INTERNATIONAL ADVANCED LEVEL

MATHEMATICS/ FURTHER MATHEMATICS/ PURE MATHEMATICS

SCHEME OF WORK

Mechanics 2

Pearson Edexcel International Advanced Subsidiary in Mathematics (XMA01)
Pearson Edexcel International Advanced Subsidiary in Further Mathematics (XFM01)
Pearson Edexcel International Advanced Subsidiary in Pure Mathematics (XPM01)
Pearson Edexcel International Advanced Level in Mathematics (YMA01)
Pearson Edexcel International Advanced Level in Further Mathematics (YFM01) Pearson Edexcel International Advanced Level in Pure Mathematics (YPM01)
First teaching September 2018
First examination from January 2019
First certification from August 2019 (International Advanced Subsidiary) and August 2020 (International Advanced Level)

A level Mathematics: Mechanics 2

Mechanics 2

Unit		Title	Estimated hours
1		**Kinematics of a particle moving in a straight line or plane**	
	a	Motion in a vertical plane under gravity; projectiles	**6**
	b	Variable acceleration (use of calculus and finding vectors $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ at a given time)	**6**
2		**Centres of mass**	
	a	Centre of mass of a discrete mass distribution in one or two dimensions, framework and uniform lamina (rectilinear shapes)	**5**
	b	Centre of mass of triangular, circular-based and composite laminas and centre of mass of a uniform circular arc	**5**
	c	Modelling equilibrium: hanging bodies and systems free to rotate (about a fixed horizontal axis)	**4**
3		**Work and energy**	
	a	Work and kinetic energy; derivation of units and formulae	**4**
	b	Potential energy, work–energy principle, conservation of mechanical energy, problem solving	**6**
	c	Power; derivation of units and formula	**4**
4		**Collisions**	
	a	Momentum as a vector (**i**, **j** problems); Impulse–momentum principle in vector form	**4**
	b	Direct impact of elastic spheres. Newton's law of restitution. Loss of kinetic energy due to impact	**6**
	c	Problem solving (including 'successive' impacts)	**4**
5		**Statics of rigid bodies:** Equilibrium and statics (including ladder problems)	**6**
			60 hours

UNIT 1: Kinematics of a particle moving in a straight line or plane

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SPECIFICATION REFERENCES

1.1 Motion in a vertical plane with constant acceleration, e.g. under gravity

1.2 Simple cases of motion of a projectile

1.3 Velocity and acceleration when the displacement is a function of time

1.4 Differentiation and integration of a vector with respect to time

PRIOR KNOWLEDGE

IGCSE/GCSE (9-1) in Mathematics at Higher Tier

- Basic trigonometry, Pythagoras and vectors

IAS Mathematics – Mechanics 1 content

- 2D Vectors in mechanics – **i**, **j** system (Unit 2 of the M1 SoW)
- Kinematics of a particle moving in a straight line and equations of motion (Unit 3 of the M1 SoW)
- Dynamics of a particle moving in a straight line – Newton's laws of motion (Unit 4 of the M1 SoW)

KEYWORDS

Projectile, range, vertical, horizontal, component, acceleration, gravity, initial velocity, vector, angle of projection, position, trajectory, parabola, distance, displacement, speed, velocity, constant acceleration, constant force, variable force, variable acceleration, retardation, deceleration, initial (t = 0), stationary (speed = 0), at rest (speed = 0), instantaneously, differentiate, integrate, turning point.

NOTES

This topic builds on the kinematics covered in IAS Mathematics – Mechanics 1 content, see M1 SoW Units 2, 3 and 4.

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1a. Motion in a vertical plane under gravity; projectiles (1.1) (1.2)	**Teaching time** 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve problems involving vertical motion under gravity;
- know how to apply the equations of motion to **i**, **j** vector problems;
- be able to find the time of flight of a projectile;
- be able to find the range and maximum height of a projectile;
- be able to derive formulae to find the greatest height, the time of flight and the horizontal range (for a full trajectory);
- know how to modify projectile equations to take account of the height of release;
- be able to derive and use the equation of the path.

TEACHING POINTS

Define a projectile as an object dropped or thrown in the air. Show a video from the internet of a golf ball being hit or a shot-putter. Explain that the path is called a parabola which is the old Greek word for throw.

Discuss the modelling assumptions: the object is treated as a particle so it does not spin and has no air resistance. Therefore the only force on the object is gravity. (Link this back to vertical motion under gravity in Mechanics 1.)

Discuss the fact that displacement, velocity and acceleration are vectors with components in the horizontal and vertical directions. These components obey the *suvat* formulae, and the horizontal and vertical directions can be treated separately.

Begin with *horizontal* projection examples and encourage students to make two lists for the motion in the horizontal and vertical directions. It is easier to start this way as the initial *vertical* velocity is zero for this type of question, hence $u = 0$ for the vertical equation of motion.

For **all** Projectile questions:

$a = 0$ (in the horizontal direction), so the horizontal velocity is constant.

$a = \pm g \text{ m s}^{-2}$ or $\pm 9.8 \text{ m s}^{-2}$ (in the vertical direction) depending on whether downwards is taken as positive or upwards is taken as positive.

The two equations of motion often will have time, t, as a common term.

Move on to projection with speed $U \text{ m s}^{-1}$ at *any* angle α (above the horizontal ground) and introduce the concept of the initial velocity having horizontal and vertical components. (It may be advisable to revise magnitude and direction, Pythagoras and basic trigonometry.)

Horizontally, $u = U\cos\alpha$ and if upwards is positive, vertically, $u = U\sin\alpha$.

Derive the formulae for the time of flight, greatest height (when the vertical velocity is zero) and horizontal range (for the maximum range you will need to use the identity $\sin 2\alpha = 2\sin\alpha\cos\alpha$ which is covered in the IAL Mathematics – Pure Mathematics 3 content, see P3 SoW Unit 2b).

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Emphasise the fact that s is displacement. So, for example, for the vertical equation of motion, we use $s = 0$ if the projectile returns to the ground, and if it is projected from a height and lands lower than its starting point, then, if upwards is positive s will be negative in the vertical direction.

Show examples with the initial velocity as an **i–j** vector (the **i** coefficient is u for the horizontal equation of motion). This actually makes it easier as the components are done for you.

Cover examples which ask for the speed, distance and direction of motion. Make sure that students can pick out the keywords, and that they realise when the answer can be left in **i**, **j** form, and when they must form a triangle and use Pythagoras and tan to calculate the magnitude and direction (e.g. when asked for the speed and direction of motion of a particle).

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

The general equation of the path, if we know the projection speed and the projection angle α, is a useful equation. It reduces to a quadratic (which you can show is a parabola with a negative x^2 term), but requires the identity $1 + \tan^2 \alpha = \sec^2 \alpha$ (IAL Mathematics – Pure Mathematics 3 content – see P3 SoW Unit 2a). It can be used to find the possible angle(s) of projection to reach any point on the trajectory. There is also a symmetry of path in the absence of air resistance.

Some graphical packages will draw the graphs using **i–j** vectors; these can be used to help students visualise the problems.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students often find projectile questions challenging, sometimes confusing the horizontal and vertical aspects of the motion, for example by including the horizontal component of velocity in an equation for the vertical motion.

Other common mistakes include considering only one component of velocity when finding speeds and making sign errors when producing quadratic equations (to find t).

NOTES

General formulae can be derived to obtain the maximum height, time of flight and range in terms of initial velocity u, acceleration g and angle α, but these only work for a full trajectory. This means learning them to use in exams should be done with caution; it is probably better for students to work from first principles, rather than learn and substitute.

Model the good practice of drawing a diagram to illustrate the situation whenever possible. This will encourage students to draw their own diagrams.

1b. Variable acceleration (use of calculus and finding vectors $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ at a given time) (1.3) (1.4)	Teaching time 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use calculus in kinematics to model motion in a straight line for a particle moving with variable acceleration;
- be able to extend techniques for motion in one dimension to two dimensions by using calculus and vector versions of equations for variable force/acceleration problems;
- understand the language and notation of kinematics appropriate to variable motion in two dimensions, i.e. knowing the notation $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ for variable acceleration in terms of time.

TEACHING POINTS

Start by stating that the *suvat* formulae (see IAS Mechanics 1 Unit 3) can only be used when acceleration is constant and the motion is in a straight line. This means the speed–time or velocity–time graphs are made up of straight lines.

Draw the graph of say, $v = 2t^2 + 2t + 1$ (for $t > 0$). This is part of a parabola where the gradient is increasing so as time passes the object is accelerating more quickly. As acceleration is not constant, the *suvat* formulae will not work for this model.

Make links (using IAL Pure Mathematics calculus) to the rate of change of velocity explaining that $\frac{\mathrm{d}v}{\mathrm{d}t}$ = gradient = acceleration. This idea that the gradient of a velocity–time graph gives acceleration should be familiar from previous work in IAS Mechanics 1 Unit 3 and also from IGCSE/GCSE (9-1) in Mathematics.

Summarise the situation by talking about, velocity as the rate of change of displacement and acceleration as the rate of change of velocity.

Express these statements in the notation of calculus: $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ and $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$.

By linking integration with the reverse of differentiation, displacement and velocity can be found by integrating expressions for velocity and acceleration respectively:

$s = \int v\,\mathrm{d}t$ and $v = \int a\,\mathrm{d}t$

Move on to explain that the constant of integration, c needs to be found by referring back to the problem and using some (usually initial) information about the body. For example knowing that the particle starts from O at rest means that when $t = 0$ (initially), $s = 0$ (at O) and $v = 0$ (at rest). These values can be substituted to calculate c.

Students will also need to relate the fact that the gradient = 0 at the max or min point to this mathematical model i.e. if $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$, then acceleration = 0, so the particle must be at max or min velocity, as it cannot accelerate (or get any faster or slower) any more at this point in time.

Motions can now be more complicated as the forces in the **i** and **j** directions can differ and be variable (i.e. $\boldsymbol{F} = m\boldsymbol{a}$). Also the notation for 2D motion replaces the displacement, s, with position vector, $\boldsymbol{r}$. Velocity, $\boldsymbol{v}$, can be defined as $\dot{\boldsymbol{r}}$ and the acceleration vector can be called $\ddot{\boldsymbol{r}}$ (rather than $\boldsymbol{a}$).

Introduce this notation to students, explaining how the dot above the $\boldsymbol{r}$ denotes how many times the $\boldsymbol{r}$ has been differentiated with respect to time. Hence $\ddot{\boldsymbol{r}}$ (representing the acceleration) effectively means $\boldsymbol{r}$ differentiated *twice* with respect to time or $\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}$.

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The other vital point to stress is when we integrate $\dot{\boldsymbol{r}}$ (or $\boldsymbol{v}$) to obtain the displacement $\boldsymbol{r}$, we have to introduce a vector constant of integration in the form $c\mathbf{i} + k\mathbf{j}$ (rather than just $+ c$). Any conditions provided in the question (e.g. the particle is initially at the point with position vector $(3\mathbf{i} + 2\mathbf{j})$ m) allow us to substitute into the expression for $\boldsymbol{r}$ and calculate the constants.

Ask questions along the lines of: Consider an aeroplane taking off. Its position is given by $\boldsymbol{r} = (80t\,\mathbf{i} + 0.5t^3\,\mathbf{j})$ m. What is its velocity and acceleration at time t? Now criticise the model. (Hint: consider motion in the x-direction.)

Reverse the process: A particle has acceleration $\boldsymbol{a} = (4t\mathbf{i} + 2\mathbf{j})$ m s^{-2} and is initially at the origin moving with velocity $2\mathbf{i}$ m s^{-1}. Find $\dot{\boldsymbol{r}}$ and $\boldsymbol{r}$ using integration. (Be careful with the constants of integration!)

Just as in the one-dimensional case, we do not need to use calculus every time; if the acceleration vector is constant, we can use vector forms of the *suvat* formulae.

Questions on this topic often ask about the direction of motion: stress that this is given by the direction of the velocity vector. To find when an object is moving due North, the East component of the velocity vector is zero and the North component positive.

Finally, a question may ask for the force acting on the particle of mass m kg. In this situation students must find the acceleration ($\ddot{\boldsymbol{r}}$) at time t and then state the force $\boldsymbol{F}$ as $\boldsymbol{F} = m\ddot{\boldsymbol{r}}$ or $\boldsymbol{F} = m\boldsymbol{a}$ (in terms of $\mathbf{i}$ and $\mathbf{j}$).

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Students need to know when to differentiate and/or integrate and how acceleration = 0 gives a maximum velocity so questions like the following are useful.

A particle moves so that its motion is modelled by the equation $v = 6t(3 - t)$ m s^{-1}.

Find: (a) the times when it is at rest, (b) its maximum velocity, (c) an expression for its acceleration, (d) the total distance it travels between the times it is stationary.

Extension: Starting with constant a, students can derive the earlier *suvat* equations. Stress the constants of integration, which produce u in $v = u + at$ and the s_0 (s when $t = 0$) in $s = ut + \frac{1}{2}at^2 + s_0$. You could extend this approach to relate double differentiation and signs of $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}$ to indicate if it is a min or max displacement.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Some common errors students make include: forgetting the constant of integration; giving the final answer as a vector when the question asked for the speed; and not being careful about changes of direction and so, for example, finding the displacement rather than the distance travelled.

NOTES

The variable is always t for this unit.

The following diagram may help students decide whether to differentiate or integrate to solve a problem. '**d**' for the **d**own arrow means '**d**ifferentiate'. Hence, down from $\boldsymbol{r}$ gives $\boldsymbol{v}$ or $\dot{\boldsymbol{r}}$ or $\frac{\mathrm{d}r}{\mathrm{d}t} = v$.

Integration is the opposite of differentiation so up is integrate. Up from $\boldsymbol{a}(\ddot{\boldsymbol{r}})$ gives $v(\dot{\boldsymbol{r}})$ or integral of $\boldsymbol{a}(\ddot{\boldsymbol{r}})$ with respect to t gives $\boldsymbol{v}(\dot{\boldsymbol{r}})$.

$\downarrow$ $\boldsymbol{s}$

diff $\downarrow$ $\boldsymbol{v}$ $(\dot{\boldsymbol{r}})$ $\uparrow$ int

$\boldsymbol{a}$ $(\ddot{\boldsymbol{r}})$ $\uparrow$

UNIT 2: Centres of mass

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SPECIFICATION REFERENCES

2.1 Centre of mass of a discrete mass distribution in one and two dimensions
2.2 Centre of mass of uniform plane figures, and simple cases of composite plane figures
2.3 Simple cases of equilibrium of a plane lamina

PRIOR KNOWLEDGE

IAS level Mathematics – Pure 2 content

- Sigma notation (See P2 SoW Unit 4c)

IAS Mathematics – Mechanics 1 content

- Modelling assumptions made throughout this course (e.g. particle, rigid, light, lamina, etc.)
- Types of forces and force diagrams (Unit 4 of the M1 SoW)
- Equilibrium and statics (Unit 5 of the M1 SoW)
- Moments (Unit 6 of the M1 SoW)

KEYWORDS

Moment, turning effect, sense, newton metre (N m), equilibrium, reaction, bar, rod, plank, uniform, centre of mass, lamina, discrete mass distribution, plane figure, median, segment, arc, composite, hanging body, axis, free to rotate, light inextensible string, freely suspended.

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2a. Centre of mass of a discrete mass distribution in one or two dimensions, framework and lamina (rectilinear shapes) (2.1) (2.2)	Teaching time 5 hours

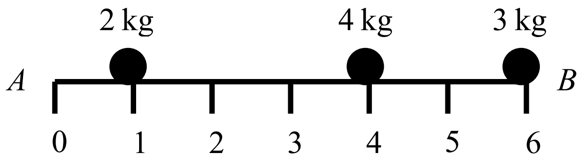
OBJECTIVES

By the end of the sub-unit, students should:

- be able to extend the principle of moments to develop an approach and formula for finding the centre of mass of a one dimensional discrete mass distribution;
- be able to find the centre of mass of a two dimensional discrete mass distribution and a framework;
- be able to find the centre of mass of a plane uniform lamina made up of rectilinear shapes e.g. L shape.

TEACHING POINTS

Revise the principle of taking moments by finding the position of the centre of mass of this light rod with the three masses placed on it.



Now generalise the process by considering a light horizontal rod *AB*, with masses m_1, m_2, m_3 placed at points distances x_1, x_2, x_3 respectively from the end *A*. The reaction *R*, acts at the point at which the rod balances and therefore acts at the centre of mass, a distance $\overline{x}$ from *A*.

Taking moments about *A* gives:

$$m_1gx_1 + m_2gx_2 + m_3gx_3 = R\overline{x}$$

Resolving vertically:

$$R = m_1g + m_2g + m_3g$$

So solving for $\overline{x}$ after substituting in *R* and cancelling *g*, gives:

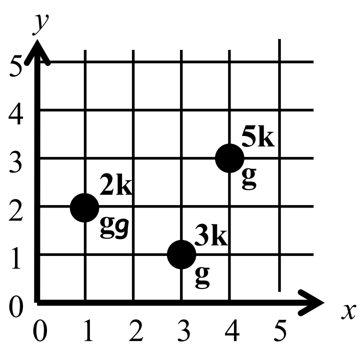
$$\overline{x} = \frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3}$$

Using the sigma notation, we get a formula to calculate the centre of mass of a rod with $i = 1$ to n masses on it.

$$\overline{x} = \frac{\sum m_i x_i}{\sum m_i}$$

This formula can be applied to a find the centre of mass of a discrete mass distribution in 2D. Some students prefer to use a tabular approach showing the information.

Consider a set of masses placed at different points (including negative coordinates) and extend the approach above, taking moments about the *y*-axis and then the *x*-axis to give the coordinates of the centre of mass of the system. Always draw a diagram and check to see if the answer is in a sensible position in terms of the system.



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It is a good idea now to give a purpose for the centre of mass. In 1D it was where we could balance the rod and now, in 2D, it is the point at which we could balance a light lamina with a system of particles attached to it. Alternatively, with a non-light lamina it would still be the point of balance. If the lamina is freely suspended from a point then the centre of mass must lie vertically below that point. This will be demonstrated in the next section to find the position of the centre of mass of a uniform triangular lamina.

The position of a centre of mass is also important in the design of structures, cars and machine parts. It has an effect on stability and the wear and tear of rotating machine parts.

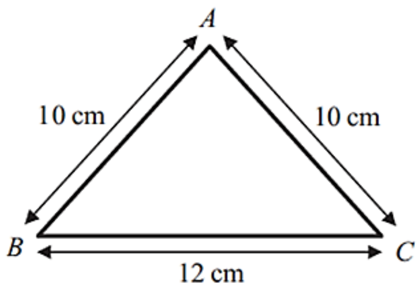
For all systems whose motion is very complicated (like a boomerang) the motion of the centre of mass is often a much simpler motion than that of the system. For example, for a spiralling object flying through the air, the motion of the centre of mass is a parabola.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The following examples are important applications of discrete mass distributions.

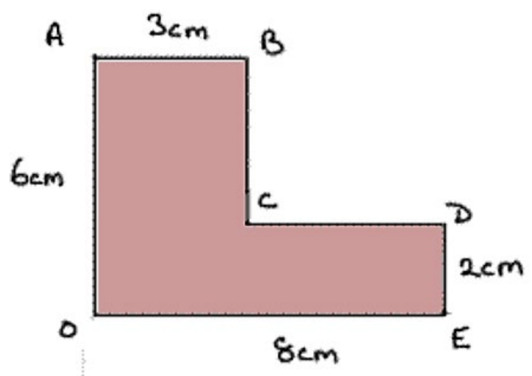
Frameworks:

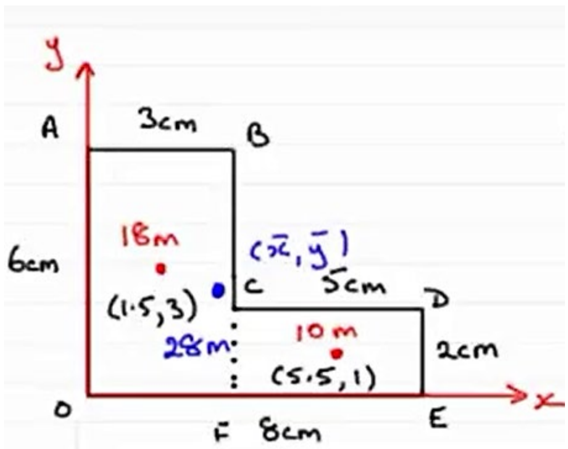
The framework below reduces to three particles, one at each of the centres of the three lengths of wire. The mass is proportional to the length of each wire. Strictly the mass does not have a unit, but we can label the mass of AB as $10m$ units (where m is mass per unit length). The m will cancel out like the g did at the start of this section. Basic trigonometry is necessary to find the distances required for the centre of mass formula or moments calculation.



Composite lamina made up of rectilinear shapes

Assuming the two rectangles are uniform, the centre of mass of each shape is in the centre of that shape (by symmetry). The mass of each shape is proportional to the area, so m will be the mass per unit area. Therefore we could almost forget about the shape and instead just concentrate on the two centre of mass coordinates, applying our formula or taking moments. A vector approach could be useful if both coordinates are required.





Finally cover questions when a section is *removed*. This means that there is a **negative** moment in the calculation and also that the mass of the object is less after removal.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Centre of mass is a topic which is generally well understood by students and one which is answered well in examinations.

Where errors do occur they are basic errors when calculating areas, treating frameworks as laminas and wasting time by calculating centres of mass rather than assuming symmetry.

NOTES

The next part will extend these methods to triangular and circular-based composite shapes.

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2b. Centre of mass of triangular, circular-based and composite laminas and centre of mass of a uniform circular arc (2.2)	Teaching time 5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know and be able to quote the position of the centre of mass for a uniform triangular lamina;
- know and be able to quote the position of the centre of mass for a uniform circular arc or sector of a circle;
- be able to adapt the given formulae for semi-circular laminas and arcs;
- be able to find the centre of mass of composite 2D shapes.

TEACHING POINTS

Begin by considering a uniform triangular lamina. If you make a triangle out of cardboard, by hanging it from each of its three vertices and drawing a vertical line from the point of suspension in each case, you can establish the position of the centre of mass from the intersection of the lines. (This links well with 'hanging bodies' in the next section).

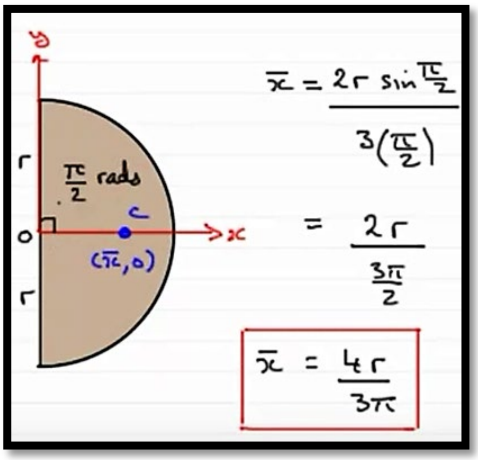
Define the three lines as the medians (i.e. lines from the vertex to the midpoint of the opposite side).

Hence the position of the centre of mass of a triangular lamina is $\frac{2}{3}$ of the median length. Often questions involve perpendicular heights which are multiples of 3. It is worth considering the special case of a right-angled triangle. It is also useful to know that the centre of mass of a uniform triangular lamina with vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3) has coordinates $\frac{1}{3}(x_1 + x_2 + x_3, y_1 + y_2 + y_3)$.

For circular-based shapes the specification states 'results given in the formulae book may be quoted without proof' so make sure students are aware of the formulae that appear in the booklet. (See below.) Students need to be clear of the difference between the use of arc and sector.

It is worth customising the formula for a sector to create a formula for a semi-circular lamina of radius r as this shape comes up often in exercises and exams.

Substituting $\alpha = \frac{\pi}{2}$ gives $\frac{4r}{3\pi}$ for the position of the centre of mass (this is quicker to learn and quote rather than have to substitute into the formula in the pressure of an exam).



Centres of mass

For uniform bodies:

Triangular lamina: $\frac{2}{3}$ along median from vertex

Circular arc, radius r, angle at centre 2α: $\frac{r \sin\alpha}{\alpha}$ from centre

Sector of circle, radius r, angle at centre 2α: $\frac{2r \sin\alpha}{3\alpha}$ from centre

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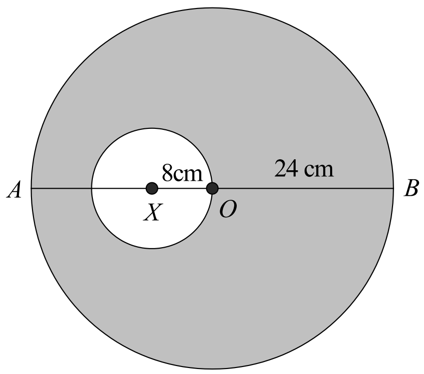
OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Questions may involve non-uniform composite plane figures/frameworks. It is a good idea to cover a wide variety of examples for composite shapes and frameworks (such as the example shown below in the common misconceptions section).

Use of an axis of symmetry, where appropriate, is acceptable, so often students only need to apply the centre of mass formula or take moments once.

For non-uniform shapes, a good example is to show a piece of paper with a corner folded up, therefore creating a triangular section in one corner. This has double the thickness, so the 'mass per unit area' is doubled. (e.g. If the area of the triangle is 6 units^2, then use $12m$ in the calculation.) Note that each part of any composite shape will be modelled as a uniform lamina; other non-uniform laminas are not part of this unit.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES



Finding the position of the centre of mass for the lamina shown illustrates some of the main errors students can make when calculating the centre of mass.

The most common error was in failing to subtract the area of the disc removed in the moments equation or in failing to subtract the moment of the disc removed. Candidates did not always choose to take moments about A, but many correct solutions were seen. A minority of candidates were either searching the formula booklet for inspiration or confused by more advanced work that they had studied, and attempted to use the formula for centre of mass of a sector of a circle.

NOTES

Proof (by integration) of those formulae stated in the formula book is not required.

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2c. Modelling equilibrium: hanging bodies and systems free to rotate (about a fixed horizontal axis) (2.3)	Teaching time 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use the position of the centre of mass for a 2D lamina to determine the orientation of the shape relative to either the vertical or horizontal when suspended freely from a point;
- be able to use the position of the centre of mass for a 2D lamina to establish a condition for equilibrium to exist when the body is free to rotate about a fixed horizontal axis;
- be able to use the position of the centre of mass for a 2D lamina to establish a condition for equilibrium to exist when the body is put on an inclined plane.

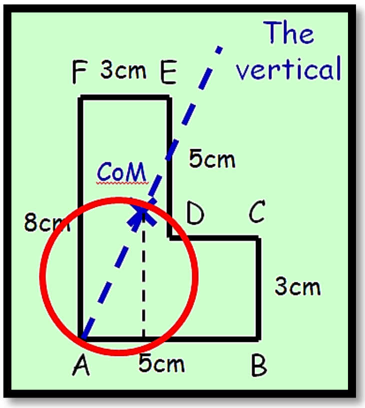
TEACHING POINTS

Finding the size of the angle that a specified line on a shape makes with either the vertical or horizontal when the shape is freely suspended from a point is usually the final part of a question; it generally follows finding the centre of mass of that shape.

As in the experiment to determine the position of the centre of mass of a triangular lamina in the previous section, the line joining the point of suspension to the centre of mass defines the vertical (since the centre of mass must lie vertically below the point of suspension).

The specified line is generally one of the sides of the shape. In the example below the lamina is freely suspended from A. It is required to find the size of the angle that the side AB makes with the vertical. That means we want to find the size of the angle that AB makes with the line joining A to the centre of mass. The diagram illustrates that the creation of a right-angled triangle is the most effective way to find the size of this angle, reading off the distances to the centre of mass to find the lengths and using 'arctan' to find the size of the required angle (usually in degrees to 1 decimal place). Note that, although when suspended the centre of mass lies vertically below A, it is not necessary to draw a new diagram.

Advise students to take care in working out the distances, as sometimes they may need to subtract the centre of mass distance from the whole length. A clearly labelled diagram should help to clarify which distances are required. They should also be aware that if the dimensions are in terms of 'a' say, then the 'a' will cancel in the 'arctan' fraction.



If a lamina rests in equilibrium on a rough inclined plane then the line of action of the weight of the lamina must pass through the side of the lamina which is in contact with the plane. If the angle of the plane is increased so that the line of action of the weight passes outside the section along which the lamina is in contact with the plane then the lamina will topple. We can usually assume that the coefficient of friction between the lamina and the plane is sufficient to prevent he lamina from slipping down the plane. An example of a question would be to find the angle that a plane makes with the horizontal when the lamina is about to topple.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Consider a shape free to rotate about a horizontal axis and hanging in equilibrium. Suppose now a mass M is attached to the shape to make it rotate about the axis and then rest in equilibrium in a different position. In this new position, the line joining the original centre of mass to the axis will no longer be vertical. One side of the shape may now be specified as being, say, horizontal.

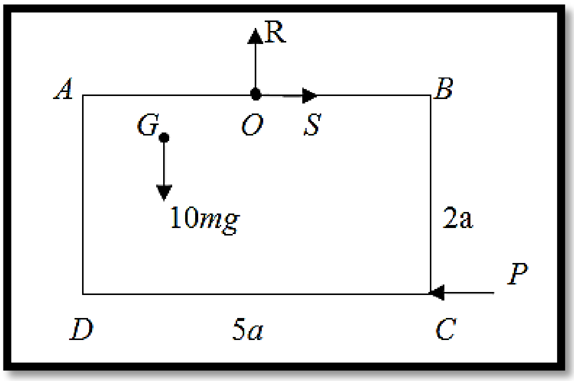
Here is the wording of a past exam question which illustrates the idea:

> The mass of the lamina is M. A particle of mass kM is attached to the lamina at D to form the system S. The system S is freely suspended from A and hangs in equilibrium with AO horizontal.

It may be required to find the value of the extra mass that has been added. This can be done by effectively treating the system as a standard moments problem. By taking moments about the point of suspension and equating the clockwise moment to the anticlockwise moment, it would be possible to find the extra mass. Another method would be to find the centre of mass of the new system and use the fact that this must lie vertically below the point of suspension.

Alternatively, rather than adding an extra mass, an extra force may be applied to the system to alter its orientation. For example, the system could rest in equilibrium with the introduction of a **horizontal** force effectively pushing it so that one side of the shape becomes horizontal as illustrated below:

> A horizontal force of magnitude P is applied at C in the direction CD. The loaded plate L remains suspended from O and rests in equilibrium with AB horizontal and C vertically below B.



From the above diagram it can be seen that, without the addition of a horizontal force P at C, the plate would hang with G (centre of mass) vertically below O. The forces R and S represent the components of the reaction at the pivot. To find the value of P we could take moments about O. To find R and S we could resolve vertically and horizontally.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When using a triangle to find the size of the required angle, students can mostly identify the correct angle, but a fairly common error is to use the distances of the centre of mass from the wrong sides i.e. not interpreting the coordinates of the centre of mass (found in the first part of the question) correctly.

Students also sometimes forget to use the centre of mass they have calculated in an earlier part of the question and instead incorrectly assume that the centre of mass is in the centre of the shape.

NOTES

When drawing the hanging body diagram, it is acceptable if students do not visualise the orientation of the body, but instead just put the vertical ‘plumb’ line from the suspension point through the centre of mass (as shown in the diagram in the teaching points section). It is the triangle which gives the solution and this is not affected by its orientation.

However, it is advisable to draw the shape hanging correctly for the equilibrium situation where an extra force or mass is added to the system; this enables the relevant distances to be calculated when taking moments about the point of suspension.

A level Mathematics: Mechanics 2

UNIT 3: Work and energy

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SPECIFICATION REFERENCES

3.1 Kinetic and potential energy, work and power. The work–energy principle. The principle of conservation of mechanical energy.

PRIOR KNOWLEDGE

IGCSE/GCSE Physics

- Intuitive idea of work and energy and S.I. units

IAS Mathematics – Mechanics 1 content

- Kinematics – motion of a particle in a straight line with constant acceleration (Unit 3 of the M1 SoW)
- Types of forces, Newton's laws, resolving forces, frictional forces, dynamics (Unit 4 of the M1 SoW)

KEYWORDS

Work, energy, power, joules, gravitational potential energy (GPE), kinetic energy (KE), energy change, resistance, force, distance, displacement, speed, velocity, conservation of mechanical energy, external force, work–energy principle, reaction, power, watts, kW, tractive (driving) force, acceleration, inclined plane, resistance, rate of working, rough/smooth surface, friction.

NOTES

Kinetic energy will also be required for Unit 4: Elastic collisions

A level Mathematics: Mechanics 2

3a. Work and kinetic energy; derivation of units and formulae (3.1)	**Teaching time** 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the derivation, units and definitions of work and energy;
- be able to define kinetic energy (KE);
- understand that work done on a body moving in a horizontal plane is the change in kinetic energy.

TEACHING POINTS

Begin with two equations familiar from IAS Mechanics 1:

$v^2 = u^2 + 2as$ and $F = ma$

Rearrange $F = ma$ to give $a = \frac{F}{m}$, substitute into the kinematics equation and rearrange to give:

$Fs = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

Note that we are assuming that F is constant.

What is the quantity mass × speed^2 ?

Its unit could be based on the Fs part of the formula above which gives newtons × metres (N m), but this is easily confused with moments, which also have the units N m.

One of the pioneers in formalising the concepts of work and energy was James Joule (1818–1889).

He defined work done as force × distance and the unit was named the joule (J)

So, $\frac{1}{2}$ × mass × speed^2 is defined as the kinetic energy (KE) of a body (i.e. energy of movement). It is a measure of a body's capacity to do work by moving and is measured in joules (J).

In practice one joule is a tiny amount of energy as it is defined as the work done in moving 1 N a distance of 1 m. Most functional applications use kJ (or 1000 J)

From the definition of work done and the equation above we have the work–energy principle for a particle moving in a horizontal plane under a constant force:

work done (force × distance) = change in kinetic energy

(Note that KE is always positive as the speed is squared.)

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Discuss situations where there are changes in kinetic energy and what types of forces cause these changes; for example, friction causing a sliding object to slow down or the driving force of an engine making a car go faster. Problems involving these scenarios will be considered in the next two sub-units.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Sign errors and confusing the different equations involving forces and energy are the main mistakes students make in these questions.

NOTES

In the next part we will consider gravitational potential energy and include this in the work–energy principle.

3b. Potential energy, work–energy principle, conservation of mechanical energy, problem solving (3.1)

Teaching time
6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the concept of gravitational potential energy (GPE);
- be able to include GPE when applying the work–energy principle;
- know the conditions for conservation of mechanical energy;
- be able to solve problems involving work and energy.

TEACHING POINTS

Consider a package of mass m kg at rest on a shelf which is h metres above the ground. The package has the potential to fall off the shelf, so it is possible to think of the energy being stored. If the package did fall (vertically), then this energy would be released and its speed would begin to increase (and so its kinetic energy would increase).

Can we measure this gravitational potential energy as the package rests on the shelf? It is the work done against gravity, which is force × distance. The force on the package is its weight, mg newtons, and the distance is the height it could potentially fall, h metres. Therefore, GPE = mgh joules.

(Note that GPE can be treated as a negative term – see Problem Solving below)

We can now extend the idea of the work–energy principle from the previous section to a particle not restricted to moving in a horizontal plane:

work done = total change in sum of KE and GPE

We generally relate a change in height of a particle to a change in its GPE rather than as work done by gravity i.e. it is included on just the right hand side of the equation.

What conditions would mean that there is no total energy change during a system's motion? This happens in the absence of any *external* forces (excluding gravity), such as a driving force, air resistance or friction.

In these situations we have conservation of (mechanical) energy:

KE + GPE (before) = KE + GPE (after)

We say mechanical energy as we are not considering, for example, heat energy or sound energy.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

When solving problems involving work and energy, it is important to understand whether mechanical energy is conserved or not. Are there any external forces like friction? It is always a good idea to read the whole question first to be able to picture the situation fully.

When a body is moving up or down an inclined plane, the GPE will be calculated using the change in *vertical* height (despite the path not being vertical). Remember that GPE can also be considered as work done by gravity; since the force of gravity acts vertically, the vertical height is used in the force × distance (mgh) formula. A particle could be moving up or down a curved path; the principle of using the change in vertical height to calculate the change in GPE still applies.

However, the work done (by friction) will be based on the *actual* distance along the plane using the force × distance formula.

Try to encourage students to avoid applying 'gain in KE = loss in GPE' unless it is absolutely clear that there are no external forces (excluding gravity) acting. In most problems, a better approach is to look at the particle at the start and at the end of the motion being considered. (Put in a dotted line to indicate the GPE level at the start of the motion – if the particle is higher at the start, still put GPE = 0 J, which will give a negative GPE at the lower point indicating loss of GPE.)

Now, calculate GPE + KE (start) and GPE + KE (end).

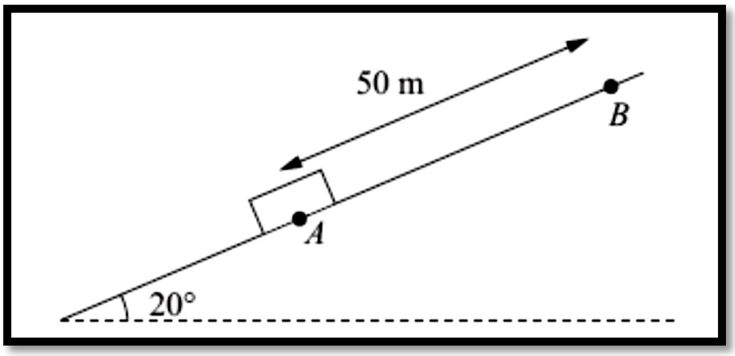
If there is no external force other than gravity, then energy is conserved so we can equate the two energy totals (often solving for a final velocity).

If the total energies cannot be equated (due to the presence of an external force), then work out the overall energy change by subtraction (loss in energy = energy 'before' – energy 'after').

If there is a total loss in energy we have work done by a resisting force. Conversely, if there is more energy at the end, then an external force has put energy into the system (e.g. cyclist pedalling up a hill).

In either case, the overall energy change must equal force × distance, where the force is the *external* force (assumed to be constant) that caused the energy change and the distance is measured in the same direction as the force.

The past exam question below illustrates when this approach would be used.



The box is released from rest at the point B and slides down the slope which is assumed to be smooth. Using conservation of mechanical energy, or otherwise, find the speed of the box as it reaches A.

Discuss what the 'or otherwise' method could refer to (use of $F = ma$ and *suvat*). Also discuss the situation if the slope were not assumed to be smooth.

It may be required to apply the work–energy principle to a whole system rather than a single particle, for example two particles connected by a string passing over a pulley as shown in the diagram below.

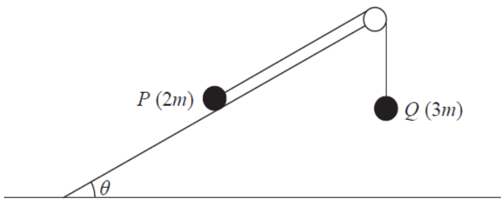


Figure 1

If the system is released from rest and the plane is smooth then both particles will gain KE, particle Q will lose GPE and particle P will gain GPE, but overall **total** mechanical energy will be conserved. If, however, the plane is not smooth then there will an overall loss in energy which, by the work energy principle, will be equal to the work done by friction as P moves up the plane.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Common errors in these sorts of questions include: sign errors, double counting, mistakenly thinking work is done against gravity when moving in a downwards direction, forgetting to include both the kinetic and gravitation potential energy terms.

A level Mathematics: Mechanics 2

In the case of two particles connected by a string over a pulley students generally realise that they need to include KE, GPE and (if appropriate) work done against friction but a common error is to omit the KE (or GPE) of one of the particles or just to write $\frac{1}{2}mv^2$ without indicating which mass is being used.

Students should also be aware that if a question states a particular method (such as using the work–energy principle) then marks will not be awarded if an alternative method (such as $F = ma$ and *suvat*) is used.

NOTES

Frictional forces and knowledge of the coefficient of friction, μ, may be required for these questions.

Potential energy in this unit has been referred to as *gravitational* potential energy (GPE) rather than just PE, to distinguish it from Elastic potential energy (EPE). Elastic potential energy is not covered in Mechanics 2.

A level Mathematics: Mechanics 2

3c. Power; derivation of units and formula (3.1)	Teaching time 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand that power in watts is the rate of doing work;
- be able to calculate the power P of a vehicle with a tractive (driving) force F, moving with velocity v;
- be able to use the formula $P = Fv$ in problem solving.

TEACHING POINTS

What is power? It is the rate at which we do work:

$$\text{Power} = \frac{\mathrm{d}}{\mathrm{d}t}(\text{work done}) = \frac{\mathrm{d}(Fs)}{\mathrm{d}t}$$

When the force is constant,

$$\text{Power} = F\frac{\mathrm{d}s}{\mathrm{d}t} = Fv$$

So, for a moving vehicle with a constant driving or tractive force, F newtons, moving at speed v ms^{-1} we have:

Power = tractive force × speed or $P = Fv$

The dimensional unit is J s^{-1}, but electrical power in SI units is measured in watts (W) (after James Watt). Therefore mechanical power is also measured in watts. (Again kW is a more common unit for heavy vehicles moving at high speed.)

Emphasise that the tractive force is *not* the resultant force, but the external force exerted by the engine of the vehicle trying to drive it forwards. None of these forces need be constant; the formula holds at any particular instant during the motion.

What is interesting is that if we have a fixed power, say 500 W for an electric car, and the tractive force is 100 N, then the speed will be 5 m s^{-1}. But, if we increase the speed to 10 m s^{-1}, then the tractive force decreases to 50 N (i.e. half the force for double the speed!) This seems counter-intuitive, but if we think about the momentum of the vehicle, this is greater at higher speed and helps to ‘keep the car moving’ and so less force is needed.

Questions may involve a vehicle travelling on a horizontal plane or on an inclined plane. It is important to note whether it is moving up or down the plane. The resistance to motion is usually taken to be constant although it could depend on v.

Some problems require, or state, that the vehicle is travelling at *maximum* or *constant* speed. This indicates that the acceleration = 0 m s^{-2} and when we resolve along the direction of motion, the resultant force is zero. It therefore follows that the tractive force of the engine and the external resistances, including components of weights if the motion is not horizontal, are then equal. This implies equilibrium, despite the fact that the car is moving. This is dynamic equilibrium, which can be thought of as the experience when flying at a constant 500 mph in an aeroplane and you cannot experience any forces other than hearing the sound of the engine.

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OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

To model everyday life, the speed of the vehicle could be given in km h^{-1} and the power in kW. This means that before starting to answer the problem, the units need changing to m s^{-1} and watts respectively.

It is also possible that the angle of an inclined plane could be given in arcsin, arctan or arccos form. The most common form is arcsin since the component of weight parallel to the plane is $mg\sin\theta$ (which means it is not necessary to evaluate the angle itself). A diagram is essential in order to identify all the forces, velocities and acceleration (where relevant).

This past exam question illustrates a typical example.

> A truck of mass of 300 kg moves along a straight horizontal road with a constant speed of 10 m s^{-1}. The resistance to motion of the truck has magnitude 120 N.
>
> **(a)** Find the rate at which the engine of the truck is working.
>
> On another occasion the truck moves at a constant speed up a hill inclined at θ to the horizontal, where $\sin\theta = \frac{1}{14}$.
>
> The resistance to motion of the truck from non-gravitational forces remains at magnitude 120 N and the rate at which the engine works is the same as in part **(a)**.
>
> **(b)** Find the speed of the truck.

Note that 'non-gravitational' forces are referred to in this question. When the truck is moving up the hill the component of weight acting parallel to the plane could be considered as acting as a resistance. To ensure that this is not included in the resistance term here the expression 'non-gravitational' forces is used.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

A common error is to miss one of the forces when writing equations. This can often be avoided if students draw a diagram of forces before they start.

Sign errors may also creep in when decelerations are involved.

There is sometimes confusion between 'tractive (driving) force' and 'resultant force'.

Students need to be aware of what is an appropriate level of accuracy to use for their final answer. For example, only 2 significant figures are appropriate following the use of $g = 9.8$ (but 3 are accepted).

NOTES

The work–energy principle can also be used to extend these power questions.

The resistance to the motion of a car when its speed is v m s^{-1} could be modelled as a variable force of magnitude, say $(200 + 2v)$ N. This is a more realistic model as the resistance to motion increases with the speed of the vehicle.

UNIT 4: Collisions

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SPECIFICATION REFERENCES

4.1 Momentum as a vector. The impulse–momentum principle in vector form. Conservation of linear momentum

4.2 Direct impact of elastic spheres. Newton's law of restitution. Loss of mechanical energy due to impact

4.3 Successive impacts of up to three particles or two particles and a smooth plane surface

PRIOR KNOWLEDGE

- Intuitive idea about collisions and S.I. units
- Mechanical energy (Unit 3)

IAS Mathematics – Pure 2 content

- Geometric Sequences & Series (including the sum to infinity) (Unit 4 of the P2 SoW)

KEYWORDS

Mass, velocity, N s, momentum, impulse, force, collisions, direct, impact, smooth, sphere, conservation, coefficient of restitution (e), Newton's (experimental) law of restitution, approach speed, separation speed, opposite direction, elastic, perfectly elastic, inelastic, plane, energy, kinetic energy, joules, 'loss' of mechanical energy.

NOTES

This topic builds on, and extends, the ideas developed in Mechanics 1 by considering collisions of spheres taking into account elasticity. It also considers how kinetic energy is changed by a collision.

4a. Momentum as a vector (i, j problems); Impulse–momentum principle in vector form (4.1)	Teaching time 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to extend the definition of linear momentum and impulse to 2D using vectors;
- be able to use the impulse–momentum principle in vector form i.e. $\mathbf{I} = m\mathbf{v} - m\mathbf{u}$.

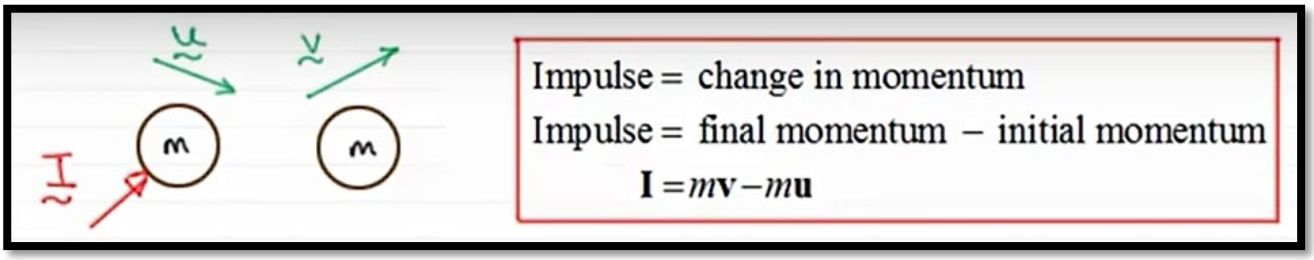
TEACHING POINTS

Linear momentum is a vector quantity (since 'momentum = mass × velocity' where mass is a scalar and velocity is a vector). Impulse is defined as 'change in momentum' so that is also a vector quantity. We apply the formula for impulse in vector form; we consider an impulse applied to a particle moving in a straight line where the impulse acts at an angle to that straight line (see diagram below).

Time is a scalar quantity, but force and velocity are vectors, therefore the formula can be written:

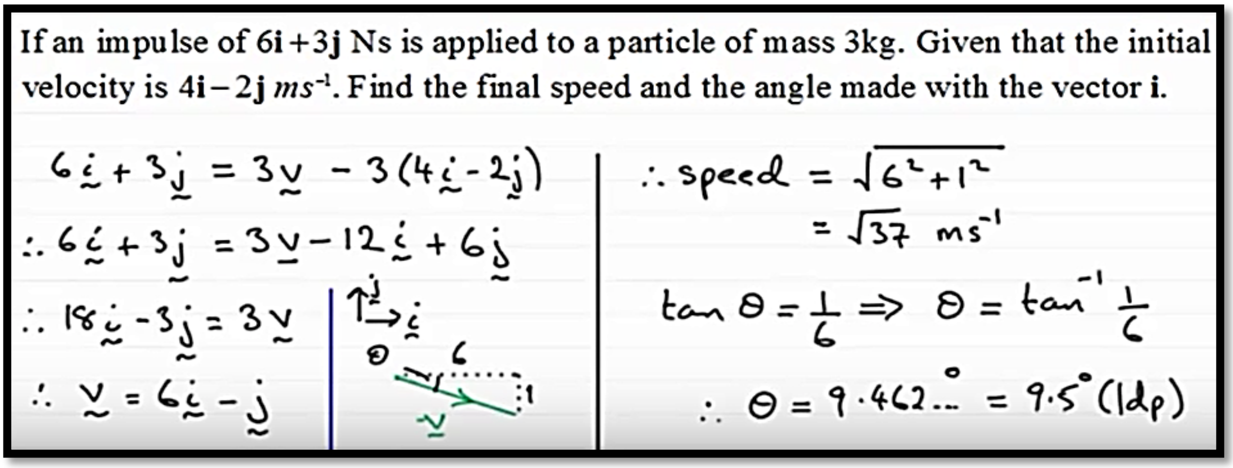
$\mathbf{F}t = m\mathbf{v} - m\mathbf{u}$ or

$\mathbf{I} = m\mathbf{v} - m\mathbf{u}$



It is good if the students can visualise the problems and draw realistic diagrams, rather than just apply the formula.

The following numerical example illustrates the use of the formula; students should be encouraged to show the relevant directions in a diagram.



Students could work in column or **i**, **j** vectors.

Working separately for the coefficients of the components **i** and **j** may lead to a simpler solution in some problems.

At other times, it is useful to use $m(\mathbf{v} - \mathbf{u}) = \mathbf{I}$ as the velocities can be simplified in terms of **i** and **j**.

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OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

This example involves use of the above formula.

> A particle of mass 0.2 kg is moving with velocity $(3a\mathbf{i} - a\mathbf{j})\,\text{ms}^{-1}$, where a is a positive constant, when it receives an impulse of magnitude $2\sqrt{5}$ N s. Immediately after receiving the impulse the particle has velocity $(a\mathbf{i} - 2a\mathbf{j})\,\text{ms}^{-1}$.
>
> (a) Find the value of a.

You should cover questions where the velocity of one of the particles is given in speed and direction form, rather than directly in components.

For example, suppose the velocity $\mathbf{u}$ is in the direction $3\mathbf{i} + 4\mathbf{j}$ and the speed is $10\,\text{m s}^{-1}$.

To find $\mathbf{u}$ the concept of a unit vector needs to be covered. The magnitude of the direction part gives a magnitude (speed) of 5.

The unit vector $\hat{\mathbf{u}}$ has a magnitude of 1, but preserves the angle properties i.e. $\hat{\mathbf{u}} = \frac{1}{5}(3\mathbf{i} + 4\mathbf{j})$. Multiplying by the speed required gives $\mathbf{u} = 10 \times \frac{1}{5}(3\mathbf{i} + 4\mathbf{j}) = (6\mathbf{i} + 8\mathbf{j})\,\text{m s}^{-1}$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

In these sorts of exam questions, the majority of errors are due to slips in the arithmetic though vectors can cause confusion for some students who combine **i** and **j** terms inappropriately.

Students must always read the question carefully so they know how to give their final answers. For example, they may unnecessarily find the magnitude of **v**, either because they do not read the question properly or they do not know the difference between speed and velocity. Similarly, they sometimes only state the velocity when the speed is actually required thereby losing marks unnecessarily.

A level Mathematics: Mechanics 2

4b. Direct impact of elastic spheres. Newton's law of restitution and loss of kinetic energy due to impact (4.2)	Teaching time 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to express the 'compressibility', 'bounciness' or 'elasticity' of an object by a value called the coefficient of restitution (e);
- know that $0 \le e \le 1$ [and that $e = 0$ means inelastic and $e = 1$ means perfectly elastic];
- know and be able to use Newton's (experimental) law of restitution for direct impacts of elastic spheres;
- be able to calculate the change in kinetic energy due to an impact.

TEACHING POINTS

When we considered collisions in Mechanics 1 we had no information about the nature of the materials from which the particles were made. What if we now consider collisions where we have this information (such as spheres made of, for example, wood and steel) and what if we could somehow include this information in our equations?

Newton performed some experiments, mainly by bouncing balls off a solid wall, and found that the ratio of the approach speed to separation speed varied according to the type of materials making up the two surfaces in contact.

If you throw a ball made up of damp mud (or a rotten tomato!) it may not even bounce off the wall, but 'splat' and stick to the wall. In this case the separation speed would be zero.

He decided to call the ratio of separation speed to approach speed the coefficient of restitution (e). It is a measure of the 'elasticity' of the objects hitting each other. Its value will be a number between 0 and 1. (For the damp mud example, the coefficient of restitution would be zero, whereas if the collision is perfectly elastic then $e = 1$.)

This definition leads to a simple formula which states that the coefficient of restitution (e) for two colliding particles is such that:

$$\boldsymbol{e} = \frac{\textbf{speed of separation of particles}}{\textbf{speed of approach of particles}}$$

This is known as Newton's (experimental) law of restitution.

Notice that we use *speed* rather than velocity (so the ratio is always positive).

For a ball bouncing off a fixed plane (either a wall or a floor) we can further simplify the formula, giving:

$$e = \frac{v}{u} \text{ or } v = eu$$

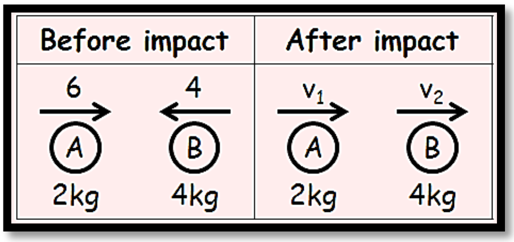
(where u is the speed with which it hits the plane and v is the rebound speed).

This shows that if $e = 1$, the ball is so 'hard' that it rebounds at the same speed as it approaches.

However, the most useful application of Newton's law of restitution is for collisions between two moving objects. Here the approach and separation speeds are relative speeds, rather than just u and v (as in situations where the wall is fixed).

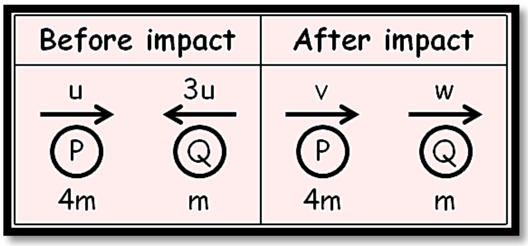
Encourage students to draw carefully labelled diagrams and to be careful when finding the approach and separation speeds. For example, for two objects moving directly towards each other (a head on collision) the approach speed is the sum of the two speeds. It is possible to think intuitively, rather than apply a formulaic approach when solving problems (see Examiner's Report below). However, students may prefer to use the formula: $v_1 - v_2 = -e\,(u_1 - u_2)$ where the velocities are all measured in the same direction so, as in the conservation of momentum equation, consideration of the signs of terms is crucial.

For questions like this one, where the final velocities of both spheres need to be found, the conservation of linear momentum still applies (using velocities rather than speeds – so directions, and consequently signs, count). This can be used along with Newton's law of restitution to form simultaneous equations which can be solved to find, for example, missing velocities or masses.



OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Include examples with algebraic velocities and masses, like the example below:



When two objects collide, there will be a bang and heat is generated. These are forms of sound energy and heat energy which indicates that some of the mechanical energy has been transformed into a different form of energy.

This means that if we consider the kinetic energy of the system before and after the impact, there will be a 'loss of KE'. This is how a past exam question worded this concept:

Given that $e = \frac{1}{2}$, find the total kinetic energy lost in the first collision between A and B.

The terms may be algebraic (often in the form mu^2) for this comparison. Note that if $e = 1$, the collision is said to be perfectly elastic and no energy is lost. Remember also that although overall energy is lost in a collision ($e < 1$) it is possible for an individual particle to gain KE.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

This quote from a recent examiner's report highlights the main sources of errors:

Most candidates understood the principles of conservation of momentum and Newton's experimental law. However, many lost a mark here because they did not pay sufficient attention to the direction of motion of the particles after the collision, leading to inconsistent signs between their two equations. Even if they had indicated directions on a diagram, this was not always consistent with their equations. It was also common to see substitution into 'template equations' rather than understanding of the equations.

The final 2 marks were very often lost because many failed to realise that the final direction of motion of P was the key to a solution. There were a few sign errors and arithmetic errors which resulted in attempts to 'fudge' the answer rather than find the source of the error.

NOTES

The next section will look at applications and more complex problems involving elastic impacts.

A level Mathematics: Mechanics 2

4c. Problem solving (including 'successive' impacts) (4.3)	**Teaching time** 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve problems of the following types involving elastic impacts:
  a) successive collisions between pairs of spheres (horizontal motion);
  b) bouncing ball (off a horizontal elastic plane);
  c) successive collisions including two spheres and sphere against a wall;
  d) determination of number of collisions or deriving the possible range of e.

TEACHING POINTS

Although this topic is essentially made up of the application of two main principles (conservation of momentum and Newton's law of restitution), there are many types of problems which could be set.

Consider three small spheres A, B, C (masses $2m$, $3m$, $4m$ respectively) lying in a straight line, with B and C stationary, and A is projected directly towards B with speed u m s^{-1}. (The coefficient of restitution for all impacts is e.) Pose the problem: A hits B, then B hits C. But will A hit B again?

The condition for the third impact is often dependent on the range of values of e, which is set up by means of an inequality originating from $v_A > v_B$ or $v_A - v_B > 0$. This inequality may require some algebraic manipulation.

Sometimes, one of the spheres will hit and rebound from a fixed vertical wall. Two such examples are below.

> Two small spheres A and B move on a smooth horizontal table. The mass of A is m and the mass of B is $4m$. Initially A is moving with speed u when it collides directly with B, which is at rest on the table. As a result of the collision, the direction of motion of A is reversed. The coefficient of restitution between the particles is e.
>
> (a) Find expressions for the speed of A and the speed of B immediately after the collision.
>
> In the subsequent motion, B strikes a fixed vertical wall and rebounds.
>
> The wall is perpendicular to the direction of motion of B.
>
> The coefficient of restitution between B and the wall is $\frac{4}{5}$.
>
> (b) Given that there is a second collision between A and B show that $\frac{1}{4} < e < \frac{9}{16}$.

> Two small spheres A and B have masses $3m$ and $2m$ respectively. They are moving towards each other in opposite directions on a smooth horizontal plane, both with speed $2u$, when they collide directly. As a result of the collision, the direction of motion of B is reversed and its speed is unchanged.
>
> (a) Find the coefficient of restitution between the spheres.
>
> Subsequently, B collides directly with another small sphere C of mass $5m$ which is at rest. The coefficient of restitution between B and C is $\frac{3}{5}$.
>
> (b) Show that, after B collides with C, there will be no further collisions between the spheres.

A level Mathematics: Mechanics 2

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Consider examples in which a small ball (mass m) is dropped from a height, h on to an elastic horizontal plane, in which the coefficient of restitution for all bounces is e.

Either kinematics or the conservation of mechanical energy can be used to find the speed of the ball immediately before the first impact, namely $\sqrt{2gh}$. However the rebound speed will be $e\sqrt{2gh}$, which is less than the speed of impact. What will be the new greatest height after this first bounce?

You will find that the height is e^2h. By repeating the process, you will find the greatest heights form the terms of a geometric sequence with common ratio e^2 and, since $0 \leq e \leq 1$, the sum of these terms will converge (see work on Geometric Series in IAL Pure 2).

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When considering multiple collisions and trying to work out whether subsequent collisions will occur, a common error is to only consider collisions between two of the three spheres. Students may also come unstuck if they do not produce a correct inequality, for example by simply considering the speed of one sphere rather than its speed relative to the other sphere.

Sign errors in equations and inequalities can be avoided by use of a clear diagram.

Students should be advised that substituting values into equations at early stages may make things easier and help avoid algebraic slips.

NOTES

Only successive impacts of up to three particles or two particles and a smooth plane surface need to be considered.

A level Mathematics: Mechanics 2

UNIT 5: Statics of rigid bodies | **Teaching time**
Equilibrium and statics (including ladder problems) (5.1) (5.2) | 6 hours

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SPECIFICATION REFERENCES

5.1 Moment of a force
5.2 Equilibrium of rigid bodies

PRIOR KNOWLEDGE

IAS Mathematics – Mechanics 1 content

- Types of forces and force diagrams
- Assumptions made throughout this course (e.g. particle, rigid, light, etc.)
- S.I. units
- Moments and frictional forces
- Resolving forces
- Kinematics (constant acceleration), Newton's laws of motion
- Basic equilibrium

KEYWORDS

Force, resultant, component, resolving, plane, parallel, perpendicular, weight, tension, thrust, friction, reaction, force diagram, equilibrium, inextensible, light, negligible, particle, rough, smooth, incline, uniform, friction, coefficient of friction, concurrent, coplanar.

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve statics problems for a system of forces which are not concurrent (e.g. ladder problems), thus applying the principle of moments for forces at any angle;
- be able to solve problems about rigid bodies resting in limiting equilibrium.

TEACHING POINTS

Start by reviewing the definition of the moment of a force from IAS Mechanics 1. Move on to ladder-type problems which will revise moments and then extend to any angle, as the forces will not be concurrent. Extend the moments formula to '*perpendicular force* × *distance*' and resolve the force to find its component at right angles to the full distance from the moments point.

Show students how to use the alternative formula '*force* × *perpendicular distance*', by measuring the perpendicular distance from the moments point to the line of action of the force.

Also make sure that students are clear about the directions of the frictional force (for examples involving rough surfaces) and the reactions at the wall and ground being labelled differently.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Extension: consider a uniform rod which has one end *freely hinged* to a wall and the other end tied to a point above the wall, making the bar horizontal. Discuss the fact that the reaction at the hinge is *not* perpendicular to the wall and that the lines of actions of all the forces in the system will all meet at one point for equilibrium. Representing the reaction at the hinge as two perpendicular forces, the 'resolving and taking moments' solution would be fairly straightforward.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students are often good at drawing force diagrams, but common errors are omitting arrowheads, incorrectly labelling (e.g. 4 kg rather than 4*g*) and missing off the normal reaction or friction forces. Students can sometimes struggle to work out the direction of the frictional force.

Common errors in questions involving moments are ignoring the weight of the ladder, sine/cosine confusion and missing a distance in one or more terms.