3

Pure 3

Revision Notes

Edexcel IAL from 2018

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Pure 3

1 Algebraic fractions

Cancelling common factors

Example: Simplify $\dfrac{x^3 - 2x^2 - 3x}{4x^2 - 36}$.

Solution: First factorise top and bottom fully –

$$\frac{x^3 - 2x^2 - 3x}{4x^2 - 36} = \frac{x(x^2 - 2x - 3)}{4(x^2 - 9)} = \frac{x(x-3)(x+1)}{4(x-3)(x+3)}$$

and now cancel all common factors, in this case $(x - 3)$ to give

Answer = $\dfrac{x(x+1)}{4(x+3)}$.

Multiplying and dividing fractions

This is just like multiplying and dividing fractions with numbers and then cancelling common factors as above.

Example: Simplify $\dfrac{9x^2 - 4}{3x^2 - 2x} \div \dfrac{3x^2 - x - 2}{x^3 + x^2}$

Solution: First turn the second fraction upside down and multiply

$$= \frac{9x^2 - 4}{3x^2 - 2x} \times \frac{x^3 + x^2}{3x^2 - x - 2}$$ factorise fully

$$= \frac{(3x-2)(3x+2)}{x(3x-2)} \times \frac{x^2(x+1)}{(3x+2)(x-1)}$$ cancel all common factors

Answer = $\dfrac{x(x+1)}{x-1}$.

Adding and subtracting fractions

Again this is like adding and subtracting fractions with numbers; **but** finding the Lowest Common Denominator can save a lot of trouble later.

Example: Simplify $\dfrac{3x}{x^2 - 7x + 12} - \dfrac{5}{x^2 - 4x + 3}$.

Solution: First factorise the denominators

$$= \frac{3x}{(x-3)(x-4)} - \frac{5}{(x-3)(x-1)}$$ we see that the L.C.D. is $(x - 3)(x - 4)(x - 1)$

$$= \frac{3x(x-1)}{(x-3)(x-4)(x-1)} - \frac{5(x-4)}{(x-3)(x-1)(x-4)}$$

$$= \frac{3x^2 - 3x - 5x + 20}{(x-1)(x-3)(x-4)} = \frac{3x^2 - 8x + 20}{(x-1)(x-3)(x-4)}$$ which cannot be simplified further.

Equations

Example: Solve $\dfrac{x}{x+1} - \dfrac{x-1}{x} = \dfrac{1}{2}$

Solution: First multiply **both sides** by the Lowest Common Denominator

$\dfrac{x}{x+1} - \dfrac{x-1}{x} = \dfrac{1}{2}$ multiply both sides by $2x(x + 1)$

$\Rightarrow$ $x \times 2x - (x - 1) \times 2(x + 1) = x(x + 1)$

$\Rightarrow$ $2x^2 - 2x^2 + 2 = x^2 + x$

$\Rightarrow$ $x^2 + x - 2 = 0,$

$\Rightarrow$ $(x + 2)(x - 1) = 0$

$\Rightarrow$ $x = -2$ or 1

Example: Simplify $\dfrac{\frac{x}{x-1} - \frac{x-3}{3x+1}}{x+3}$

Solution: First multiply top and bottom by $(x-1)(3x+1)$

$$= \frac{\frac{x}{x-1} - \frac{x-3}{3x+1}}{x+3} \times \frac{(x-1)(3x+1)}{(x-1)(3x+1)}$$

$$= \frac{x(3x+1) - (x-3)(x-1)}{(x+3)(x-1)(3x+1)}$$

$$= \frac{3x^2 + x - x^2 + 4x - 3}{(x+3)(x-1)(3x+1)}$$

$$= \frac{2x^2 + 5x - 3}{(x+3)(x-1)(3x+1)}$$

$$= \frac{(2x-1)(x+3)}{(x+3)(x-1)(3x+1)}$$

$$= \frac{(2x-1)}{(x-1)(3x+1)}$$

2 Functions

A function is an expression (often in x) which has **only one** value for each value of x.

Notation

$y = x^2 - 3x + 7$, $\quad f(x) = x^2 - 3x + 7 \quad$ and $\quad f : x \to x^2 - 3x + 7$

are all ways of writing the same function.

Domain, range and graph

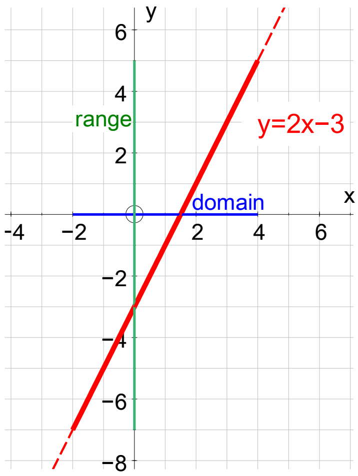
The *domain* is the set of values which x can take:

this is sometimes specified in the definition
sometimes is evident from the function: e.g. for $\sqrt{x}$, x can only take positive x or zero values.

The *range* is that part of the y–axis which is used.

Example: Find the range of the function

$f : x \to 2x - 3$ with domain $x \in \Re$: $-2 < x \leq 4$.



Solution: First sketch the graph for values of x between -2 and 4 (the domain), and we can see that we are only using the y–axis from -7 to 5,

not including $y = -7$ (since $x \neq -2$),
but including $y = 5$ (since x can equal 4)
and so the range is
$y \in \Re$: $-7 < y \leq 5$.

Example: Find the largest possible domain and the range for the function

$f : x \to \sqrt{x-3} + 1$. Note that $\sqrt{x}$ is defined as positive or zero **only**.

Solution: First notice that we cannot have the square root of a negative number and so

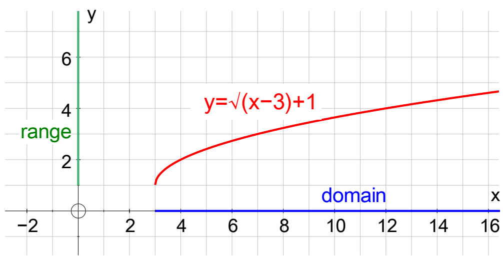
$x - 3$ cannot be negative

$\Rightarrow \quad x - 3 \geq 0$

$\Rightarrow \quad$ largest possible domain is $x \in \Re$: $x \geq 3$.

To find the range we first sketch the graph and we see that the graph will cover all of the y-axis from 1 upwards

and so the range is $y \in \Re$: $y \geq 1$.



Example: Find the largest possible domain and the range for the function

$f : x \to \dfrac{2x}{x+1}$. Give the equations of its asymptotes.

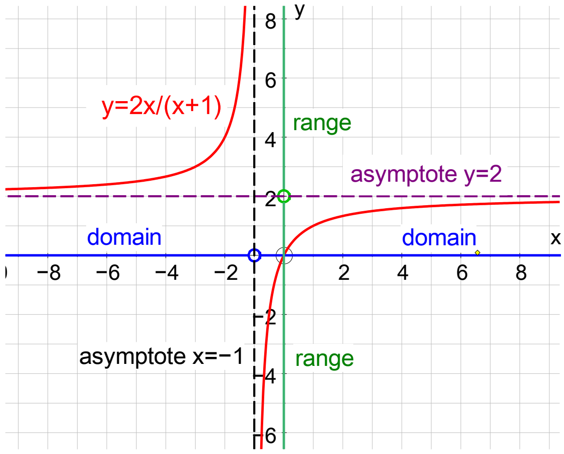
Solution: The only problem occurs when the denominator is 0, and so x cannot be –1.

Thus the largest domain is $x \in \Re$: $x \neq -1$.

To find the range we sketch the graph and we see that y can take any value except 2,

so the range is $y \in \Re$: $y \neq 2$.

Asymptotes are $x = -1$ and $y = 2$



Defining functions

Some mappings can be made into functions by restricting the domain.

Examples:

1) The mapping $x \to \sqrt{x}$ where $x \in \Re$ is not a function as $\sqrt{-9}$ is not defined, but if we restrict the domain to positive or zero real numbers then $f : x \to \sqrt{x}$ where $x \in \Re, x \geq 0$ **is** a function.

2) $x \to \dfrac{1}{x-3}$ where $x \in \Re$ is not a function as the image of $x = 3$ is not defined,

   but $f : x \to \dfrac{1}{x-3}$ where $x \in \Re$, $x \neq 3$ **is** a function.

Composite functions

To find the composite function fg we must do g first.

Example: $f : x \to 3x - 2$ and $g : x \to x^2 + 1$. Find fg and gf.

Solution: Think of f and g as 'rules'

f is $\boxed{\text{multiply by } 3} \to \boxed{\text{subtract } 2}$

g is $\boxed{\text{square}} \to \boxed{\text{add } 1}$

$\Rightarrow$ fg is $\boxed{\text{square}} \to \boxed{\text{add } 1} \to \boxed{\text{multiply by } 3} \to \boxed{\text{subtract } 2}$

giving $(x^2 + 1) \times 3 - 2 = 3x^2 + 1$

$\Rightarrow$ $fg : x \to 3x^2 + 1$ or $fg(x) = 3x^2 + 1$.

gf is $\boxed{\text{multiply by } 3} \to \boxed{\text{subtract } 2} \to \boxed{\text{square}} \to \boxed{\text{add } 1}$

giving $(3x - 2)^2 + 1 = 9x^2 - 12x + 5$

$\Rightarrow$ $gf : x \to 9x^2 - 12x + 5$ or $gf(x) = 9x^2 - 12x + 5$.

Note that fg and gf are **not** the same.

Inverse functions and their graphs

The inverse of f is the 'opposite' of f:

thus the inverse of 'multiply by 3' is 'divide by 3'

and the inverse of 'square' is 'square root'.

The inverse of f is written as f^{-1}: note that this does **not** mean '1 over f'.

The *graph* of $y = f^{-1}(x)$ is the reflection in $y = x$ of the graph of $y = f(x)$.

To find the inverse of a function $x \to y$

(i) interchange x and y

(ii) find y in terms of x.

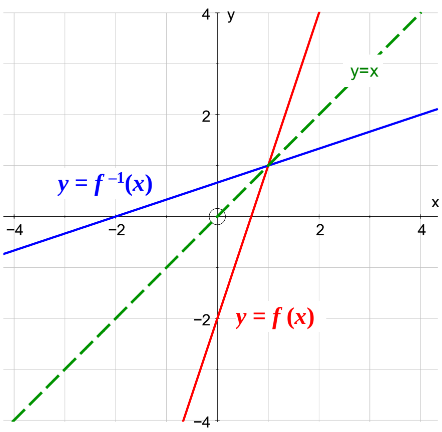
Example: Find the inverse of $f : x \to 3x - 2$.

Solution: We have $x \to y = 3x - 2$

(i) interchanging x and y $\Rightarrow$ $x = 3y - 2$

(ii) solving for y $\Rightarrow$ $y = \dfrac{x+2}{3}$

$\Rightarrow$ $f^{-1} : x \to \dfrac{x+2}{3}$



Example: Find the inverse of $g : x \to \dfrac{x+3}{2x-5}$.

Solution: We have $x \to y = \dfrac{x+3}{2x-5}$

(i) interchanging x and y

$\Rightarrow x = \dfrac{y+3}{2y-5}$

(ii) solving for y

$\Rightarrow$ $x(2y - 5) = y + 3$

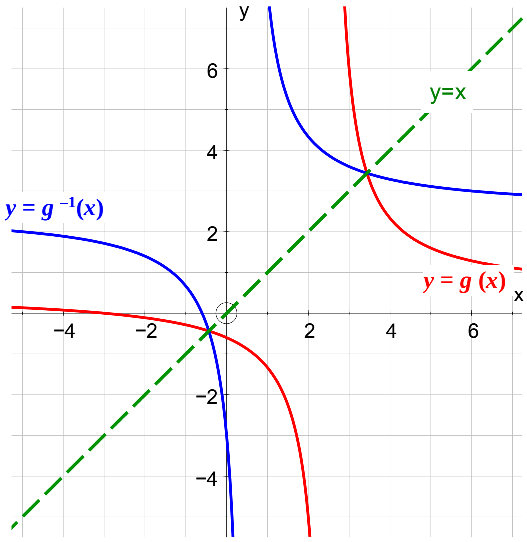
$\Rightarrow$ $2xy - 5x = y + 3$

$\Rightarrow$ $2xy - y = 5x + 3$

$\Rightarrow$ $y(2x - 1) = 5x + 3$

$\Rightarrow$ $y = \dfrac{5x+3}{2x-1}$

$\Rightarrow$ $g^{-1} : x \to \dfrac{5x+3}{2x-1}$



Note that $f f^{-1}(x) = f^{-1} f(x) = x$.

Domain and range of inverse functions

Note that the domain of $f(x)$ is the range of $f^{-1}(x)$,
and that the range of $f(x)$ is the domain of $f^{-1}(x)$.
This is because the graph of $y = f^{-1}(x)$ is that of $y = f(x)$ after a reflection in the line $y = x$.

Example: $f(x) = (x - 3)^2 + 4, \quad x \in \Re, \ x \le 3.$

(*a*) Sketch the graph of $y = f(x)$, and state its range.

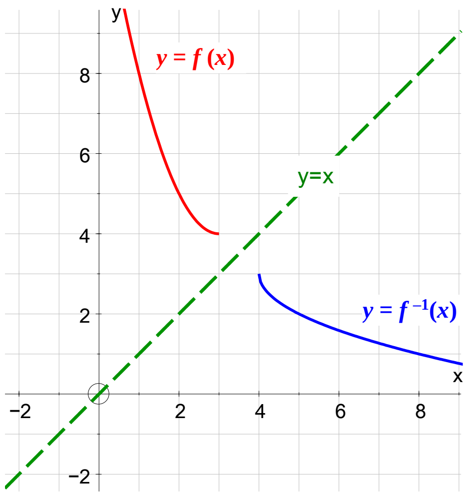
(*b*) Find the inverse function, $f^{-1}(x)$ and sketch its graph on the same diagram. Show the line $y = x$ on your diagram.

(*c*) State the domain and range of $f^{-1}(x)$.

Solution:

(*a*) As the domain of f is $x \le 3$, we only have the 'left' part of the parabola.

The range is $f(x) \ge 4, \ f(x) \in \Re$.



(*b*) To find the inverse, swap x and y, then find y.

$$x = (y - 3)^2 + 4$$

$$\Rightarrow \quad y - 3 = \pm\sqrt{x - 4}\,.$$

From the reflection of $y = f(x)$ in $y = x$, we can see that we want the *negative* sign

$$\Rightarrow \quad y = 3 - \sqrt{x - 4}.$$

(*c*) The domain of $f^{-1}(x)$ is $x \ge 4$ and the range of $f(x)$ is $f(x) > 4$.

The range of $f^{-1}(x)$ is $f^{-1}(x) \le 3$ and the domain of $f(x)$ is $x \le 3$.

Modulus functions

Modulus functions $y = |f(x)|$

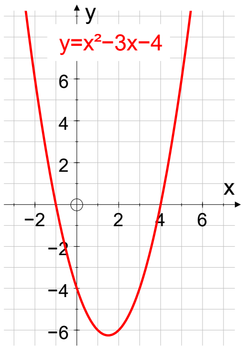
$|f(x)|$ is the 'positive value of $f(x)$',

so to sketch the graph of $y = |f(x)|$ first sketch the graph of $y = f(x)$ and then reflect the part(s) below the x–axis to above the x–axis.

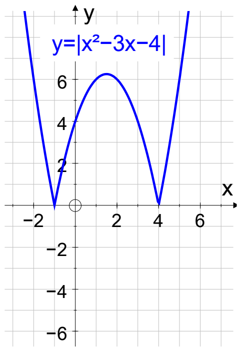
Example: Sketch the graph of $y = |x^2 - 3x - 4|$.

Solution:

First sketch the graph of $y = x^2 - 3x - 4$



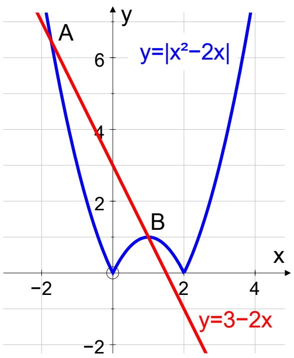
Then reflect the portion below the x-axis in the x–axis to give



Example: (*a*) Solve the equation $|x^2 - 2x| = 3 - 2x$.

(*b*) Find the range of values of x which satisfy $|x^2 - 2x| < 3 - 2x$.

Solution: First sketch the graphs of $y = |x^2 - 2x|$ and $y = 3 - 2x$



The graphs meet at A and B.

A is the intersection of $y = x^2 - 2x$ and $y = 3 - 2x$

$\Rightarrow \quad x^2 - 2x = 3 - 2x \;\Rightarrow\; x = \pm\sqrt{3}$

From the graph we see that $x = -\sqrt{3}$ only.

B is the intersection of $y = x^2 - 2x$ and $y = -(3 - 2x)$

$\Rightarrow \quad x^2 - 2x = -3 + 2x \quad \Rightarrow \quad x^2 - 4x + 3 = 0$

$\Rightarrow \quad (x - 1)(x - 3) = 0 \quad \Rightarrow \quad x = 1$ or 3

From the graph we see that $x = 1$ only.

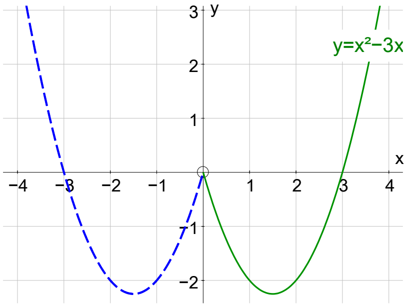
(*b*) $|x^2 - 2x| < 3 - 2x$ is when the curve lies below the line

$\Rightarrow \quad -\sqrt{3} < x < 1$.

Modulus functions $y = f(|x|)$

In this case $f(-3) = f(3)$, $f(-5) = f(5)$, $f(-8.7) = f(8.7)$ etc. and so the graph on the left of the y–axis must be the reflection of the graph on the right of the y–axis,

so to sketch the graph, first sketch the graph for **positive values of x only**, then reflect the graph sketched in the y–axis.



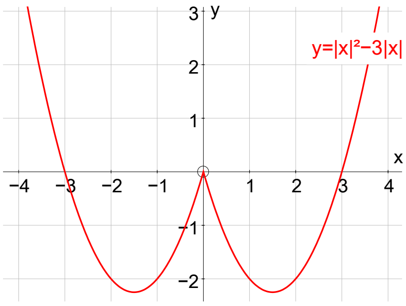
Example: Sketch the graph of $y = |x|^2 - 3|x|$.

Solution: $f(x) = x^2 - 3x$

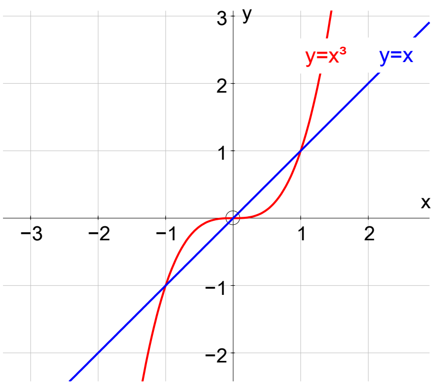
$\Rightarrow \quad f(|x|) = |x|^2 - 3|x|$

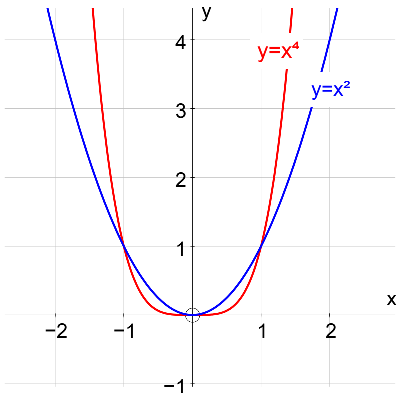
First sketch the graph of $y = x^2 - 3x$ for positive values of x only.

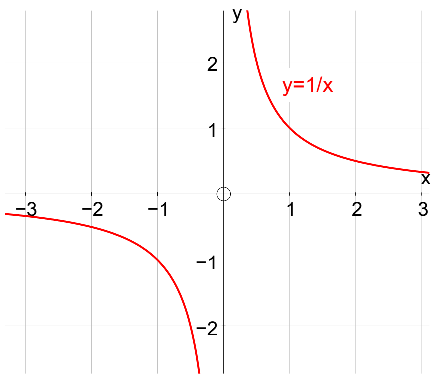
Then reflect your graph in the y–axis to complete the sketch.

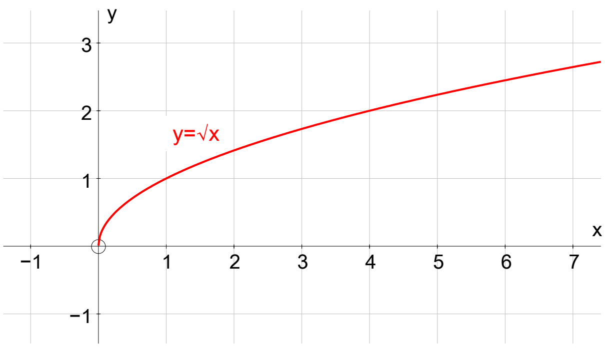


Standard graphs









Combinations of transformations of graphs

We know the following transformations of graphs:

$y = f(x)$

translated through $\left(\frac{x}{a}\right)$ becomes $y = f(x - a) + b$

stretched factor a in the y-direction becomes $y = a \times f(x)$

stretched factor a in the x-direction becomes $y = f\left(\frac{x}{a}\right)$

reflected in the x-axis becomes $y = -f(x)$

reflected in the y-axis becomes $y = f(-x)$

We can combine these transformations:

Examples:

1) $y = 2f(x - 3)$ is the image of $y = f(x)$ under a stretch in the y-axis of factor 2 followed by a translation $\binom{3}{0}$, or the translation followed by the stretch.

2) $y = 3x^2 + 6$ is the image of $y = x^2$ under a stretch in the y-axis of factor 3 followed by a translation $\binom{0}{6}$,

   BUT these transformations can**not** be done in the reverse order.

   To do a translation before a stretch we have to notice that

   $3x^2 + 6 = 3(x^2 + 2)$ which is the image of $y = x^2$ under a translation of $\binom{0}{2}$ followed by a stretch in the y-axis of factor 3.

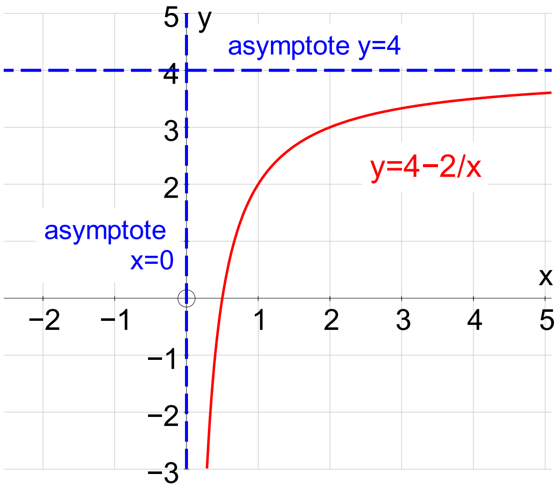
3) $y = -\sin(x + \pi)$ is the image of $y = \sin x$ under a reflection in the x-axis followed by a translation of $\binom{-\pi}{0}$, or the translation followed by the reflection.

Sketching curves

When sketching curves, show the coordinates of the intercepts with the axes, and the equations of any asymptotes – show the asymptotes with dotted lines.

Example: Sketch the curve $y = 4 - \frac{2}{x}$, $x > 0$

Solution:



Note that the domain is $x > 0$, so no graph to the left of the y-axis.

$x \neq 0 \Rightarrow$ curve does not meet the x-axis

$y = 0 \Rightarrow x = \frac{1}{2}$

Thinking of $y = \frac{-2}{x}$ translated up 4,

the horizontal asymptote is $y = 4$.

Do not forget that the y-axis, $x = 0$, is also an asymptote.

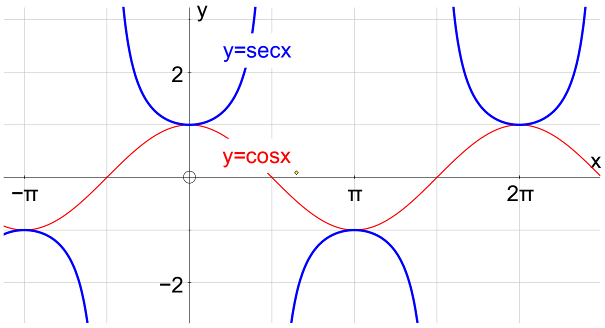
3 Trigonometry

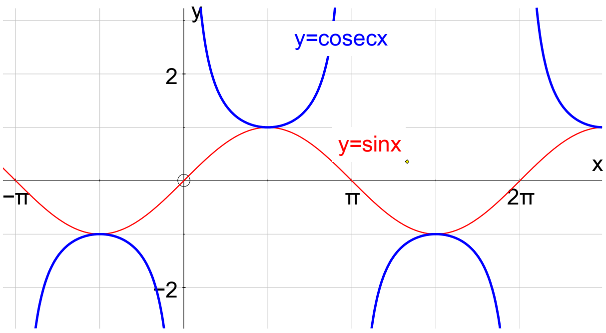
Sec, cosec and cot

Secant is written $\sec x = \dfrac{1}{\cos x}$; *cosecant* is written $\operatorname{cosec} x = \dfrac{1}{\sin x}$

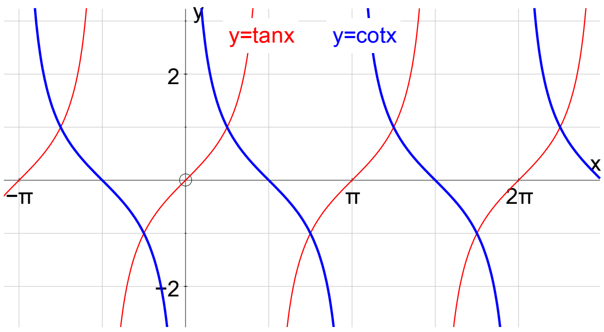
cotangent is written $\cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$

Graphs





Notice that your calculator does not have sec, cosec and cot buttons so to solve equations involving sec, cosec and cot, change them into equations involving sin, cos and tan and then use your calculator as usual.



Example: Find cosec 35°.

Solution: $\operatorname{cosec} 35^\circ = \dfrac{1}{\sin 35^\circ} = \dfrac{1}{0 \cdot 53576...} = 1 \cdot 743$ to 4 S.F.

Example: Solve $\sec x = 3{\cdot}2$ for $0 \le x \le 2\pi$

Solution: $\sec x = 3{\cdot}2 \quad \Rightarrow \quad \dfrac{1}{\cos x} = 3 \cdot 2 \Rightarrow \Rightarrow \quad \cos x = \dfrac{1}{3 \cdot 2} = 0 \cdot 3125$

$\Rightarrow \quad x = 1{\cdot}25$ or $2\pi - 1{\cdot}25 = 5{\cdot}03$ radians to 3 S.F.

Inverse trigonometrical functions

The inverse of $\sin x$ is written as $\arcsin x$ or $\sin^{-1} x$ and in order that there should only be **one value** of the function for one value of x we restrict the domain to $-{}^{\pi}/_{2} \le x \le {}^{\pi}/_{2}$.

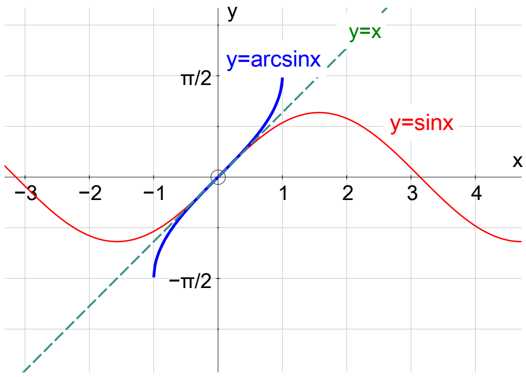
Note that $\sin^{-1} x$ does **not** mean '1 over $\sin x$'

Note that the graph of $y = \arcsin x$ is the reflection of part of the graph of $y = \sin x$ in the line $y = x$.

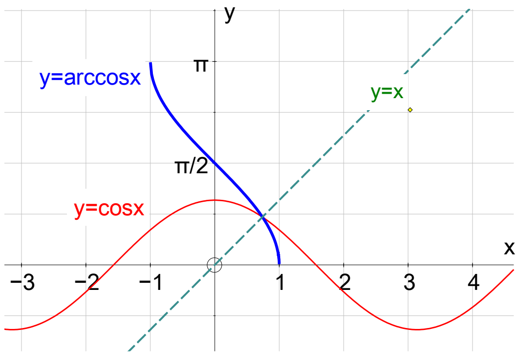
Similarly for the inverses of $\cos x$ and $\tan x$, as shown below.

Graphs

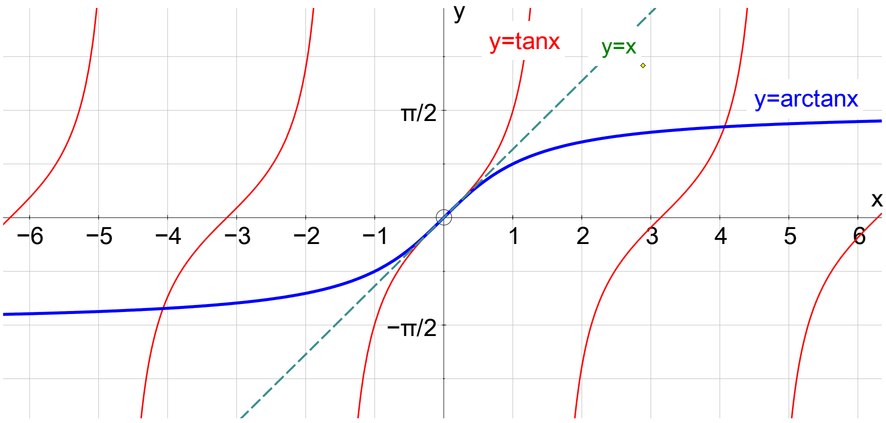
$y = \arcsin x$



$y = \arccos x$



$y = \arctan x$



Trigonometrical identities

You should **learn** these

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \text{cosec}^2\theta$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\sin 2A = 2\sin A\cos A$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\cos 2A = \cos^2 A - \sin^2 A$

$\quad = 2\cos^2 A - 1$

$\quad = 1 - 2\sin^2 A$

$\sin^2 A = ½(1 - \cos 2A)$

$\cos^2 A = ½(1 + \cos 2A)$

$\sin^2 ½\theta = ½(1 - \cos\theta)$

$\cos^2 ½\theta = ½(1 + \cos\theta)$

$\sin 3A = 3\sin A - 4\sin^3 A$

$\cos 3A = 4\cos^3 A - 3\cos A$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

$$\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$-2\sin A\sin B = \cos(A+B) - \cos(A-B)$

The last four formulae, $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ etc., are **not** in the formula booklet, and should be **learnt**.

You should know the proofs of the four formulae $\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$, etc.

Proof of sin *P* + sin *Q* = 2 sin [(*P* + *Q*)/2] cos[(*P* –*Q*)/2]

We know that $\quad \sin(A+B) = \sin A\cos B + \cos A\sin B$

and $\quad \underline{\sin(A-B) = \sin A\cos B - \cos A\sin B}$

$\Rightarrow \quad \text{in}(A+B) + \sin(A-B) = 2\sin A\cos B$

Now put $P = A + B$, and $Q = A - B$

$\Rightarrow \quad P + Q = 2A$, and $P - Q = 2B$, $\quad \Rightarrow \quad A = \frac{P+Q}{2}$, and $B = \frac{P-Q}{2}$

$\Rightarrow \quad \sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$.

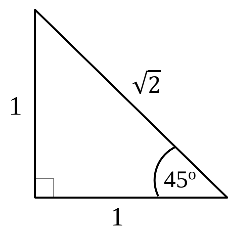
The other formulae can be proved in a similar way.

Finding exact values

When finding exact values you may **not** use calculators.

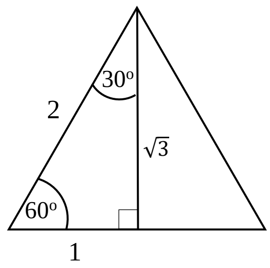
Reminder – 30º, 45º and 60º.

Isosceles right angle triangle



$\sin 45^\circ = \frac{1}{\sqrt{2}}$, $\cos 45^\circ = \frac{1}{\sqrt{2}}$, $\tan 45^\circ = 1$

Equilateral triangle



$\sin 60^\circ = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = \frac{1}{2}$, $\tan 60^\circ = \sqrt{3}$

$\sin 30^\circ = \frac{1}{2}$, $\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\tan 30^\circ = \frac{1}{\sqrt{3}}$

Example: Find the exact value of $\cos 15^\circ$

Solution: We know the exact values of $\sin 45^\circ$, $\cos 45^\circ$ and $\sin 30^\circ$, $\cos 30^\circ$

so we consider $\cos 15 = \cos(45 - 30) = \cos 45 \cos 30 + \sin 45 \sin 30$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

Example: Given that A is obtuse and that B is acute, and $\sin A = {}^3/_5$ and $\cos B = {}^5/_{13}$ find the exact value of $\sin(A + B)$.

Solution: **You cannot draw a 90º triangle with an obtuse or reflex angle!!!**

We know that $\sin(A + B) = \sin A \cos B + \cos A \sin B$ so we must first find $\cos A$ and $\sin B$.

Using $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow$ $\cos^2 A = 1 - {}^9/_{25} = {}^{16}/_{25}$ and $\sin^2 B = 1 - {}^{25}/_{169} = {}^{144}/_{169}$

$\Rightarrow$ $\cos A = \pm\, {}^4/_5$ and $\sin B = \pm\, {}^{12}/_{13}$

But A is obtuse so $\cos A$ is negative, and B is acute so $\sin B$ is positive

$\Rightarrow$ $\cos A = -\,{}^4/_5$ and $\sin B = {}^{12}/_{13}$

$\Rightarrow$ $\sin(A + B) = {}^3/_5 \times {}^5/_{13} + -\,{}^4/_5 \times {}^{12}/_{13} = {}^{-33}/_{65}$

Note that you can use a 90º triangle to find sin B since b is acute.

Proving identities.

Start with one side, usually the L.H.S., and fiddle with it until it equals the other side.

Do **not** fiddle with both sides at the same time.

Example: Prove that $\frac{\cos 2A+1}{1-\cos 2A} = \cot^2 A$.

Solution: L.H.S. $= \frac{\cos 2A+1}{1-\cos 2A} = \frac{2\cos^2 A-1+1}{1-(1-2sin^2 A)} \; \frac{2\cos^2 A}{2\sin^2 A} = \cot^2 A$. Q.E.D.

Eliminating a variable between two equations

Example: Eliminate θ from the parametric equations $x = \sec\theta - 1$, $y = \tan\theta$.

Solution: We remember that $\tan^2\theta + 1 = \sec^2\theta$

$\Rightarrow$ $\sec^2\theta - \tan^2\theta = 1$.

$\sec\theta = x + 1$ and $\tan\theta = y$

$\Rightarrow$ $(x + 1)^2 - y^2 = 1$

$\Rightarrow$ $y^2 = (x + 1)^2 - 1 = x^2 + 2x$.

Solving equations

Here you have to select the 'best' identity to help you solve the equation.

Example: Solve the equation $\sec^2 A = 3 - \tan A$, for $0 \leq A \leq 360^{o.}$

Solution: We know that $\tan^2 A + 1 = \sec^2 A$

$\Rightarrow$ $\tan^2 A + 1 = 3 - \tan A$

$\Rightarrow$ $\tan^2 A + \tan A - 2 = 0$, factorising gives

$\Rightarrow$ $(\tan A - 1)(\tan A + 2) = 0$

$\Rightarrow$ $\tan A = 1$ or $\tan A = -2$

$\Rightarrow$ $A = 45, 225,$ or $116{\cdot}6, 296{\cdot}6$.

Example: Solve $\sin 3x - \sin 5x = 0$ for $0^o \leq x \leq 90^o$.

Solution: Using the formula $\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$\Rightarrow \quad 2\cos 4x \sin(-x) = 0 \quad \Rightarrow \quad \cos 4x \sin x = 0$

$\Rightarrow \quad \cos 4x = 0, \quad$ or $\sin x = 0$

$\Rightarrow \quad 4x = 90, 270, (450), \ldots$ or $x = 0, (180), \ldots$

$\Rightarrow \quad x = 0^o, 22{\cdot}5^o$ or $67{\cdot}5^o$.

R cos(x + α)

An alternative way of writing $a\cos x \pm b\sin x$ using one of the formulae listed below

(1) $R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$

(2) $R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$

(3) $R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

(4) $R\sin(x-\alpha) = R\sin x\cos\alpha - R\cos x\sin\alpha$

To keep *R* positive and α acute, we select the formula with corresponding + and – signs.

The technique is the same which ever formula we choose.

Example: Solve the equation $12\sin x - 5\cos x = 6$ for $0^o \leq x \leq 360^0$.

Solution: First re–write in the above form:

notice that the $\sin x$ is positive and the $\cos x$ is negative so we need formula (4).

$R\sin(x-\alpha) = R\sin x\cos\alpha - R\cos x\sin\alpha = 12\sin x - 5\cos x$

Equating coefficients of $\sin x$, $\Rightarrow R\cos\alpha = 12$ **I**

Equating coefficients of $\cos x$, $\Rightarrow R\sin\alpha = 5$ **II**

By choosing the correct formula we ensure that there are no minus signs in **I** and **II**

Squaring and adding **I** and **II** $\Rightarrow R^2\cos^2\alpha + R^2\sin^2\alpha = 12^2 + 5^2$

$\Rightarrow \quad R^2(\cos^2\alpha + \sin^2\alpha) = 144 + 25$ but $\cos^2\alpha + \sin^2\alpha = 1$

$\Rightarrow \quad R^2 = 169 \quad \Rightarrow R = \pm 13$

But choosing the correct formula means that R is positive $\Rightarrow R = +13$

I I $\div$ **I** $\Rightarrow \quad \tan\alpha = {}^5/_{12}$

$\Rightarrow \quad \alpha = 22{\cdot}620\ldots^o$ or $337{\cdot}379\ldots^o$ or ……

and choosing the correct formula means that α is acute

$\Rightarrow \quad \alpha = 22{\cdot}620\ldots^o$.

$\Rightarrow \quad 12\sin x - 5\cos x = 13\sin(x - 22{\cdot}620\ldots)$

To solve $\quad 12 \sin x - 5 \cos x = 6$

$\Rightarrow \quad 12 \sin x - 5 \cos x = 13 \sin(x - 22{\cdot}620\ldots) = 6$

$\Rightarrow \quad \sin(x - 22{\cdot}620\ldots) = {}^{6}/_{13}$

$\Rightarrow \quad x - 22{\cdot}620\ldots = 27{\cdot}486\ldots$ or $180 - 27{\cdot}486\ldots = 152{\cdot}514\ldots$

$\Rightarrow \quad x = 50{\cdot}1^{\circ}$ or $175{\cdot}1^{\circ}$.

Example: Find the maximum value of $24 \sin x - 10 \cos x$ and the smallest positive value of x for which it occurs.

Solution: From the above example

$24 \sin x - 10 \cos x = 2(12 \sin x - 5 \cos x) = 2 \times 13 \sin(x - 22.6) = 26 \sin(x - 22.6)$.

The maximum value of $\sin(\textit{anything})$ is 1 and occurs when the angle is 90, 450, 810 etc. i.e. $90 + 360n$

$\Rightarrow$ the max value of $26 \sin(x - 22{\cdot}6)$ is 26

when $x - 22{\cdot}6 = 90 + 360n \Rightarrow x = 112{\cdot}6 + 360n^{\circ}$,

$\Rightarrow$ smallest positive value of x is $112{\cdot}6^{\circ}$.

Example: Find the minimum value of

$$h = \frac{450}{11 + (12 \sin x - 5 \cos x)^2}$$

Solution: The minimum value of h will occur when the denominator is a maximum, in other words when $(12 \sin x - 5 \cos x)^2 = (13 \sin(x - 22 \cdot 6))^2$ is a maximum.

$13 \sin(x - 22 \cdot 6)$ varies between -13 and $+13$

$\Rightarrow \quad (13 \sin(x - 22 \cdot 6))^2$ varies between 0 and 13^2

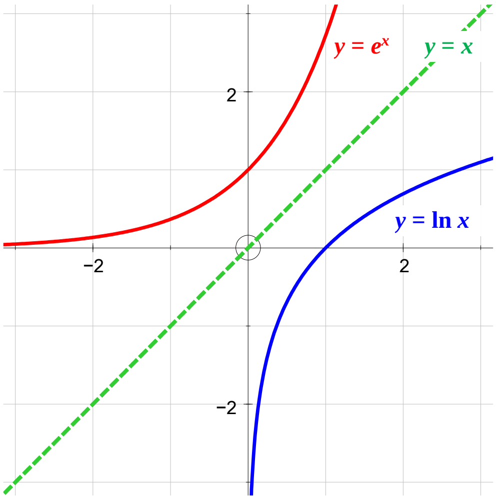
$\Rightarrow$ maximum value of the denominator is $11 + 13^2 = 180$

$\Rightarrow$ minimum value of h is $\frac{450}{180} = 2{\cdot}5$.

4 Exponentials and logarithms

Natural logarithms

Definition and graph



$e \approx 2{\cdot}7183$ and logs to base e are called *natural logarithms*.

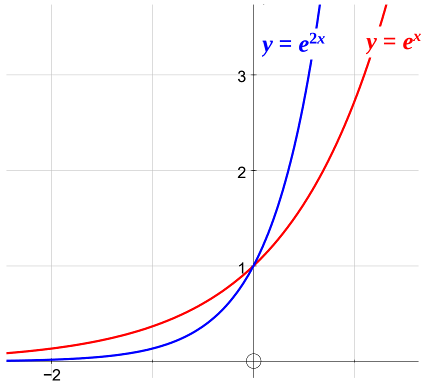
$\log_e x$ is usually written $\ln x$.

Note that $y = e^x$ and $y = \ln x$ are inverse functions

$\Rightarrow$ $\ln e^x = x$ and $e^{\ln x} = x$

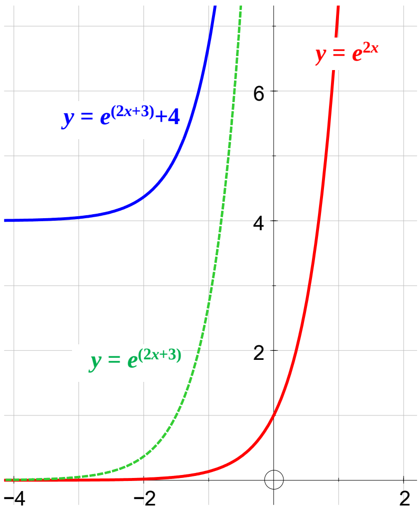
and that the graph of one is the reflection of the other in the line $y = x$.

Graph of $y = e^{(ax+b)} + c$.



The graph of $y = e^{2x}$ is the graph of $y = e^x$ stretched by a factor of $^1/_2$ in the direction of the x-axis.

$y = e^{2x}$ is above $y = e^x$ for $x > 0$, and below for $x < 0$.



The graph of $y = e^{(2x+3)}$ is the graph of $y = e^{2x}$ translated through $\begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix}$ since

$2x + 3 = 2(x - \frac{-3}{2})$

and the graph of $y = e^{(2x+3)} + 4$ is that of $y = e^{(2x+3)}$ translated through $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

Equations of the form $e^{ax+b} = p$

Example: Solve $e^{2x+3} = 5$.

Solution: Take the natural logarithm of each side, remember that $\ln x$ is the inverse of e^x

$\Rightarrow$ $\ln(e^{2x+3}) = \ln 5$ $\Rightarrow$ $2x + 3 = \ln 5$

$\Rightarrow$ $x = \frac{\ln 5 - 3}{2} = -0{\cdot}695$ to 3 S.F.

Example: Solve $\ln(3x - 5) = 4$.

Solution:

(i) From the definition of logs $\log_a x = y \Leftrightarrow x = a^y$

$\ln(3x - 5) = 4$

$\Rightarrow$ $3x - 5 = e^4$

$\Rightarrow$ $x = \frac{e^4 + 5}{3} = 19{\cdot}9$ to 3 S.F.

5 Curves into straight lines

1) When y and x are related by an equation of the form $y = ax^n$, we can obtain a straight line graph by taking logarithms of both sides

   $\Rightarrow$ $\log y = \log a + n \log x$

   and if we plot $\log y$ against $\log x$,
   we get a straight line with gradient n and intercept on the $\log y$ axis of $\log a$.

2) When y and x are related by an equation of the form $y = kb^x$, we can obtain a straight line graph by taking logarithms of both sides

   $\Rightarrow$ $\log y = \log k + x \log b$

   and if we plot $\log y$ against x,
   we get a straight line with gradient $\log b$ and intercept on the $\log y$ axis of $\log k$.

6 Differentiation

Chain rule

If y is a composite function like $y = (5x^2 - 7)^9$

think of y as $y = u^9$, where $u = 5x^2 - 7$

then the chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = 9u^8 \times \frac{du}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = 9(5x^2 - 7)^8 \times (10x) = 90x(5x^2 - 7)^8.$$

The rule is very simple, just differentiate the function of u and multiply by $\dfrac{du}{dx}$.

Example: $y = \sqrt{(x^3 - 2x)}$. Find $\dfrac{dy}{dx}$.

Solution: $y = \sqrt{(x^3 - 2x)} = (x^3 - 2x)^{1/2}$. Put $u = x^3 - 2x$

$$\Rightarrow \quad y = u^{1/2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \tfrac{1}{2}u^{-1/2} \times \frac{du}{dx} = \tfrac{1}{2}(x^3 - 2x)^{-1/2} \times (3x^2 - 2)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2 - 2}{2(x^3 - 2x)^{1/2}}.$$

Product rule

If y is the product of two functions, u and v, then

$$y = uv \quad \Rightarrow \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example: Differentiate $y = x^2 \times \sqrt{(x - 5)}$.

Solution: $y = x^2 \times \sqrt{(x - 5)} = x^2 \times (x - 5)^{1/2}$

so put $u = x^2$ and $v = (x - 5)^{1/2}$

$$\Rightarrow \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= x^2 \times \tfrac{1}{2}(x - 5)^{-1/2} + (x - 5)^{1/2} \times 2x$$

$$= \frac{x^2}{2\sqrt{x - 5}} + 2x\sqrt{x - 5}.$$

Quotient rule

If y is the quotient of two functions, u and v, then

$$y = \frac{u}{v} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Example: Differentiate $y = \dfrac{2x-3}{x^2+5x}$

Solution: $y = \dfrac{2x-3}{x^2+5x}$, so put $u = 2x - 3$ and $v = x^2 + 5x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x^2+5x)\times 2 \ - \ (2x-3)\times(2x+5)}{(x^2+5x)^2}$$

$$= \frac{2x^2+10x \ - \ (4x^2+4x-15)}{(x^2+5x)^2} = \frac{-2x^2+6x+15}{(x^2+5x)^2}.$$

Example: If $y = \dfrac{3x-2}{\sqrt{x-1}}$, find $\dfrac{dy}{dx}$, expressing your answer as a single algebraic fraction in its simplest form.

Solution: $y = \dfrac{3x-2}{(x-1)^{\frac{1}{2}}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x-1)^{\frac{1}{2}} \times 3 \ - \ (3x-2) \times \frac{1}{2}(x-1)^{\frac{-1}{2}}}{x-1}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x-1)^{\frac{1}{2}} \times 3 \ - \ \dfrac{(3x-2)}{2(x-1)^{1/2}}}{x-1}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x-1)^{\frac{1}{2}} \times 3 \ - \ \dfrac{(3x-2)}{2(x-1)^{1/2}}}{x-1} \times \frac{2(x-1)^{1/2}}{2(x-1)^{1/2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6(x-1) - (3x-2)}{2(x-1)^{3/2}} = \frac{6x-6-3x+2}{2(x-1)^{3/2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x-4}{2(x-1)^{3/2}}$$

Derivatives of e^x and $\log_e x \equiv \ln x$.

$y = e^x \quad \Rightarrow \quad \frac{dy}{dx} = e^x$

$y = \ln x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$

$y = \ln kx \quad \Rightarrow \quad y = \ln k + \ln x \quad \Rightarrow \quad \frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$

or $y = \ln kx \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{kx} \times k = \frac{1}{x}$ using the chain rule

$y = \ln x^k \quad \Rightarrow \quad y = k \ln x \quad \Rightarrow \quad \frac{dy}{dx} = k \times \frac{1}{x} = \frac{k}{x}$

or $y = \ln x^k \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x^k} \times kx^{k-1} = \frac{k}{x}$ using the chain rule

Example: Find the derivative of $f(x) = x^3 - 5e^x$ at the point where $x = 2$.

Solution: $f(x) = x^3 - 5e^x$

$\Rightarrow \quad f'(x) = 3x^2 - 5e^x$

$\Rightarrow \quad f'(2) = 12 - 5e^2 = -24{\cdot}9$

Example: Differentiate the function $f(x) = \ln 3x - \ln x^5$

Solution: $f(x) = \ln 3x - \ln x^5$

$\Rightarrow \quad f'(x) = \frac{1}{3x} \times 3 - \frac{1}{x^5} \times 5x^4 = \frac{-4}{x}$

Example: Find the derivative of $f(x) = \log_{10} 3x$.

Solution: $f(x) = \log_{10} 3x = \dfrac{\ln 3x}{\ln 10}$ using change of base formula

$= \frac{\ln 3 + \ln x}{\ln 10} = \frac{\ln 3}{\ln 10} + \frac{\ln x}{\ln 10}$

$\Rightarrow \quad f'(x) = 0 + \frac{\frac{1}{x}}{\ln 10} = \frac{1}{x \ln 10}.$

Example: $y = e^{x^2}$. Find $\dfrac{dy}{dx}$.

Solution: $y = e^u$, where $u = x^2$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^u \times \frac{du}{dx} = e^u \times 2x = 2x\,e^{x^2}.$$

Example: $y = \ln 7x^3$. Find $\dfrac{dy}{dx}$.

Solution: $y = \ln u$, where $u = 7x^3$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{u} \times \frac{du}{dx} = \frac{1}{7x^3} \times 21x^2 = \frac{3}{x}.$$

(dy/dx) = 1/(dx/dy)

Using the chain rule we can see that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$, $\quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Example: $x = \sin^2 3y$. Find $\frac{dy}{dx}$.

Solution: First find $\frac{dx}{dy}$ as this is easier.

$$\frac{dx}{dy} = 2 \sin 3y \cos 3y \times 3 = 6 \sin 3y \cos 3y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{6 \sin 3y \cos 3y} = \frac{1}{3 \sin 6y} = \frac{1}{3} \operatorname{cosec} 6y$$

Derivative of a^x

$$y = a^x \quad \Rightarrow \quad \frac{dy}{dx} = a^x \ln a$$

Proof:

$$y = a^x$$

$$\Rightarrow \quad \ln y = \ln a^x = x \ln a$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \ln a$$

$$\Rightarrow \quad \frac{dy}{dx} = y \ln a = a^x \ln a$$

You should **know** this proof.

Example: If $y = 5^{x+2}$, find $\frac{dy}{dx}$.

Solution 1: $y = 5^x \times 5^2 = 25 \times 5^x$

$\Rightarrow \quad \frac{dy}{dx} = 25 \times 5^x \ln 5$

Example: $y = 7^{x^2}$, find $\frac{dy}{dx}$

Solution: $y = 7^u \quad \Rightarrow \quad \frac{dy}{dx} = 7^u \ln 7 \times \frac{du}{dx} = 7^{x^2} \ln 7 \times 2x$ using the chain rule

Trigonometric differentiation

x **must be in RADIANS when differentiating trigonometric functions.**

$f(x)$	$f'(x)$	*important formulae*
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	$\sin^2 x + \cos^2 x = 1$
$\tan x$	$\sec^2 x$	
$\sec x$	$\sec x \tan x$	$\tan^2 x + 1 = \sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$	
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$1 + \cot^2 x = \operatorname{cosec}^2 x$

Chain rule – further examples

Example: $y = \sin^4 x$. Find $\frac{dy}{dx}$.

Solution: $y = \sin^4 x$. Put $u = \sin x$

$\Rightarrow \quad y = u^4$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = 4u^3 \times \frac{du}{dx} = 4\sin^3 x \times \cos x$.

Example: $y = e^{\sin x}$. Find $\dfrac{dy}{dx}$.

Solution: $y = e^u$, where $u = \sin x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^u \times \frac{du}{dx} = e^{\sin x} \times \cos x = \cos x \times e^{\sin x}.$$

Example: $y = \ln(\sec x)$. Find $\dfrac{dy}{dx}$.

Solution: $y = \ln u$, where $u = \sec x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{u} \times \frac{du}{dx} = \frac{1}{\sec x} \times \sec x \tan x = \tan x.$$

Trigonometry and the product and quotient rules

Example: Differentiate $y = x^2 \times \operatorname{cosec} 3x$.

Solution: $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$y = x^2 \times \operatorname{cosec} 3x$$

Put $u = x^2$ and $v = \operatorname{cosec} 3x$

$$\frac{dy}{dx} = x^2 \times (-3\operatorname{cosec} 3x \cot 3x) + \operatorname{cosec} 3x \times 2x$$

$$= -3x^2 \operatorname{cosec} 3x \cot 3x + 2x \operatorname{cosec} 3x.$$

Example: Differentiate $y = \dfrac{\tan 2x}{7x^3}$

Solution: $y = \dfrac{u}{v} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$y = \frac{\tan 2x}{7x^3}$$

Put $u = \tan 2x$ and $v = 7x^3$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{7x^3 \times 2\sec^2 2x \;-\; \tan 2x \times 21x^2}{(7x^3)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{14x^3 \sec^2 2x \;-\; 21x^2 \tan 2x}{49x^6}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x\sec^2 2x \;-\; 3\tan 2x}{7x^4}$$

7 Integration

Integrals of a^x, e^x and $1/x$

We know that $\frac{d}{dx}(a^x) = a^x \log a$

$$\Rightarrow \quad \int a^x \, dx = \frac{a^x}{\log a} + c$$

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$

for a further treatment of this result, see the appendix

Example: Find $\displaystyle\int \frac{x^3 + 3x}{x^2} \, dx$

Solution: $\displaystyle\int \frac{x^3 + 3x}{x^2} \, dx = \int x + \frac{3}{x} \, dx = ½\, x^2 + 3 \ln|x| + c.$

Standard integrals

***x* must be in RADIANS when integrating trigonometric functions.**

$f(x)$	$\int f(x) \, dx$	$f(x)$	$\int f(x) \, dx$
x^n	$\frac{x^{n+1}}{n+1}$	$\sin x$	$-\cos x$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\cos x$	$\sin x$
e^x	e^x	$\sec x \tan x$	$\sec x$
e^{kx}	$\frac{e^{kx}}{k}$	$\sec^2 x$	$\tan x$
		$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x$
		$\operatorname{cosec}^2 x$	$-\cot x$

Integration using trigonometric identities

Example: Find $\int \cot^2 x \quad dx$.

Solution: $\cot^2 x = \operatorname{cosec}^2 x - 1$

$$\Rightarrow \quad \int \cot^2 x \quad dx = \int \operatorname{cosec}^2 x - 1 \quad dx$$

$$= \quad -\cot x - x + c.$$

Example: Find $\int \sin^2 x \, dx$

Solution: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\Rightarrow \quad \int \sin^2 x\,dx = \frac{1}{2}\int 1 - \cos 2x\,dx$

$= \quad \frac{1}{2}x - \frac{1}{4}\sin 2x + c.$

You **cannot** change x to $3x$ in the above result to find $\int \sin^2 3x\,dx$ see next example

Example: Find $\int \sin^2 3x\,dx$.

Solution: $\sin^2 3x = \frac{1}{2}(1 - \cos 2 \times 3x) = \frac{1}{2}(1 - \cos 6x)$

$\Rightarrow \quad \int \sin^2 3x\,dx = \frac{1}{2}\int 1 - \cos 6x\,dx = \frac{1}{2}x - \frac{1}{12}\sin 6x + c.$

Example: Find $\int \sin 3x \cos 5x\;dx$.

Solution: Using the formula $2\sin A\cos B = \sin(A + B) + \sin(A - B)$

This formula is NOT in the formula booklet – you can use the formulae for sin($A \pm B$) and add them

$\int \sin 3x \cos 5x\;dx$

$= \quad \frac{1}{2}\int \sin 8x + \sin(-2x)\;dx \quad = \quad \frac{1}{2}\int \sin 8x - \sin 2x\;dx$

$= \quad -\frac{1}{16}\cos 8x + \frac{1}{4}\cos 2x + c.$

Integration by 'reverse chain rule'

Some integrals which are not standard functions can be integrated by thinking of the chain rule for differentiation.

Example: Find $\int \sin^4 3x \cos 3x\,dx$.

Solution: $\int \sin^4 3x \cos 3x\,dx$

If we think of $u = \sin 3x$, then the integrand looks like $u^4\frac{du}{dx}$ if we ignore the constants, which would integrate to give $\frac{1}{5}u^5$

so we differentiate $u^5 = \sin^5 3x$

to give $\frac{d}{dx}(\sin^5 3x) = 5(\sin^4 3x) \times 3\cos 3x = 15\sin^4 3x\cos 3x$

which is 15 times what w

e want and so

$\int \sin^4 3x \cos 3x\,dx = \frac{1}{15}\sin^5 3x \quad + \quad c$

Example: Find $\int \frac{x}{(2x^2-3)}\,dx$

Solution: $\int \frac{x}{(2x^2-3)}\,dx$

If we think of $u = (2x^2 - 3)$, then the integrand looks like $\frac{1}{u}\,\frac{du}{dx}$ if we ignore the constants, which would integrate to $\ln|u|$

so we differentiate $\ln|u| = \ln|2x^2 - 3|$

to give $\frac{d}{dx}(\ln|2x^2-3|) = \frac{1}{2x^2-3} \times 4x = \frac{4x}{(2x^2-3)}$

which is 4 times what we want and so

$\int \frac{x}{(2x^2-3)}\,dx = \frac{1}{4}\ln|2x^2-3| + c.$

In general $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

Example: Find $\int xe^{2x^2}\,dx$

Solution: First consider $\frac{d}{dx}\left(e^{2x^2}\right) = 4x\,e^{2x^2}$, which is 4 × the integrand

$$\Rightarrow \quad \int xe^{2x^2}\,dx = \frac{e^{2x^2}}{4} + c$$

Example: Find $\int 5^{3x}\,dx$

Solution: We know that $\frac{d}{dx}(5^{3x}) = 5^{3x}\ln 5 \times 3,$ using the chain rule

$$\Rightarrow \quad \int 5^{3x}\,dx = \frac{5^{3x}}{3\ln 5} + c$$

Integrals of tan *x* and cot *x*

$$\int \tan x\;dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx,$$

and we now have $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

$$\Rightarrow \quad \int \tan x\;dx = -\ln|\cos x| + c$$

$$\Rightarrow \quad \int \tan x\;dx = \ln|\sec x| + c$$

cot *x* can be integrated by a similar method to give

$$\int \cot x\;dx = \ln|\sin x| + c$$

Integrals of sec *x* and cosec *x*

$$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

The top is now the derivative of the bottom

and we have $\displaystyle\int \frac{f\,'(x)}{f(x)}\, dx = \ln\left|f(x)\right| + c$

$\Rightarrow \quad \int \sec x\, dx = \ln\left|\sec x + \tan x\right| + c$

and similarly

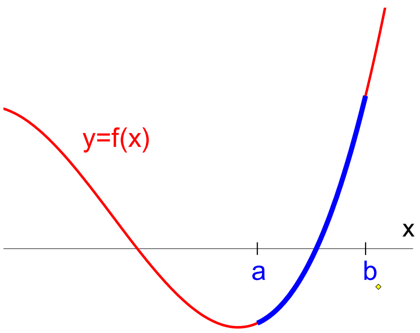
$$\int \operatorname{cosec} x\, dx = -\ln\left|\operatorname{cosec} x + \cot x\right| + c$$

8 Numerical methods

Locating the roots of f(x) = 0

A quick sketch of the graph of $y = f(x)$ can give a rough idea of the roots of $f(x) = 0$.

If $y = f(x)$ **changes sign** between $x = a$ and $x = b$ **and** if $f(x)$ is **continuous** in this region then a root of $f(x) = 0$ lies between $x = a$ and $x = b$.



The iteration $x_{n+1} = g(x_n)$

Example:

(*a*) Show that a root, α, of the equation $f(x) = x^3 - 8x - 7 = 0$ lies between 3 and 4.

(*b*) Show that the equation $x^3 - 8x - 7 = 0$ can be re–arranged as $x = \sqrt[3]{8x+7}$.

(*c*) Starting with $x_1 = 3$, use the iteration $x_{n+1} = \sqrt[3]{8x_n + 7}$ to find the first four iterations for x.

(*d*) Show that your value of x_4 is correct to 3 S.F.

Solution:

(*a*) $f(3) = 27 - 24 - 7 = -4$, and $f(4) = 64 - 32 - 7 = +25$

Thus $f(x)$ **changes sign** and $f(x)$ is **continuous** $\Rightarrow$ there is a root between 3 and 4.

(*b*) $x^3 - 8x - 7 = 0 \quad \Rightarrow \quad x^3 = 8x + 7 \quad \Rightarrow \quad x = \sqrt[3]{8x+7}$.

(*c*) $x_1 = 3$

$\Rightarrow \quad x_2 = \sqrt[3]{8 \times 3 + 7}$ you **must** do the first iteration **with** the numbers

$\Rightarrow \quad = 3{\cdot}14138065239$

$\Rightarrow \quad x_3 = 3{\cdot}17912997899$

$\Rightarrow \quad x_4 = 3{\cdot}18905898325$

(*d*) $x_4 = 3{\cdot}19$ to 3 S.F.

$f(3{\cdot}185) = 3{\cdot}185^3 - 8 \times 3{\cdot}185 - 7 = -0{\cdot}17 \ldots$

$f(3{\cdot}195) = 3{\cdot}195^3 - 8 \times 3{\cdot}195 - 7 = +0{\cdot}05 \ldots$

$f(x)$ **changes sign** and $f(x)$ is **continuous**

$\Rightarrow$ there is a root in the interval [3·185, 3·195]

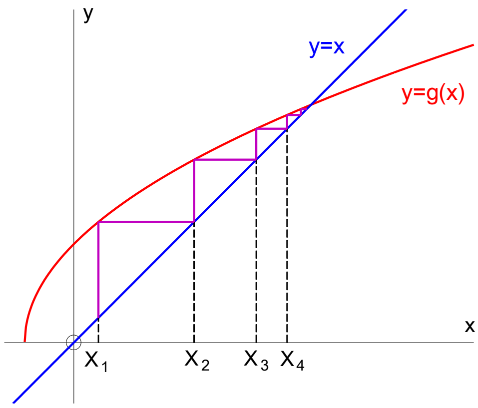
$\Rightarrow \quad \alpha = 3{\cdot}19$ to 3 S.F.

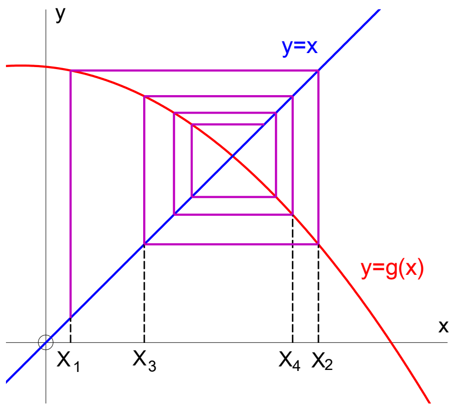
Conditions for convergence

If an equation is rearranged as $x = g(x)$ and if there is a root $x = \alpha$

then the iteration $x_{n+1} = g(x_n)$, starting with an approximation near $x = \alpha$

(i) will converge if $-1 < g'(\alpha) < 1$



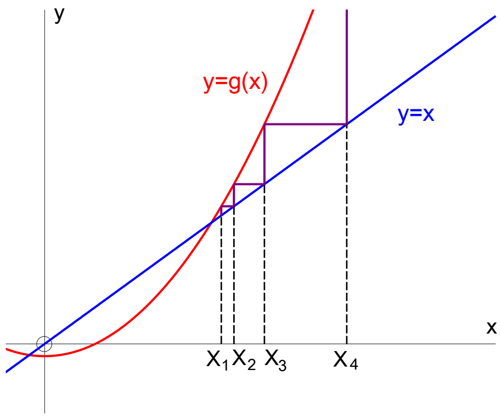


(a) will converge without oscillating if $0 < g'(\alpha) < 1$,

(b) will oscillate and converge if $-1 < g'(\alpha) < 0$,

(ii) will diverge if

$g'(\alpha) < -1$ or $g'(\alpha) > 1$.

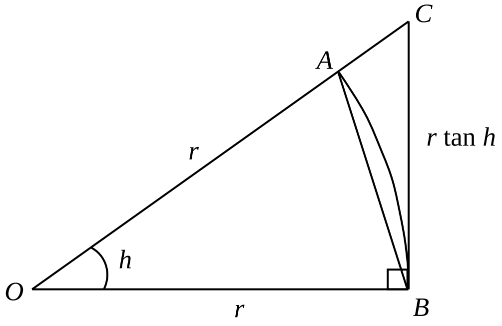


9 Appendix

Derivatives of sin x and cos x

limit of (sin h)/h as h tends to 0

$\displaystyle \lim_{h\to 0}\left(\frac{\sin h}{h}\right) = 1$



OAB is a sector of a circle with centre *O* and radius *r*.

The area of the *triangle OAB*, $\frac{1}{2}r^2\sin h$,

is less than the area of the *sector OAB*, $\frac{1}{2}r^2h$

$\Rightarrow \quad \frac{1}{2}r^2\sin h < \frac{1}{2}r^2h$

$\Rightarrow \quad \frac{\sin h}{h} < 1$ …….**I**

Also the area of the *sector OAB*, $\frac{1}{2}r^2h$

r tan *h*

is less than the area of the *triangle OBC*, $\frac{1}{2}r^2\tan h$,

$\Rightarrow \quad \frac{1}{2}r^2h < \frac{1}{2}r^2\tan h \qquad \Rightarrow \qquad \cos h < \frac{\sin h}{h}$ ……..**II**

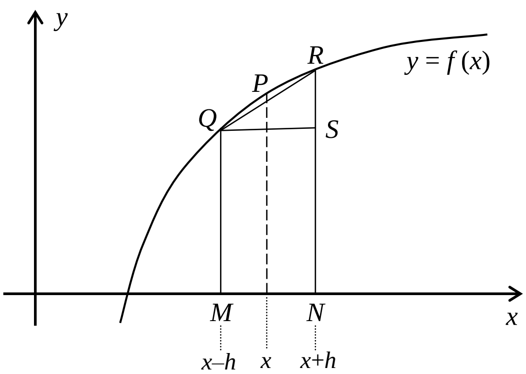
I and **II** $\Rightarrow \cos h < \frac{\sin h}{h} < 1$

and as $h \to 0$, $\lim_{h\to 0}\cos h = 1$

$\Rightarrow 1 < \lim_{h\to 0}\left(\frac{\sin h}{h}\right) < 1 \qquad \Rightarrow \lim_{h\to 0}\left(\frac{\sin h}{h}\right) = 1$

***h* must be in RADIANS**, as the formula for the area of sector is only true if *h* is in **radians.**

Alternative formula for derivative



The gradient of the curve at *P* will be nearly equal to the gradient of the line *QR*.

$QM = f(x - h)$ and $RN = f(x + h)$

$\Rightarrow \quad RS = f(x + h) - f(x - h)$

$QS = MN = 2h$

$\Rightarrow \quad$ gradient of $QR = \frac{f(x+h) - f(x-h)}{2h}$

and as $h \to 0$, the gradient of $QR \to f'(x)$ = gradient of the curve at *P*

$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$

Preiously we used the formula $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Derivatives of sin *x* and cos *x*

$f(x) = \sin x$ and $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x-h)}{2h}$

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x-h)}{2h}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - (\sin x \cos h - \cos x \sin h)}{2h}$$

$$= \lim_{h \to 0} \frac{2\cos x \sin h}{2h}$$

$$= \lim_{h \to 0} \cos x \frac{\sin h}{h}$$

but $\lim_{h \to 0} \left(\dfrac{\sin h}{h}\right) = 1$

$$\Rightarrow f'(x) = \frac{d}{dx}(\sin x) = \cos x$$

Similarly, we can show that $\dfrac{d}{dx}(\cos x) = -\sin x$

***x* must be in RADIANS.**

Integral of 1/x for negative limits

We know that the 'area' under any curve, from $x = a$ to $x = b$ is approximately

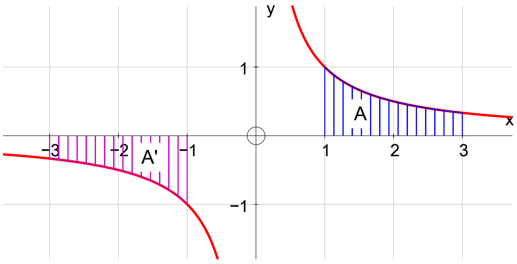
$$\sum_a^b y\ \delta x \quad \to \quad \int_a^b y\ dx, \quad as\ \delta x \to 0$$

If the curve is above the x-axis, all the y values are positive, and if $a < b$ then all values of δx are positive, and so the integral is positive.

This gives us a way of defining

$$\int_{-a}^{-b} \frac{1}{x}\ dx = \Big[\ln|x|\Big]_{-a}^{-b} = \ln b - \ln a \qquad \text{for positive } a \text{ and } b$$

Example: Find $\int_{-1}^{-3} \frac{1}{x}\, dx$.



Solution: The integral wanted is shown as A' in the diagram.

By symmetry $|A'| = A$ (A positive)

and we need to decide whether the integral is $+A$ or $-A$.

From $x = -1$ to $x = -3$, we are going in the direction of x **decreasing**

$\Rightarrow$ all δx are negative.

And the graph is below the x-axis,

$\Rightarrow$ the y values are negative, $\Rightarrow$ $y\,\delta x$ is **positive**

$\Rightarrow$ $\sum_{-1}^{-3} y\ \delta x > 0 \ = \ \int_{-1}^{-3} y\ dx > 0$

$\Rightarrow$ the integral is **positive** and equal to A**.**

The integral, $A' = A = \int_1^3 \frac{1}{x}\, dx = \left[\ln x\right]_1^3 = \ln 3 - \ln 1 = \ln 3$

$\Rightarrow$ $A' = \int_{-1}^{-3} \frac{1}{x}\, dx = \ln 3$

Notice that this is what we get if we write $\ln|x|$ in place of $\ln x$

$$\int_{-1}^{-3} \frac{1}{x}\, dx = \left[\ln|x|\right]_{-1}^{-3} = \ln 3 - \ln 1 = \ln 3$$

As it will always be possible to use symmetry in this way, since we can never have one positive and one negative limit (because there is a discontinuity at $x = 0$), it is correct to write $\ln|x|$ for the integral of ${}^{1}/_{x}$.

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