

1

Pure
Further Mathematics

F1

Revision Notes

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Further Pure 1

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1 Complex Numbers

Definitions and arithmetical operations

$i = \sqrt{-1}$, so $\sqrt{-16} = 4i$, $\sqrt{-11} = \sqrt{11} i$, etc.

These are called *imaginary* numbers

Complex numbers are written as $z = a + bi$, where a and $b \in \mathbb{R}$.
 a is the *real part* and b is the *imaginary part*.

$+$, $-$, $\times$ are defined in the ‘sensible’ way; division is more complicated.

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i \\(a + bi) - (c + di) &= (a - c) + (b - d)i \\(a + bi) \times (c + di) &= ac + bdi^2 + adi + bci \\&= (ac - bd) + (ad + bc)i\end{aligned}\quad \text{since } i^2 = -1$$

$$\begin{aligned}\text{So } (3 + 4i) - (7 - 3i) &= -4 + 7i \\ \text{and } (4 + 3i)(2 - 5i) &= 23 - 14i\end{aligned}$$

Division – this is just rationalising the denominator.

$$\begin{aligned}\frac{3+4i}{5+2i} &= \frac{3+4i}{5+2i} \times \frac{5-2i}{5-2i} && \text{multiply top and bottom by the complex conjugate} \\&= \frac{23+14i}{25+4} = \frac{23}{29} + \frac{14}{29}i\end{aligned}$$

Complex conjugate

$$z = a + bi$$

The *complex conjugate* of z is $z^* = a - bi$

Properties

If $z = a + bi$ and $w = c + di$, then

$$\begin{aligned}(\text{i}) \quad \{(a + bi) + (c + di)\}^* &= \{(a + c) + (b + d)i\}^* \\&= \{(a + c) - (b + d)i\} \\&= (a - bi) + (c - di) \\&= (a + bi)^* + (c + di)^*\end{aligned}$$

$$\Leftrightarrow (z + w)^* = z^* + w^*$$

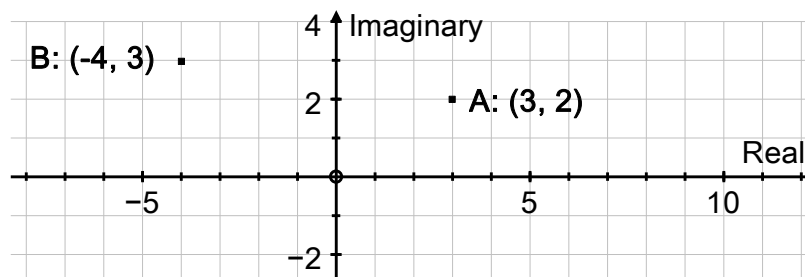
$$\begin{aligned}
 \text{(ii)} \quad \{(a + bi)(c + di)\}^* &= \{(ac - bd) + (ad + bc)i\}^* \\
 &= \{(ac - bd) - (ad + bc)i\} \\
 &= (a - bi)(c - di) \\
 &= (a + bi)^*(c + di)^*
 \end{aligned}$$

$$\Leftrightarrow (zw)^* = z^* w^*$$

Complex number plane, or Argand diagram

We can represent complex numbers as points on the complex number plane:

$3 + 2i$ as the point $A(3, 2)$, and $-4 + 3i$ as the point $B(-4, 3)$.

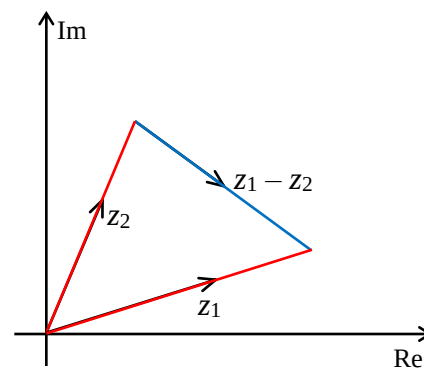
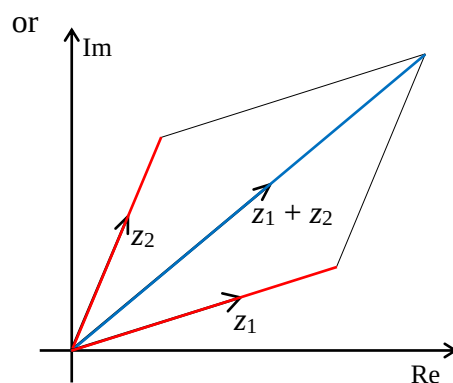


Complex numbers and vectors

Complex numbers under addition (or subtraction) behave just like vectors under addition (or subtraction). We can show complex numbers on the Argand diagram as **either** points **or** vectors.

$$(a + bi) + (c + di) = (a + c) + (b + d)i \quad \Leftrightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i \quad \Leftrightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a - c \\ b - d \end{pmatrix}$$

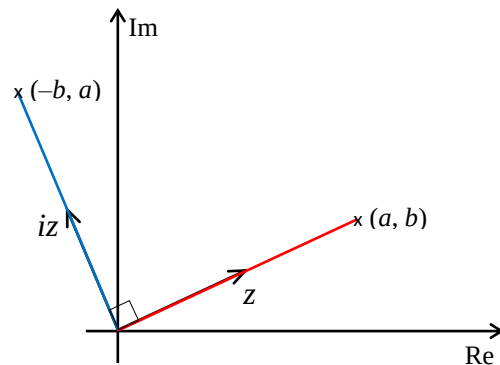


Multiplication by i

$i(3 + 4i) = -4 + 3i$ – on an Argand diagram this would have the effect of a rotation through $\frac{\pi}{2}$ about the origin.

In general;

$$i(a + bi) = -b + ai$$



Modulus of a complex number

This is just like polar co-ordinates.

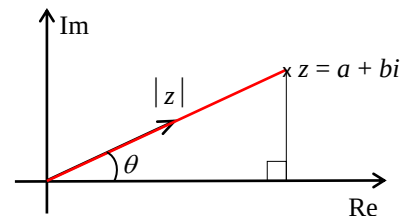
The modulus of z is $|z|$ and

is the length of the complex number

$$|z| = \sqrt{a^2 + b^2}.$$

$$z z^* = (a + bi)(a - bi) = a^2 + b^2$$

$$\Rightarrow z z^* = |z|^2.$$



Argument of a complex number

The argument of z is $\arg z$ = the angle made by the complex number with the positive x-axis.

Your textbook takes $-\pi < \arg z \leq \pi$, but you are allowed to take $0 \leq \arg z < 2\pi$.

N.B. Always draw a diagram when finding $\arg z$.

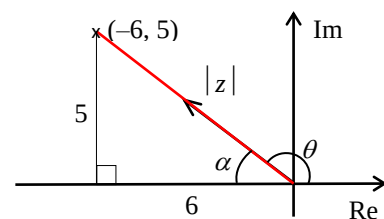
Example: Find the modulus and argument of $z = -6 + 5i$.

Solution: First sketch a diagram (it is easy to get the argument wrong if you don't).

$$|z| = \sqrt{6^2 + 5^2} = \sqrt{61}$$

$$\text{and } \tan \alpha = \frac{5}{6} \Rightarrow \alpha = 0.694738276$$

$$\Rightarrow \arg z = \theta = \pi - \alpha = 2.45 \quad \text{to 3 S.F.}$$



Equality of complex numbers

$$a + bi = c + di \quad \Rightarrow \quad a - c = (d - b)i$$

$$\Rightarrow (a - c)^2 = (d - b)^2 i^2 = -(d - b)^2 \quad \text{squaring both sides}$$

$$\text{But } (a - c)^2 \geq 0 \quad \text{and} \quad -(d - b)^2 \leq 0$$

$$\Rightarrow (a - c)^2 = -(d - b)^2 = 0$$

$$\Rightarrow a = c \quad \text{and} \quad b = d$$

$$\text{Thus } a + bi = c + di$$

$$\Rightarrow \text{real parts are equal } (a = c), \text{ and imaginary parts are equal } (b = d).$$

Square roots

Example: Find the square roots of $5 + 12i$, in the form $a + bi$, $a, b \in \mathbb{R}$.

Solution: Let $\sqrt{5 + 12i} = a + bi$

$$\Rightarrow 5 + 12i = (a + bi)^2 = a^2 - b^2 + 2abi$$

$$\text{Equating real parts} \quad \Rightarrow \quad a^2 - b^2 = 5, \quad \mathbf{I}$$

$$\text{equating imaginary parts} \quad \Rightarrow \quad 2ab = 12 \quad \Rightarrow \quad a = \frac{6}{b}$$

$$\text{Substitute in } \mathbf{I} \quad \Rightarrow \quad \left(\frac{6}{b}\right)^2 - b^2 = 5$$

$$\Rightarrow 36 - b^4 = 5b^2 \quad \Rightarrow \quad b^4 + 5b^2 - 36 = 0$$

$$\Rightarrow (b^2 - 4)(b^2 + 9) = 0 \quad \Rightarrow \quad b^2 = 4$$

$$\Rightarrow b = \pm 2, \text{ and } a = \pm 3$$

$$\Rightarrow \sqrt{5 + 12i} = 3 + 2i \quad \text{or} \quad -3 - 2i.$$

Roots of equations

(a) **Any** polynomial equation with complex coefficients has a complex solution.

This is *The Fundamental Theorem of Algebra*, and is too difficult to prove at this stage.

Corollary: Any complex polynomial can be factorised into linear factors over the complex numbers.

- (b) **If** $z = a + bi$ is a root of $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \alpha_{n-2} z^{n-2} + \dots + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$,
and if all the α_i are **real**,
 then the conjugate, $z^* = a - bi$ is also a root.

The proof of this result is in the appendix.

- (c) For any polynomial with zeros $a + bi$, $a - bi$,
 $(z - (a + bi))(z - (a - bi)) = z^2 - 2az + a^2 + b^2$ will be a quadratic factor in which the coefficients are all **real**.
- (d) Using (a), (b), (c) we can see that any polynomial with **real** coefficients can be factorised into a mixture of linear and quadratic factors, all of which have **real** coefficients.

Example: Show that $3 - 2i$ is a root of the equation $z^3 - 8z^2 + 25z - 26 = 0$.
 Find the other two roots.

Solution: Put $z = 3 - 2i$ in $z^3 - 8z^2 + 25z - 26$

$$= (3 - 2i)^3 - 8(3 - 2i)^2 + 25(3 - 2i) - 26$$

$$= 27 - 54i + 36i^2 - 8(9 - 12i + 4i^2) + 75 - 50i - 26$$

$$= 27 - 54i - 36 + 8i - 72 + 96i + 32 + 75 - 50i - 26$$

$$= 27 - 36 - 72 + 32 + 75 - 26 + (-54 + 8 + 96 - 50)i$$

$$= 0 + 0i$$

$$\Rightarrow 3 - 2i \text{ is a root}$$

$$\Rightarrow \text{the conjugate, } 3 + 2i, \text{ is also a root} \quad \text{since all coefficients are **real**}$$

$$\Rightarrow (z - (3 + 2i))(z - (3 - 2i)) = z^2 - 6z + 13 \text{ is a factor.}$$

Factorising, by inspection,

$$z^3 - 8z^2 + 25z - 26 = (z^2 - 6z + 13)(z - 2) = 0$$

$\Rightarrow$ roots are $z = 3 \pm 2i$, or 2

Example: Given that $z = 2 + i$ is a root of $z^4 - 2z^3 - z^2 + 2z + 10 = 0$, find all the roots.

Solution: $2 + i$ is a root

$\Rightarrow$ the complex conjugate $2 - i$ is also a root since all coefficients are **real**

$\Rightarrow (z - (2 + i))(z - (2 - i)) = z^2 - 4z + 5$ is a factor.

$\Rightarrow (z^2 - 4z + 5)(z^2 + 2z + 2) = 0$ by inspection

$\Rightarrow z^2 - 4z + 5 = 0$ or $z^2 + 2z + 2 = (z + 1)^2 + 1 = 0$

$\Rightarrow z = 2 + i, 2 - i, -1 + i$ or $-1 - i$.

2 Numerical solutions of equations

Accuracy of solution

When asked to show that a solution is accurate to n D.P., you must look at the value of $f(x)$ ‘half’ below and ‘half’ above, and conclude that

there is a **change of sign** in the **interval**, and the function is **continuous**, therefore there is a **solution in the interval correct to n D.P.**

Example: Show that $\alpha = 2.0946$ is a root of the equation
 $f(x) = x^3 - 2x - 5 = 0$, accurate to 4 D.P.

Solution:

$$f(2.09455) = -0.0000165\dots, \text{ and } f(2.09465) = +0.00997$$

There is a **change of sign** and f is **continuous**

$\Rightarrow$ there is a **root** in **[2.09455, 2.09465]** $\Rightarrow$ **root is $\alpha = 2.0946$ to 4 D.P.**

Interval bisection

(i) Find an interval $[a, b]$ which contains the root of an equation $f(x) = 0$.

(ii) $x = \frac{a+b}{2}$ is the mid-point of the interval $[a, b]$

Find $f\left(\frac{a+b}{2}\right)$ to decide whether the root lies in $\left[a, \frac{a+b}{2}\right]$ or $\left[\frac{a+b}{2}, b\right]$.

(iii) Continue finding the mid-point of each subsequent interval to narrow the interval which contains the root.

Example: (i) Show that there is a root of the equation
 $f(x) = x^3 - 2x - 7 = 0$ in the interval $[2, 3]$.
(ii) Find an interval of width 0.25 which contains the root.

Solution: (i) $f(2) = 8 - 4 - 7 = -3$, and $f(3) = 27 - 6 - 7 = 14$

There is a **change of sign** and f is **continuous** $\Rightarrow$ there is a root in $[2, 3]$.

(ii) Mid-point of $[2, 3]$ is $x = 2.5$, and $f(2.5) = 15.625 - 5 - 7 = 3.625$

$\Rightarrow$ change of sign between $x = 2$ and $x = 2.5$

$\Rightarrow$ root in $[2, 2.5]$

Mid-point of $[2, 2.5]$ is $x = 2.25$,
 and $f(2.25) = 11.390625 - 4.5 - 7 = -0.109375$

$\Rightarrow$ change of sign between $x = 2.25$ and $x = 2.5$

$\Rightarrow$ root in $[2.25, 2.5]$, which is an interval of width 0.25

Linear interpolation

To solve an equation $f(x)$ using linear interpolation.

First, find an interval which contains a root,

second, assume that the curve is a straight line and use similar triangles to find where the line crosses the x -axis,

third, repeat the process as often as necessary.

- Example:* (i) Show that there is a root, α , of the equation
 $f(x) = x^3 - 2x - 9 = 0$ in the interval $[2, 3]$.
 (ii) Use linear interpolation once to find an approximate value of α .
 Give your answer to 3 D.P.

Solution: (i) $f(2) = 8 - 4 - 9 = -5$, and $f(3) = 27 - 6 - 9 = 12$

There is a **change of sign** and f is **continuous** $\Rightarrow$ there is a root in $[2, 3]$.

- (ii) From (i), curve passes through $(2, -5)$ and $(3, 12)$, and we assume that the curve is approximately a straight line between these two points.

Let the line cross the x -axis at $(\alpha, 0)$

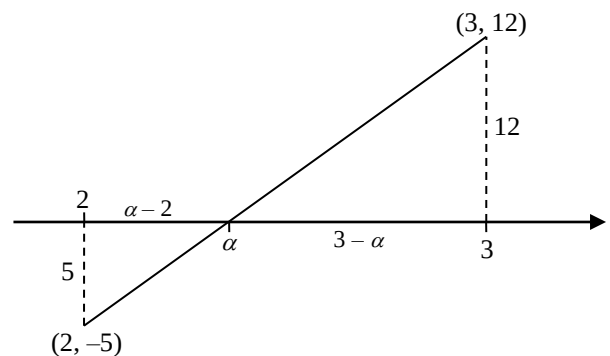
Using similar triangles

$$\frac{3-\alpha}{\alpha-2} = \frac{12}{5}$$

$$\Rightarrow 15 - 5\alpha = 12\alpha - 24$$

$$\Rightarrow \alpha = \frac{39}{17} = 2\frac{5}{17}$$

$$\Rightarrow \alpha = 2.294 \text{ to 3 D.P.}$$



Repeating the process will improve accuracy.

Newton-Raphson

Suppose that the equation $f(x) = 0$ has a root at $x = \alpha$, $\Rightarrow f(\alpha) = 0$

To find an approximation for this root, we first find a value $x = a$ near to $x = \alpha$ (decimal search).

In general, the point where the tangent at P , $x = a$, meets the x -axis, $x = b$, will give a better approximation.

At P , $x = a$, the gradient of the tangent is $f'(a)$,

and the gradient of the tangent is also $\frac{PM}{NM}$.

$$PM = y = f(a) \text{ and } NM = a - b$$

$$\Rightarrow f'(a) = \frac{PM}{NM} = \frac{f(a)}{a-b} \Rightarrow b = a - \frac{f(a)}{f'(a)}.$$

Further approximations can be found by repeating the process, which would follow the dotted line converging to the point $(\alpha, 0)$.

This formula can be written as the iteration $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Example: (i) Show that there is a root, α , of the equation

$$f(x) = x^3 - 2x - 5 = 0 \text{ in the interval } [2, 3].$$

- (ii) Starting with $x_0 = 2$, use the Newton-Raphson formula to find x_1 , x_2 and x_3 , giving your answers to 3 D.P. where appropriate.
- (iii) Show that $\alpha = 2.095$ is accurate to 3 D.P.

Solution: (i) $f(2) = 8 - 4 - 5 = -1$, and $f(3) = 27 - 6 - 5 = 16$

There is a **change of sign** and f is **continuous** $\Rightarrow$ there is a root in $[2, 3]$.

$$(ii) \quad f(x) = x^3 - 2x - 5 \quad \Rightarrow \quad f'(x) = 3x^2 - 2$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{8-4-5}{12-2} = 2.1$$

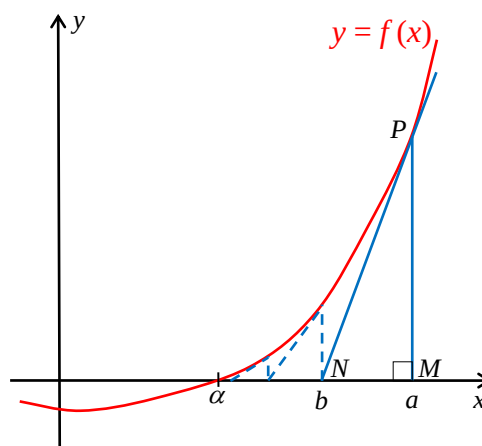
$$\Rightarrow x_2 = 2.094568121 = 2.095$$

$$\Rightarrow x_3 = 2.094551482 = 2.095$$

$$(iii) \quad f(2.0945) = -0.000574... \\ f(2.0955) = +0.010592...$$

There is a **change of sign** and f is **continuous** $\Rightarrow$ there is a root in the interval

$$\Rightarrow \alpha = 2.095 \text{ to 3 D.P.}$$



3 Coordinate systems

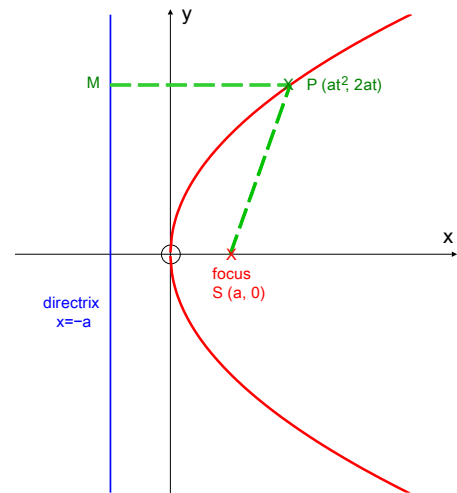
Parabolas

$y^2 = 4ax$ is the equation of a parabola which passes through the origin and has the x-axis as axis of symmetry.

Parametric form

$x = at^2$, $y = 2at$ satisfy the equation for all values of t . t is a parameter, and these equations are the parametric equations of the parabola $y^2 = 4ax$.

Note that $y^2 = (2at)^2 = 4a^2t^2 = 4ax$.



Focus and directrix

The point $S(a, 0)$ is the *focus*, and

the line $x = -a$ is the *directrix*.

A parabola is defined as the locus of a point P which is equidistant from the focus and the directrix, $PM = PS$.

Let P be (x, y) then $PM^2 = PS^2 \Rightarrow (x + a)^2 = (x - a)^2 + y^2$

$\Rightarrow y^2 = 4ax$

Gradient

For the parabola $y^2 = 4ax$, with general point $P, (at^2, 2at)$, we can find the gradient in two ways:

1. $y^2 = 4ax$
 $\Rightarrow 2y \frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$, which we can write as $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$
2. At $P, x = at^2, y = 2at$
 $\Rightarrow \frac{dy}{dt} = 2a, \frac{dx}{dt} = 2at$
 $\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$

Note that there can be disadvantages in using $y = \sqrt{4ax}$ so don't!

Tangents and normals

Example: Find the equations of the tangents to $y^2 = 8x$ at the points where $x = 18$, and show that the tangents meet on the x -axis.

Solution: $x = 18 \Rightarrow y^2 = 8 \times 18 \Rightarrow y = \pm 12$

$$2y \frac{dy}{dx} = 8 \Rightarrow \frac{dy}{dx} = \pm \frac{1}{3} \quad \text{since } y = \pm 12$$

$$\Rightarrow \text{tangents are } y - 12 = \frac{1}{3}(x - 18) \Rightarrow x - 3y + 18 = 0 \quad \text{at } (18, 12)$$

$$\text{and } y + 12 = -\frac{1}{3}(x - 18) \Rightarrow x + 3y + 18 = 0. \quad \text{at } (18, -12)$$

To find the intersection, add the equations to give

$$2x + 36 = 0 \Rightarrow x = -18 \Rightarrow y = 0$$

$\Rightarrow$ tangents meet at $(-18, 0)$ on the x -axis.

Example: Find the equation of the normal to the parabola given by $x = 3t^2$, $y = 6t$.

Solution: $x = 3t^2$, $y = 6t \Rightarrow \frac{dx}{dt} = 6t$, $\frac{dy}{dt} = 6$,

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6}{6t} = \frac{1}{t}$$

$$\Rightarrow \text{gradient of the normal is } \frac{-1}{\frac{1}{t}} = -t$$

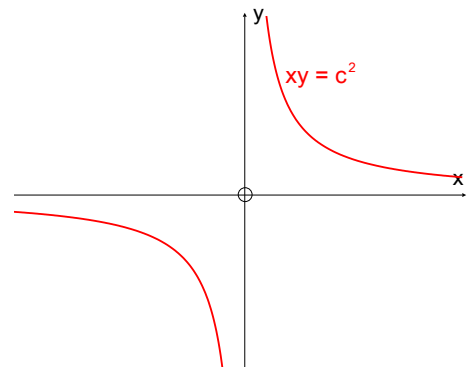
$$\Rightarrow \text{equation of the normal is } y - 6t = -t(x - 3t^2).$$

Notice that this ‘general equation’ gives the equation of the normal for any particular value of t :— when $t = -3$ the normal is $y + 18 = 3(x - 27) \Leftrightarrow y = 3x - 99$.

Rectangular hyperbolas

A *rectangular* hyperbola is a hyperbola in which the asymptotes meet at 90° .

$xy = c^2$ is the equation of a rectangular hyperbola in which the x -axis and y -axis are perpendicular asymptotes.



Parametric form

$x = ct, y = \frac{c}{t}$ are parametric equations of the hyperbola $xy = c^2$.

Tangents and normals

Example: Find the equation of the tangent to the hyperbola $xy = 36$ at the point where $x = 3$.

Solution: $x = 3 \Rightarrow 3y = 36 \Rightarrow y = 12$

$$y = \frac{36}{x} \Rightarrow \frac{dy}{dx} = -\frac{36}{x^2} = -4 \quad \text{when } x = 3$$

$$\Rightarrow \text{tangent is } y - 12 = -4(x - 3)$$

$$\Rightarrow 4x + y - 24 = 0.$$

Example: Find the equation of the normal to the hyperbola given by $x = 3t, y = \frac{3}{t}$.

Solution: $x = 3t, y = \frac{3}{t} \Rightarrow \frac{dx}{dt} = 3, \frac{dy}{dt} = \frac{-3}{t^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-3}{t^2}}{3} = \frac{-1}{t^2}$$

$$\Rightarrow \text{gradient of the normal is } \frac{-1}{\frac{-1}{t^2}} = t^2$$

$$\Rightarrow \text{equation of the normal is } y - \frac{3}{t} = t^2(x - 3t)$$

$$\Rightarrow t^3x - ty = 3t^4 - 3.$$

4 Matrices

You must be able to add, subtract and multiply matrices.

Order of a matrix

An $r \times c$ matrix has r rows and c columns;

the first number is the number of Rows

the second number is the number of Columns.

Identity matrix

The identity matrix is $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note that $MI = IM = M$ for any matrix M .

Determinant and inverse

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the *determinant* of M is

$$\text{Det } M = |M| = ad - bc.$$

To find the *inverse* of $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\text{Note that } M^{-1}M = MM^{-1} = I$$

- (i) Find the determinant, $ad - bc$.

If $ad - bc = 0$, there is no inverse.

- (ii) Interchange a and d (the leading diagonal)
Change sign of b and c , (the other diagonal)
Divide all elements by the determinant, $ad - bc$.

$$\Rightarrow M^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Check:

$$M^{-1}M = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} da - bc & 0 \\ 0 & -cb + ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Similarly we could show that $MM^{-1} = I$.

Example: $M = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$ and $MN = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$. Find N .

Solution: Notice that $M^{-1}(MN) = (M^{-1}M)N = IN = N$ multiplying on the **left** by M^{-1}

But $MNM^{-1} \neq IN$ we cannot multiply on the **right** by M^{-1}

First find M^{-1}

$$\text{Det } M = 4 \times 3 - 2 \times 5 = 2 \Rightarrow M^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}$$

Using $M^{-1}(MN) = IN = N$

$$\Rightarrow N = \frac{1}{2} \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -7 & 4 \\ 13 & -6 \end{pmatrix} = \begin{pmatrix} -3.5 & 2 \\ 6.5 & -3 \end{pmatrix}.$$

Singular and non-singular matrices

If $\det A = 0$, then A is a *singular matrix*, and A^{-1} does not exist.

If $\det A \neq 0$, then A is a *non-singular matrix*, and A^{-1} exists

Linear Transformations

A matrix can represent a transformation, but the point must be written as a column vector before multiplying by the matrix.

Example: The image of $(2, 3)$ under $T = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}$ is given by $\begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 23 \\ 8 \end{pmatrix}$

$\Rightarrow$ the image of $(2, 3)$ is $(23, 8)$.

Note that the image of $(0, 0)$ is always $(0, 0)$

$\Leftrightarrow$ the **origin never moves** under a matrix (linear) transformation

Base vectors

The vectors $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are called *base vectors*, and are particularly important in describing the geometrical effect of a matrix, and in finding the matrix for a particular geometric transformation.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} a \\ c \end{pmatrix}, \text{ the first column, and } \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} b \\ d \end{pmatrix}, \text{ the second column}$$

This is a more important result than it seems!

Finding the geometric effect of a matrix transformation

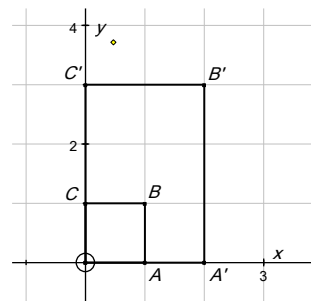
We can easily write down the images of $\mathbf{i}$ and $\mathbf{j}$, sketch them and find the geometrical transformation.

Example: Find the transformation represented by the matrix $\mathbf{T} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

Solution: Find images of $\mathbf{i}, \mathbf{j}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and show on a sketch. Make sure that you letter the points

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 3 & 3 \end{pmatrix}$$

From sketch we can see that the transformation is a two-way stretch, of factor 2 parallel to the x -axis with the y -axis invariant, and of factor 3 parallel to the y -axis with the x -axis invariant.



Finding the matrix of a given transformation.

Example: Find the matrix for a reflection in $y = -x$.

Solution: First find the images of $\mathbf{i}$ and $\mathbf{j}$. These will be the two columns of the matrix.

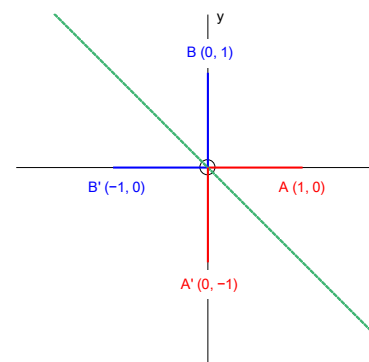
$$A \rightarrow A' \Rightarrow \mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

This will be the first column of the matrix $\begin{pmatrix} 0 & * \\ -1 & * \end{pmatrix}$

$$B \rightarrow B' \Rightarrow \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

This will be the second column of the matrix $\begin{pmatrix} * & -1 \\ * & 0 \end{pmatrix}$

$$\Rightarrow \text{Matrix of the reflection is } \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$



Rotation matrix

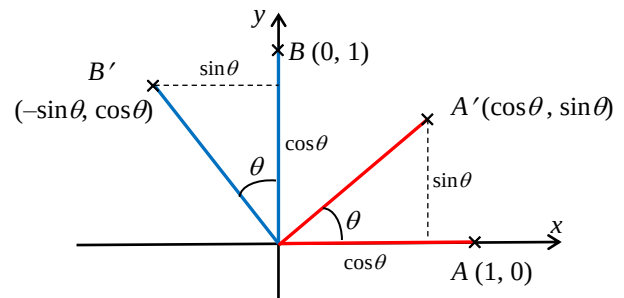
From the diagram we can see that

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix},$$

$$\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$$

These will be the first and second columns of the matrix

$$\Rightarrow \text{matrix is } R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$



Determinant and area factor

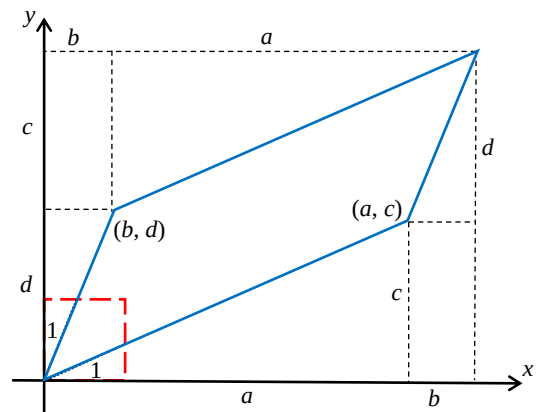
For the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$\text{and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

$\Rightarrow$ the unit square is mapped on to the parallelogram as shown in the diagram.

The area of the unit square = 1.



The area of the parallelogram = $(a+b)(c+d) - 2 \times (bc + \frac{1}{2}ac + \frac{1}{2}bd)$

$$= ac + ad + bc + bd - 2bc - ac - bd$$

$$= ad - bc = \det A.$$

All squares of the grid are mapped onto congruent parallelograms

$\Rightarrow$ area factor of the transformation is $\det A = ad - bc$.

5 Roots of Equations

Roots of quadratic equations

Sum and Product

Suppose that the equation $ax^2 + bx + c = 0$ has roots α and β .

$\Rightarrow (x - \alpha)$ and $(x - \beta)$ are factors

There might also be a constant factor k

$$\Rightarrow ax^2 + bx + c \equiv k(x - \alpha)(x - \beta)$$

$$\Rightarrow ax^2 + bx + c \equiv kx^2 - k(\alpha + \beta)x + k\alpha\beta$$

As this is an identity we can equate coefficients

$$x^2: a = k$$

$$x: b = -k(\alpha + \beta) \quad \Rightarrow \quad \alpha + \beta = \frac{-b}{k} = \frac{-b}{a}$$

$$x^0: c = k\alpha\beta \quad \Rightarrow \quad \alpha\beta = \frac{c}{a}.$$

$$\Rightarrow \text{If } \alpha \text{ and } \beta \text{ are roots of } ax^2 + bx + c = 0 \text{ then } \alpha + \beta = \frac{-b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

Example: α and β are the roots of $2x^2 - 3x - 6 = 0$.

Write down the values of $\alpha + \beta$ and $\alpha\beta$

and find the values of (i) $\alpha^2 + \beta^2$, (ii) $(\alpha - \beta)^2$ and (iii) $\alpha^3 + \beta^3$.

$$\text{Solution: } \alpha + \beta = \frac{-b}{a} = \frac{3}{2} \text{ and } \alpha\beta = \frac{c}{a} = \frac{-6}{2} = -3$$

$$\begin{aligned} \text{(i)} \quad & \text{We know that } (\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 \\ \Rightarrow \quad & \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(\frac{3}{2}\right)^2 - 2 \times (-3) = 8\frac{1}{4} \end{aligned}$$

$$\text{(ii)} \quad (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \left(\frac{3}{2}\right)^2 - 4 \times (-3) = 14\frac{1}{4}$$

$$\begin{aligned} \text{(iii)} \quad & (\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 \\ \Rightarrow \quad & \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) \\ & = \left(\frac{3}{2}\right)^3 - 3 \times (-3) \times \frac{3}{2} = 16\frac{7}{8} \end{aligned}$$

Example: α and β are the roots of a quadratic equation, where $\alpha + \beta = \frac{7}{2}$ and $\alpha\beta = \frac{-5}{3}$.

Find the quadratic equation with integer coefficients.

Solution: $\alpha + \beta = \frac{7}{2} = \frac{-b}{a} \Rightarrow \frac{b}{a} = \frac{-7}{2}$, and $\alpha\beta = \frac{-5}{3} = \frac{c}{a}$

We can write $ax^2 + bx + c = 0$ as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

$$\Rightarrow x^2 + \frac{-7}{2}x + \frac{-5}{3} = 0$$

$$\Rightarrow 6x^2 - 21x - 10 = 0$$

Example: α and β are the roots of $x^2 - 2x - 6 = 0$.

Find the equation whose roots are $\gamma = \alpha - 4$ and $\delta = \beta - 4$.

Solution:

Method 1:

$$\alpha + \beta = \frac{-b}{a} = 2 \text{ and } \alpha\beta = \frac{c}{a} = -6$$

$$\begin{aligned} \Rightarrow \gamma + \delta &= \alpha - 4 + \beta - 4 \\ &= \alpha + \beta - 8 = 2 - 8 = -6 \end{aligned}$$

$$\begin{aligned} \text{and } \gamma\delta &= (\alpha - 4)(\beta - 4) \\ &= \alpha\beta - 4(\alpha + \beta) + 16 = -6 - 4 \times 2 + 16 = 2 \end{aligned}$$

$$\Rightarrow \text{new equation is } x^2 + 6x + 2 = 0$$

Method 2:

From $\gamma = \alpha - 4$ and $\delta = \beta - 4$

we can see that if we write $y = x - 4$

$$\Rightarrow x = y + 4 \text{ in the equation we get } (y + 4)^2 - 2(y + 4) - 6 = 0$$

and that this equation will have roots which are 4 less than those of the original equation

$$\Rightarrow y^2 + 8y + 16 - 2y - 8 - 6 = 0$$

$$\Rightarrow y^2 + 6y + 2 = 0 \quad \text{which agrees with method 1.}$$

Example: α and β are the roots of $x^2 + 2x - 5 = 0$.

Find the equation whose roots are $\gamma = \alpha^2$ and $\delta = \beta^2$.

Solution:

Method 1:

$$\alpha + \beta = \frac{-b}{a} = -2 \quad \text{and} \quad \alpha\beta = \frac{c}{a} = -5$$

$$\begin{aligned} \Rightarrow \quad \gamma + \delta &= \alpha^2 + \beta^2 \\ &= (\alpha + \beta)^2 - 2\alpha\beta = 4 + 10 = 14 \end{aligned}$$

$$\begin{aligned} \text{and} \quad \gamma\delta &= \alpha^2\beta^2 \\ &= (\alpha\beta)^2 = 25 \end{aligned}$$

$$\Rightarrow \quad \text{new equation is } x^2 - 14x + 25 = 0$$

Method 2:

From $\gamma = \alpha^2$ and $\delta = \beta^2$

we can see that if we write $y = x^2 \Rightarrow x = \sqrt{y}$ in the equation $x^2 + 2x - 5 = 0$

we get $(\sqrt{y})^2 + 2(\sqrt{y}) - 5 = 0$

and that this equation will have roots which are the squares of those of the original equation

$$\Rightarrow \quad y - 5 = -2(\sqrt{y})$$

$$\Rightarrow \quad (y - 5)^2 = 4y$$

$$\Rightarrow \quad y^2 - 10y + 25 = 4y$$

$$\Rightarrow \quad y^2 - 14y + 25 = 0$$

which agrees with method 1.

Method 2 will work for any *symmetric* function of the roots, and can be a big advantage for more complicated expressions and for equations of a higher degree.

Example: α and β are the roots of $x^2 + 4x + 5 = 0$.

Find the equation whose roots are $\gamma = \frac{1}{\alpha+1}$ and $\delta = \frac{1}{\beta+1}$.

Solution:

Method 1:

$$\alpha + \beta = \frac{-b}{a} = -4 \quad \text{and} \quad \alpha\beta = \frac{c}{a} = 5$$

$$\begin{aligned} \Rightarrow \quad \gamma + \delta &= \frac{1}{\alpha+1} + \frac{1}{\beta+1} \\ &= \frac{\beta+1+\alpha+1}{(\alpha+1)(\beta+1)} = \frac{\alpha+\beta+2}{\alpha\beta+(\alpha+\beta)+1} = \frac{-4+2}{5-4+1} = -1 \end{aligned}$$

$$\begin{aligned} \text{and} \quad \gamma\delta &= \frac{1}{\alpha+1} \times \frac{1}{\beta+1} \\ &= \frac{1}{\alpha\beta+(\alpha+\beta)+1} = \frac{1}{5-4+1} = \frac{1}{2} \end{aligned}$$

$$\Rightarrow \quad \text{new equation is } x^2 + x + \frac{1}{2} = 0$$

$$\Rightarrow \quad 2x^2 + 2x + 1 = 0$$

Method 2:

$$\text{From } \gamma = \frac{1}{\alpha+1} \text{ and } \delta = \frac{1}{\beta+1}$$

we can see that if we write $y = \frac{1}{x+1} \Rightarrow x = \frac{1}{y} - 1$ in the equation $x^2 + 4x + 5 = 0$

$$\text{we get } \left(\frac{1}{y} - 1\right)^2 + 4\left(\frac{1}{y} - 1\right) + 5 = 0$$

and that this equation will have the required roots

$$\Rightarrow \quad \frac{1}{y^2} - \frac{2}{y} + 1 + \frac{4}{y} - 4 + 5 = 0$$

$$\Rightarrow \quad 2y^2 + 2y + 1 = 0$$

which agrees with method 1.

6 Series

You need to know the following sums

$$\sum_{r=1}^n r = 1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^n r^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2$$

$$= \left(\frac{1}{2}n(n+1)\right)^2 = \left(\sum_{r=1}^n r\right)^2$$

a fluke, but it helps to remember the formula

Example: Find $\sum_{r=1}^n r(r^2 - 3)$.

$$\begin{aligned} \text{Solution: } \sum_{r=1}^n r(r^2 - 3) &= \sum_{r=1}^n r^3 - 3 \sum_{r=1}^n r \\ &= \frac{1}{4}n^2(n+1)^2 - 3 \times \frac{1}{2}n(n+1) \\ &= \frac{1}{4}n(n+1)\{n(n+1) - 6\} \\ &= \frac{1}{4}n(n+1)(n+3)(n-2) \end{aligned}$$

Example: Find $S_n = 2^2 + 4^2 + 6^2 + \dots + (2n)^2$.

$$\begin{aligned} \text{Solution: } S_n &= 2^2 + 4^2 + 6^2 + \dots + (2n)^2 = 2^2(1^2 + 2^2 + 3^2 + \dots + n^2) \\ &= 4 \times \frac{1}{6}n(n+1)(2n+1) = \frac{2}{3}n(n+1)(2n+1) . \end{aligned}$$

Example: Find $\sum_{r=5}^{n+2} r^2$

$$\begin{aligned} \text{Solution: } \sum_{r=5}^{n+2} r^2 &= \sum_{r=1}^{n+2} r^2 - \sum_{r=1}^4 r^2 && \text{notice that the top limit is } 4 \text{ not } 5 \\ &= \frac{1}{6}(n+2)(n+2+1)(2(n+2)+1) - \frac{1}{6} \times 4 \times 5 \times 9 \\ &= \frac{1}{6}(n+2)(n+3)(2n+5) - 30. \end{aligned}$$

7 Proof by induction

1. Show that the result/formula is true for $n = 1$ (and sometimes $n = 2, 3 \dots$)
“therefore the result/formula is true for $n = 1$ ”.
2. Make induction assumption
“Assume that the result/formula is true for $n = k$ ”.
Show that the result/formula must then be true for $n = k + 1$
“therefore the result/formula is true for $n = k + 1$ ”.
3. Belt and Braces
“We have shown that the result/formula is true for $n = 1$, and that IF it is true when $n = k$, then it must also be true when $n = k + 1$ ”
4. Final conclusion
“therefore the result/formula is true for all positive integers, n , by induction”.

Summation

Example: Use mathematical induction to prove that

$$S_n = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

Solution: When $n = 1$, $S_1 = 1^2 = 1$ and $S_1 = \frac{1}{6} \times 1(1+1)(2 \times 1 + 1) = \frac{1}{6} \times 1 \times 2 \times 3 = 1$

$$\Rightarrow S_n = \frac{1}{6}n(n+1)(2n+1) \text{ is true for } n = 1.$$

Assume that the formula is true for $n = k$

$$\Rightarrow S_k = 1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{1}{6}k(k+1)(2k+1)$$

$$\begin{aligned}\Rightarrow S_{k+1} &= 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= \frac{1}{6}(k+1)\{k(2k+1) + 6(k+1)\} \\ &= \frac{1}{6}(k+1)\{2k^2 + 7k + 6\} = \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)\{(k+1) + 1\}\{2(k+1) + 1\}\end{aligned}$$

$$\Rightarrow \text{The formula is true for } n = k + 1$$

We have shown that the statement is true for $n = 1$, and that IF it is true when $n = k$, then it must also be true when $n = k + 1$

$$\Rightarrow S_n = \frac{1}{6}n(n+1)(2n+1) \text{ is true for all positive integers, } n, \text{ by induction.}$$

Recurrence relations

Example: A sequence, 4, 9, 19, 39, ... is defined by the recurrence relation

$$u_1 = 4, \quad u_{n+1} = 2u_n + 1. \text{ Prove that } u_n = 5 \times 2^{n-1} - 1.$$

Solution: When $n = 1$, $u_1 = 4$, and $u_1 = 5 \times 2^{1-1} - 1 = 5 - 1 = 4$, $\Rightarrow$ **formula true for $n = 1$.**

Assume that the formula is true for $n = k$, $\Rightarrow u_k = 5 \times 2^{k-1} - 1$.

From the recurrence relation,

$$u_{k+1} = 2u_k + 1 = 2(5 \times 2^{k-1} - 1) + 1$$

$$\Rightarrow u_{k+1} = 5 \times 2^k - 2 + 1 = 5 \times 2^{(k+1)-1} - 1$$

$\Rightarrow$ **the formula is true for $n = k + 1$**

We have shown that the statement is true for $n = 1$, and that IF it is true when $n = k$, then it must also be true when $n = k + 1$

$\Rightarrow$ **the formula is true for all positive integers, n , by induction.**

Example: A sequence is defined by the recurrence relation

$$u_1 = 2, \quad u_2 = 10, \quad u_{n+2} = 5u_{n+1} - 6u_n.$$

Prove that $u_n = 2 \times 3^n - 2^{n+1}$.

Solution: $u_1 = 2 \times 3 - 2^2 = 2$, and $u_2 = 2 \times 3^2 - 2^3 = 10$

$\Rightarrow$ **formula true when $n = 1$ and $n = 2$**

Assume that the formula is true when $n = k$ and $n = k + 1$

$$\Rightarrow u_k = 2 \times 3^k - 2^{k+1} \text{ and } u_{k+1} = 2 \times 3^{k+1} - 2^{k+2}$$

$$\begin{aligned} \Rightarrow u_{k+2} &= 5u_{k+1} - 6u_k = 5(2 \times 3^{k+1} - 2^{k+2}) - 6(2 \times 3^k - 2^{k+1}) \\ &= 10 \times 3^{k+1} - 5 \times 2^{k+2} - 12 \times 3^k + 6 \times 2^{k+1} \\ &= 6 \times 3^{k+1} - 2 \times 2^{k+2} \end{aligned}$$

$$\Rightarrow u_{k+2} = 2 \times 3^{k+2} - 2^{k+3}$$

$\Rightarrow$ **formula true when $n = k + 2$**

We have shown that the formula is true for $n = 1$ and for $n = 2$, and that IF it is true when $n = k$ and $n = k + 1$, then it must also be true when $n = k + 2$

$\Rightarrow$ **the formula is true for all positive integers, n , by induction.**

Divisibility problems

Example: Prove by induction that $f(n) = n^3 - 7n + 3$ is divisible by 3 for all positive integers n .

Solution: $f(1) = 1^3 - 7 \times 1 + 3 = -3$, which is divisible by 3

$\Rightarrow$ statement is true for $n = 1$

Assume that the statement is true for $n = k$,

$$\begin{aligned}f(k) &= k^3 - 7k + 3 \\f(k+1) - f(k) &= (k+1)^3 - 7(k+1) + 3 - (k^3 - 7k + 3) \\&= k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3 - k^3 + 7k - 3 \\&= 3k^2 + 3k - 9 \\&= 3(k^2 + k - 3) \text{ which is a multiple of 3}\end{aligned}$$

$\Rightarrow$ the statement is true for $n = k + 1$

We have shown that the statement is true for $n = 1$, and that IF it is true when $n = k$, then it must also be true when $n = k + 1$

$\Rightarrow f(n) = n^3 - 7n + 3$ is divisible by 3 for all positive integers, n , by induction.

In some problems considering $f(k+1) - f(k)$, will lead to a proof which sometimes has hidden difficulties, and a more reliable way is to consider $f(k+1) - m \times f(k)$, where m is chosen to eliminate the exponential term.

Example: Prove that $f(n) = 5^n - 4n - 1$ is divisible by 16 for all positive integers, n .

Solution: When $n = 1$, $f(1) = 5^1 - 4 - 1 = 0$, which is divisible by 16,

and **so $f(n)$ is divisible by 16 when $n = 1$.**

Assume that the result is true for $n = k$, $\Rightarrow f(k) = 5^k - 4k - 1$ is divisible by 16.

Considering $f(k+1) - 5 \times f(k)$ we will **eliminate the 5^k term.**

$$\begin{aligned}f(k+1) - 5 \times f(k) &= (5^{k+1} - 4(k+1) - 1) - 5 \times (5^k - 4k - 1) \\&= 5^{k+1} - 4k - 4 - 1 - 5^{k+1} + 20k + 5 = 16k\end{aligned}$$

you must now make $f(k+1)$ the subject of the formula

$$\Rightarrow f(k+1) = 5 \times f(k) + 16k$$

Since $f(k)$ is divisible by 16 (induction assumption), and $16k$ is divisible by 16, then $f(k+1)$ must be divisible by 16,

$$\Rightarrow f(n) = 5^n - 4n - 1 \text{ is divisible by 16 for } n = k + 1$$

We have shown that the statement is true for $n = 1$, and that IF it is true when $n = k$, then it must also be true when $n = k + 1$

$\Rightarrow f(n) = 5^n - 4n - 1$ is divisible by 16 for all positive integers, n , by induction.

Example: Prove that $f(n) = 2^{2n+3} + 3^{2n-1}$ is divisible by 5 for all positive integers n .

Solution: When $n = 1$, $f(1) = 2^{2+3} + 3^{2-1} = 32 + 3 = 35 = 5 \times 7$,

and **so the result is true for $n = 1$.**

Assume that the result is true for $n = k$

$$\Rightarrow f(k) = 2^{2k+3} + 3^{2k-1} \text{ is divisible by 5}$$

Now eliminate a power; we could consider either the power of 2 or the power of 3 (it does not matter which)

$$f(k+1) - 2^2 \times f(k), \text{ which would eliminate the } 2^{2k+3} \text{ term} \quad \mathbf{I}$$

$$\text{or } f(k+1) - 3^2 \times f(k), \text{ which would eliminate the } 3^{2k-1} \text{ term} \quad \mathbf{II}$$

$$\begin{aligned} \mathbf{I} \Rightarrow f(k+1) - 2^2 \times f(k) &= 2^{2(k+1)+3} + 3^{2(k+1)-1} - 2^2 \times (2^{2k+3} + 3^{2k-1}) \\ &= 2^{2k+5} + 3^{2k+1} - 2^{2k+5} - 2^2 \times 3^{2k-1} \end{aligned}$$

$$\Rightarrow f(k+1) - 4 \times f(k) = 9 \times 3^{2k-1} - 4 \times 3^{2k-1} = 5 \times 3^{2k-1}$$

you must now make $f(k+1)$ the subject of the formula

$$\Rightarrow f(k+1) = 4 \times f(k) + 5 \times 3^{2k-1}$$

Since $f(k)$ is divisible by 5 (induction assumption), and $5 \times 3^{2k-1}$ is divisible by 5, **then $f(k+1)$ must be divisible by 5.**

We have shown that the statement is true for $n = 1$, and that IF it is true when $n = k$, then it must also be true when $n = k + 1$

$\Rightarrow f(n) = 2^{2n+3} + 3^{2n-1}$ is divisible by 5 for all positive integers, n , by induction.

There will be times when you are unable to eliminate a power. In these cases consider $f(k+1) - f(k)$ and hope!

Example: Prove by induction that $f(n) = (3n+2) \times 4^n + 7$ is divisible by 9 for all positive integers n .

Solution: When $n = 1$ $f(1) = (3 \times 1 + 2) \times 4^1 + 7 = 27 = 9 \times 3$, divisible by 9

and **so the result is true for $n = 1$.**

Assume that the result is true for $n = k$

$$\Rightarrow f(k) = (3k+2) \times 4^k + 7 = 9m$$

Here we cannot eliminate the power of 4 so we try

$$\begin{aligned} f(k+1) - f(k) &= (3k+5) \times 4^{k+1} + 7 - (3k+2) \times 4^k - 7 \\ &= 12k \times 4^k + 20 \times 4^k + 7 - 3k \times 4^k - 2 \times 4^k - 7 \\ &= 9k \times 4^k + 18 \times 4^k \\ &= 9 \times 4^k (k+2) \end{aligned}$$

you must now make $f(k+1)$ the subject of the formula

$$\begin{aligned} \Rightarrow f(k+1) &= f(k) + 9 \times 4^k (k+2) = 9m + 9 \times 4^k (k+2) \\ &= 9(m + 9 \times 4^k (k+2)) \quad \text{which is a multiple of 9} \end{aligned}$$

$\Rightarrow f(k+1)$ is divisible by 9.

We have shown that the statement is true for $n = 1$, and that IF it is true when $n = k$, then it must also be true when $n = k + 1$

$\Rightarrow f(n) = (3k+2) \times 4^k + 7$ is divisible by 9 for all positive integers, n , by induction.

Powers of matrices

Example: If $M = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$, prove that $M^n = \begin{pmatrix} 2^n & 1 - 2^n \\ 0 & 1 \end{pmatrix}$ for all positive integers n .

Solution: When $n = 1$, $M^1 = \begin{pmatrix} 2^1 & 1 - 2^1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = M$

$\Rightarrow$ the formula is true for $n = 1$.

Assume the formula is true for $n = k \Rightarrow M^k = \begin{pmatrix} 2^k & 1 - 2^k \\ 0 & 1 \end{pmatrix}$.

$$M^{k+1} = MM^k = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2^k & 1 - 2^k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 \times 2^k & 2 - 2 \times 2^k - 1 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow M^{k+1} = \begin{pmatrix} 2^{k+1} & 1 - 2^{k+1} \\ 0 & 1 \end{pmatrix} \Rightarrow \text{The formula is true for } n = k + 1$$

We have shown that the formula is true for $n = 1$, and that IF it is true when $n = k$, then it must also be true when $n = k + 1$

$\Rightarrow M^n = \begin{pmatrix} 2^n & 1 - 2^n \\ 0 & 1 \end{pmatrix}$ is true for all positive integers, n , by induction.

8 Appendix

Complex roots of a real polynomial equation

Preliminary results:

$$\begin{aligned} \text{I} \quad & (z_1 + z_2 + z_3 + z_4 + \dots + z_n)^* = z_1^* + z_2^* + z_3^* + z_4^* + \dots + z_n^*, \\ & \text{by repeated application of } (z + w)^* = z^* + w^* \\ \text{II} \quad & (z^n)^* = (z^*)^n \\ & (zw)^* = z^*w^* \\ \Rightarrow & (z^n)^* = (z^{n-1}z)^* = (z^{n-1})^*(z)^* = (z^{n-2}z)^*(z)^* = (z^{n-2})^*(z)^*(z)^* \dots = (z^*)^n \end{aligned}$$

Theorem: If $z = a + bi$ is a root of $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \alpha_{n-2} z^{n-2} + \dots + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$,
and if all the α_i are real,
 then the conjugate, $z^* = a - bi$ is also a root.

Proof: If $z = a + bi$ is a root of the equation $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \dots + \alpha_1 z + \alpha_0 = 0$

$$\text{then } \alpha_n z^n + \alpha_{n-1} z^{n-1} + \dots + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$$

$$\Rightarrow (\alpha_n z^n + \alpha_{n-1} z^{n-1} + \dots + \alpha_2 z^2 + \alpha_1 z + \alpha_0)^* = 0 \quad \text{since } 0^* = 0$$

$$\Rightarrow (\alpha_n z^n)^* + (\alpha_{n-1} z^{n-1})^* + \dots + (\alpha_2 z^2)^* + (\alpha_1 z)^* + (\alpha_0)^* = 0 \quad \text{using I}$$

$$\Rightarrow \alpha_n^* (z^n)^* + \alpha_{n-1}^* (z^{n-1})^* + \dots + \alpha_2^* (z^2)^* + \alpha_1^* (z)^* + \alpha_0^* = 0 \quad \text{since } (zw)^* = z^*w^*$$

$$\Rightarrow \alpha_n (z^n)^* + \alpha_{n-1} (z^{n-1})^* + \dots + \alpha_2 (z^2)^* + \alpha_1 (z)^* + \alpha_0 = 0 \quad \alpha_i \text{ real} \Rightarrow \alpha_i^* = \alpha_i$$

$$\Rightarrow \alpha_n (z^*)^n + \alpha_{n-1} (z^*)^{n-1} + \dots + \alpha_2 (z^*)^2 + \alpha_1 (z^*) + \alpha_0 = 0 \quad \text{using II}$$

$$\Rightarrow z^* = a - bi \text{ is also a root of the equation.}$$

Formal definition of a linear transformation

A linear transformation T has the following properties:

$$(i) \quad T \begin{pmatrix} kx \\ ky \end{pmatrix} = kT \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(ii) \quad T \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) = T \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

It can be shown that **any** matrix transformation is a linear transformation, and that **any** linear transformation can be represented by a matrix.

Roots of higher degree equations

Roots of cubic equations

Symmetric functions of the roots

Suppose that the equation $ax^3 + bx^2 + cx + d = 0$ has roots α , β and γ .

$\Rightarrow (x - \alpha), (x - \beta)$ and $(x - \gamma)$ are factors

There might also be a constant factor k

$$\Rightarrow ax^3 + bx^2 + cx + d \equiv k(x - \alpha)(x - \beta)(x - \gamma)$$

$$\Rightarrow ax^3 + bx^2 + cx + d \equiv kx^3 - k(\alpha + \beta + \gamma)x^2 + k(\alpha\beta + \beta\gamma + \gamma\alpha)x - k\alpha\beta\gamma$$

As this is an identity we can equate coefficients

$$x^3: a = k$$

$$x^2: b = -k(\alpha + \beta + \gamma) \Rightarrow \sum \alpha = \alpha + \beta + \gamma = \frac{-b}{k} = \frac{-b}{a}$$

$$x: c = k(\alpha\beta + \beta\gamma + \gamma\alpha) \Rightarrow \sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$x^0: d = -k\alpha\beta\gamma \Rightarrow \alpha\beta\gamma = \frac{-d}{k} = \frac{-d}{a}$$

Roots of quartic equations

Symmetric functions of the roots

Suppose that the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ has roots α , β , γ and δ .

In a similar way, we can show that for the symmetric functions of the roots

$$\sum \alpha = \alpha + \beta + \gamma + \delta = \frac{-b}{a}$$

$$\sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta = \frac{c}{a}$$

$$\sum \alpha\beta\gamma = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = \frac{-d}{a}$$

$$\alpha\beta\gamma\delta = \frac{e}{a}$$

Note that in a symmetric function, any pair of letters can be interchanged without changing the expression:

thus, interchanging β and δ in $\alpha + \beta + \gamma + \delta$ gives $\alpha + \delta + \gamma + \beta$, which leaves the expression unchanged,

and interchanging α and β in $\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta$ gives $\beta\alpha\gamma + \alpha\gamma\delta + \gamma\delta\beta + \delta\beta\alpha$, which leaves the expression unchanged.

The symmetric functions of the roots of higher degree equations follow a similar pattern.

Derivative of x^n , for any integer

We can use proof by induction to show that $\frac{d}{dx}(x^n) = nx^{n-1}$, for any integer n .

1) We know that the derivative of x^0 is 0 which equals $0x^{-1}$,

since $x^0 = 1$, and the derivative of 1 is 0

$$\Rightarrow \frac{d}{dx}(x^n) = nx^{n-1} \text{ is true for } n = 0.$$

2) We know that the derivative of x^1 is 1 which equals $1 \times x^{1-1}$

$$\Rightarrow \frac{d}{dx}(x^n) = nx^{n-1} \text{ is true for } n = 1$$

Assume that the result is true for $n = k$

$$\Rightarrow \frac{d}{dx}(x^k) = kx^{k-1}$$

$$\Rightarrow \frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \times x^k) = x \times \frac{d}{dx}(x^k) + 1 \times x^k \quad \text{product rule}$$

$$\Rightarrow \frac{d}{dx}(x^{k+1}) = x \times kx^{k-1} + x^k = kx^k + x^k = (k+1)x^k$$

$$\Rightarrow \frac{d}{dx}(x^n) = nx^{n-1} \text{ is true for } n = k+1$$

$$\Rightarrow \frac{d}{dx}(x^n) = nx^{n-1} \text{ is true for all positive integers, } n, \text{ by mathematical induction.}$$

3) We know that the derivative of x^{-1} is $-x^{-2}$ which equals $-1 \times x^{-1-1}$

$$\Rightarrow \frac{d}{dx}(x^n) = nx^{n-1} \text{ is true for } n = -1$$

Assume that the result is true for $n = k$

$$\Rightarrow \frac{d}{dx}(x^k) = kx^{k-1}$$

$$\Rightarrow \frac{d}{dx}(x^{k-1}) = \frac{d}{dx}\left(\frac{x^k}{x}\right) = \frac{x \times \frac{d}{dx}(x^k) - x^k \times 1}{x^2} \quad \text{quotient rule}$$

$$\Rightarrow \frac{d}{dx}(x^{k+1}) = \frac{x \times kx^{k-1} - x^k}{x^2} = \frac{(k-1)x^k}{x^2} = (k-1)x^{k-2} = (k-1)x^{(k-1)-1}$$

$$\Rightarrow \frac{d}{dx}(x^n) = nx^{n-1} \text{ is true for } n = k-1$$

We are going backwards (**from** $n = k$ **to** $n = k-1$), and, since we started from $n = -1$,

$$\Rightarrow \frac{d}{dx}(x^n) = nx^{n-1} \text{ is true for all negative integers, } n, \text{ by mathematical induction.}$$

Putting 1), 2) and 3), we have proved that

$$\frac{d}{dx}(x^n) = nx^{n-1}, \text{ for any integer } n.$$

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