{(-4)^2 + (4)^2} \\ &= \sqrt{32} \\ &= 4\sqrt{2} \end{aligned}$$

Your notes

z is in the second quadrant so the argument is positive and obtuse
Use trigonometry to first find α in radians

$$\tan\alpha = \frac{4}{4}$$
$$\alpha = \tan^{-1}(1)$$
$$\alpha = \frac{\pi}{4}$$

Then find θ by subtracting α from 180° (π radians)

$$\theta = \pi - \alpha$$
$$= \pi - \frac{\pi}{4}$$
$$= \frac{3\pi}{4}$$

Write the final answer in the form $r(\cos\theta + \mathrm{i}\sin\theta)$

$$z = 4\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right)$$

Leave the modulus and argument exact

Your notes

Your notes

Equating Real & Imaginary Parts

How do I equate real and imaginary parts?

- If two complex expressions are **equal**, then
  - their **real** parts are **equal**
  - and their **imaginary** parts are **equal**
- If $a + bi = 8 - 10i$ where a and b are real
  - then $a = 8$ (equating real parts)
  - and $b = -10$ (equating imaginary parts)

How do I solve equations using real and imaginary parts?

- **Introduce** $z = x + yi$ for expressions in z
- For example, solve $z + 2z^* = 12 - i$
  - Let $z = x + yi$
  - **Substitute** in
    - $(x + yi) + 2(x - yi) = 12 - i$
  - **Expand** and **collect** terms
    - $3x - yi = 12 - i$
  - **Equate real** and **imaginary** parts
    - $3x = 12$ gives $x = 4$
    - $-y = -1$ gives $y = 1$
  - **Substitute back** into z
    - $z = 4 + i$

How do I use real and imaginary parts with modulus signs?

- If $z = x + yi$ then $|z| = \sqrt{x^2 + y^2}$
- Helpful **modulus rules** are:
  - $|z_1 z_2| = |z_1||z_2|$

Your notes

- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $|z| = |z^*|$
- $zz^* = z^*z = |z|^2$
  - Proved using $z = x + y\text{i}$ and $z^* = x - y\text{i}$
- For example, find p if $z_1 = p + p\text{i}$, $z_2 = 2\text{i}$ and $|z_1 z_2| = 6\sqrt{2}$
  - Use $|z_1 z_2| = |z_1||z_2|$
    - $|p + p\text{i}||2\text{i}| = 6\sqrt{2}$
  - Use $|z| = \sqrt{x^2 + y^2}$
    - $\sqrt{p^2 + p^2} \times 2 = 6\sqrt{2}$ so $2\sqrt{2p^2} = 6\sqrt{2}$
  - **Square both sides** and simplify
    - $p^2 = 9$
    - $p = \pm 3$

Examiner Tips and Tricks

Harder exam questions may not tell you to write z as $x + y\text{i}$ (you have to spot it yourself).

Worked Example

Let z and w be complex numbers.
It is known that $w = 5 + 2\text{i}$ and that $z^2 + 10w = zz^*$.

Find the two possible values of z.

Write z in the form $x + y\text{i}$

$$z = x + y\text{i}$$

Substitute this, and w, into the equation

$$(x + y\text{i})^2 + 10(5 + 2\text{i}) = (x + y\text{i})(x - y\text{i})$$

Expand the brackets and use $\text{i}^2 = -1$

$$x^2 + 2xy\text{i} + y^2\text{i}^2 + 50 + 20\text{i} = x^2 - y^2\text{i}^2$$
$$x^2 + 2xy\text{i} - y^2 + 50 + 20\text{i} = x^2 + y^2$$

Equate the real parts and solve for y

$$x^2 - y^2 + 50 = x^2 + y^2$$
$$50 = 2y^2$$
$$25 = y^2$$
$$y = \pm 5$$

Now equate the imaginary parts and solve for x
Note that there are no imaginary parts on the right

$$2xy + 20 = 0$$
$$xy = -10$$
$$x = -\frac{10}{y}$$

When $y = 5$, then $x = -2$
When $y = -5$, then $x = 2$
Substitute these back in to get z

$-2 + 5\text{i}$ or $2 - 5\text{i}$

Your notes